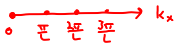
<u>Free-electron gas in 1D</u>

$$\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} x\right)$$

$$E_n = \frac{\hbar^2}{2m} \frac{n^2\pi^2}{L^2} = \frac{\hbar^2}{2m} k^2 \qquad \vec{k} = \frac{n\pi}{L} \hat{i}$$



For $\frac{N_f}{2}$ total pairs we populate $\frac{N_f}{2}\frac{\pi}{L}$ volume of k-space at $T \to 0$.

Then: $k_{max} = \frac{N_f}{2}\frac{\pi}{L} = k_F \Rightarrow E_F = \frac{\hbar^2}{2m}\frac{N_f^2}{4}\frac{\pi^2}{L^2}$

$$E_{tot} = \int dE = \int E\, dN = \int E \underbrace{\rho_k}_{\frac{2L}{\pi}} \overbrace{dV}^{dk} = \int_0^{k_F} \frac{\hbar^2}{2m} k^2 \frac{2L}{\pi} dk$$

$$= \frac{L\hbar^2}{3m\pi} k_F^3$$

$$= \frac{\hbar^2\pi^2}{24m} \frac{1}{L^2} N_f^3 \qquad \Rightarrow P = -\frac{dE_{tot}}{dL}$$

<u>For bosons</u>

$$E_{tot} = N_b \frac{k^2}{2m} \frac{\pi^2}{L^2}$$

In 1D: $E^b_{tot} \sim N_b \qquad E^f_{tot} \sim N_f^3$

3D: $E^b_{tot} \sim N_b \qquad E^f_{tot} \sim N_f^{5/3}$

In higher dimensions the distribution of states will "spread" out in k-space.

$$E_{tot} \propto k^2 \Rightarrow E_{tot}^{\frac{1}{2}} \propto k$$

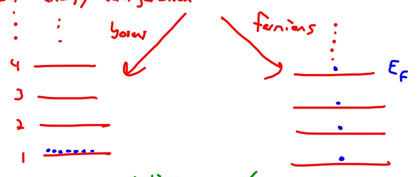
N (# of states) $\propto k^3$ in 3D $\Rightarrow N^{\frac{1}{3}} \propto k \Rightarrow E_{tot} \propto N^{\frac{2}{3}}$

$\propto k$ in 1D $\Rightarrow N \propto k \Rightarrow \underbrace{E_{tot} \propto N^2}$

difference in behavior btwn bosons and fermions

Quantum Statistical Mechanics

As $T \to 0$ a multi-particle system goes to lowest energy configuration



$N_1 = N$

$N_{n>1} = 0$

occupation numbers

$N_1 = 1$

$N_2 = 1$

$\vdots$

$N_F = 1$

$N_{n>F} = 0$

What happens when $T \neq 0$?

Basic postulate of statistical mechanics.

For a given E_{tot}, every microscopic configuration compatible w/ E_{tot} is equally likely.

So if the total # of micro. config. is Y then the probability of any one of those configurations is $P_i = \frac{1}{Y}$.

But if a set of configurations corresponds to the same set of occupation #'s, then $P_{set} = \frac{1}{Y} + \frac{1}{Y} \cdots$.

So the most probable set of occupation #'s corresponds to the set of N_n's that can be realized in the most ways.

# of microstates for a choice of occupation #'s N_n, is $Q(N_1, N_2, \cdots)$

$N = N_1 + N_2 + N_3 + N_4 + N_5$

$Q(0, 0, \overset{\text{Easy Ex.}}{N}, 0, 0) = 1$

$Q(N_1, N_2, N_3, N_4, N_5) \neq 1$

Today: We will calculate Q's

Next time: We extremize the Qs to find most probable occupation #'s.

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Counting (Combinatorics)

To calculate $Q(N_1, N_2, \cdots)$ we start w/ a set of particles N.

Now choose a set of occupation numbers $(N_1, N_2, \cdots)$ comp. w/ $N = N_1 + N_2 + \cdots$

The reason this is nontrivial is due to degeneracy.

E_1	E_2	E_3	E_4	$\cdots$
$d_1 = 3$	$d_2 = 3$	$d_3 = 2$	$d_4 = 1$	$\cdots$
N_1	N_2	N_3	N_4	$\cdots$

2 step method:
1. Choose N_1 of the N total particles.
2. Place N_1 particles in E_1 bins

Distinct Particles:

1. N ways to choose particle 1
   $N-1$ " " " " 2
   $N-2$ " " " " 3
   $N-N_1+1$ " " " N_1

   $N(N-1)(N-2)\cdots(N-N_1+1)$ ways to choose but these are ordered. To remove ordering divide by $N_1!$.

   In total the number of ways to select N_1 particles from N w/out ordering is
   $$\frac{N(N-1)\cdots(N-N_1+1)}{N_1!} = \frac{N!}{N_1!(N-N_1)!} \equiv \binom{N}{N_1}$$

2. We have N_1 particles to distribute to d_1 bins.
   Particle 1 can go into d_1 bins.
   Particle 2 " " " d_1 bins.
   $d_1^{N_1}$ ways to distribute

Total # of ways to choose N_1 particles and distribute among d_1 bins is: $\binom{N}{N_1} d_1^{N_1}$

For E_2, choose N_2 from $N-N_1$

$$\binom{N-N_1}{N_2} d_2^{N_2}$$

The total # of ways of populating $N_1, N_2, \cdots$

$$Q_d(N_1, N_2, \cdots) = \frac{N!\, d_1^{N_1}}{N_1!(N-N_1)!} \cdot \frac{(N-N_1)!\, d_2^{N_2}}{N_2!(N-N_1-N_2)!} \cdots$$

$$= N! \prod_{n=1}^{\infty} \frac{d_n^{N_n}}{N_n!}$$

Identical Particles: 1. How many ways can I choose N_1 particles from N.

$\Rightarrow$ Only 1 way.

Fermions: N_1 particles to distribute among d_1 bins.

We need $d_1 > N_1$

So for d_1 bins just pick N_1 of them.

$$Q_f(N_1, N_2, \cdots) = \prod_{n=1}^{\infty} \binom{d_n}{N_n}$$

Example of counting w/ distinguishable particles.

(a b c d) $N=4$ total

$N=4$

$N_1=3$

choose 3 to go in E_1 level

$4 \cdot 3 \cdot 2 = 24$

acd, adc

abc, acb

abd, adb

bac, bca

bad, bda

bcd, bdc

cdb, cbd

cab, cba

cad, cda

dab, dba

dac, dca

dbc, dcb

If I just choose 3 out of 4, there are 24 possibilities (or $\underset{\text{4 ways to choose 1st}}{4} \cdot \underset{\text{3 ways to choose 2nd}}{3} \cdot \underset{\text{2 ways to choose 3rd}}{2}$)

But we don't want to treat $\overbrace{\text{abc, acb, cab, cba, bca, bac}}^{6}$ as separate choices. So divide $24/6 = 4$ ways.

counts # of ways to order 3 objects

In general, for N_1 objects you can order them in $N_1!$ ways, so to remove ordering you divide by $N_1!$