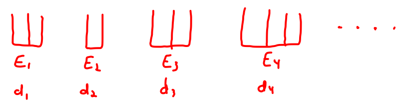
Quantum Stat Mech cont.

N_1 ⁘ N_2 ⁘ N_3 ⁘ N_4 ⁘ $\sum_n N_n = N$



$Q(N_1, N_2, \cdots) \equiv$ # of ways to choose and place $N = N_1 + N_2 + \cdots$ particles into bins.

0. Do each energy level, then multiply results $\prod_{n=1}^{\infty}$

1. First count the ways of choosing into N_n particles from N.

2. Then count # of ways to put N_n particles into E_n w/ d_n.

$\binom{a}{b} \equiv \frac{a!}{b!(a-b)!}$ = # of ways of choosing "b" distinct elements of "a", where we <u>don't</u> care about ordering.

e.g. $\uparrow, \downarrow, \rightarrow$ $a = 3$ For $b = 2$: $\overbrace{\uparrow\downarrow \sim \downarrow\uparrow}^{6 \text{ total}}$

$\left.\begin{array}{l} \uparrow\downarrow \sim \downarrow\uparrow \\ \uparrow\rightarrow \sim \rightarrow\uparrow \\ \downarrow\rightarrow \sim \rightarrow\downarrow \end{array}\right\}$ 3 distinct choices w/out ordering

$\frac{3!}{2!(1!)} = 3$

<u>Note</u>: Arises differently

Used in step 1 for distinguishable particles. $\binom{N}{N_1}, \binom{N-N_1}{N_2}, \cdots$

Used in step 2 for fermions. $\binom{d_n}{N_n}$ $d_n > N_n$

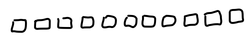
		<u>Step 1</u>	<u>Step 2</u>
<u>Dist</u>	$Q_d(N_1, N_2, \cdots) = N! \prod_{n=1}^{\infty} \frac{d_n^{N_n}}{N_n!}$	$\binom{N - \sum_{i<n} N_i}{N_n}$	$d_n^{N_n}$
<u>Fermions</u>	$Q_f(N_1, N_2, \cdots) = \prod_{n=1}^{\infty} \binom{d_n}{N_n}$	1	$\binom{d_n}{N_n}$
<u>Bosons</u>	$Q_b(N_1, N_2, \cdots) = \prod_{n=1}^{\infty} \binom{N_n + d_n - 1}{N_n}$	1	$\binom{N_n + d_n - 1}{N_n}$

Identical Bosons: $\underbrace{\bullet\ \bullet}_{\text{bin}} \times \underbrace{\bullet}_{\text{bin}} \times \underbrace{\bullet\bullet\bullet\bullet}_{\text{bin}} \times \underbrace{\bullet}_{\text{bin}}$

$\bullet$'s $= N_n$

X's $= d_n - 1$

Consider:



$\times\ \times\ \times\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$

$N_n + d_n - 1$ total elements

Distributing X and $\bullet$ gives $(N_n + d_n - 1)!$

To ignore order of X's $\frac{1}{(d_n - 1)!}$

To ignore order of $\bullet$'s $\frac{1}{N_n!}$

$\frac{(N_n + d_n - 1)!}{N_n! \underbrace{(d_n - 1)}_{N_n + d_n - 1 - N_n}!}$

Dist $Q_d(N_1, N_2, \cdots) = N! \prod_{n=1}^{\infty} \frac{d_n^{N_n}}{N_n!}$

Fermion $Q_f(N_1, N_2, \cdots) = \prod_{n=1}^{\infty} \binom{d_n}{N_n}$

Bosons $Q_b(N_1, N_2, \cdots) = \prod_{n=1}^{\infty} \binom{N_n + d_n - 1}{N_n}$

$d_n = 1$

$Q_d = N! \prod_{n=1}^{\infty} \frac{1}{N_n!}$

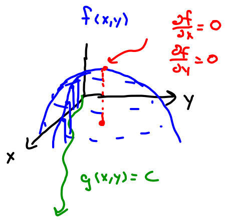
$Q_f = 1$

$Q_b = 1$

We got Q's ... what now?

Recall: Most probable set of occupation numbers $(N_1, N_2, \cdots)$ comes from maximizing $Q(N_1, N_2, \cdots)$ subject to $N_1 + N_2 + \cdots = N_{tot}$ and $N_1E_1 + N_2E_2 + \cdots = E_{tot}$.

$\Rightarrow$ Lagrange Multipliers



To extremize $f(x,y)$ subject to $g(x,y) = c$

build: $G(x,y,\lambda) = f(x,y) + \lambda(g(x,y) - c)$

Now find stationary points of G w.r.t. x, y, λ.

separates products into sums

$$G = \ln Q + \alpha\left[N - \sum_n N_n\right] + \beta\left[E - \sum_n N_n E_n\right]$$

Then: $\dfrac{\partial G}{\partial N_n} = 0$

<u>For bosons:</u>

$$G_b = \sum_{n=1}^{\infty}\left[\ln\{(N_n+d_n-1)!\} - \ln(N_n!) - \ln\{(d_n-1)!\}\right.$$
$$+ \alpha\left[N - \sum_{n=1}^{\infty} N_n\right] + \beta\left[E - \sum_{n=1}^{\infty} N_n E_n\right]$$

$$Q_b = \prod_{n=1}^{\infty}\binom{N_n+d_n-1}{N_n} = \prod_{n=1}^{\infty}\frac{(N_n+d_n-1)!}{N_n!\,(d_n-1)!}$$

$\ln ab = \ln a + \ln b$
$\ln \frac{a}{b} = \ln a - \ln b$

Use Stirling's approx.: $\boxed{\ln(z!) \simeq z\ln(z) - z \quad \text{if } z \gg 1}$

$$G_b \simeq \alpha N + \beta E + \sum_{n=1}^{\infty}\left\{(N_n+d_n-1)\ln(N_n+d_n-1) - N_n - d_n + 1\right.$$
$$- N_n\ln(N_n) + N_n - (d_n-1)\ln(d_n-1)$$
$$\left. + d_n - 1 - \alpha N_n - \beta N_n E_n\right\}$$

$$\frac{\partial G_b}{\partial N_n} = \ln(N_n+d_n-1) - \ln(N_n) - \alpha - \beta E_n = 0$$

$$N_n = \frac{d_n - 1}{e^{(\alpha+\beta E_n)} + 1}$$