ReVIeW: 2 particles (1,2) in 2 states (a,b)

In QM

Distinguishable Particles: e.g. (electron, muon)

$\psi_{tot} = \psi_a(x_1)\psi_b(x_2) \Rightarrow P_{12}\psi_{tot} = \psi_{new}$

Indistinguishable Particles: e.g. (electron, electron)

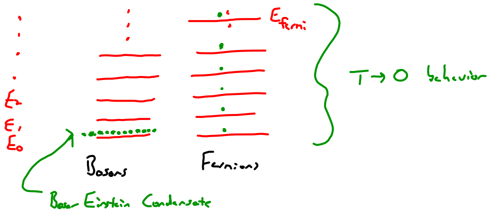
"Bosons" $\psi_{tot} = \frac{1}{\sqrt{2}}\left(\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) + \psi_b(\vec{r}_1)\psi_a(\vec{r}_2)\right) \Rightarrow \underbrace{P_{12}\psi_{tot} = \psi_{tot}}_{\text{Symmetric under part. exchange}}$

"Fermions" $\underbrace{\psi_{tot} = \frac{1}{\sqrt{2}}\left(\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) - \psi_b(\vec{r}_1)\psi_a(\vec{r}_2)\right)}_{} \Rightarrow \underbrace{P_{12}\psi_{tot} = -\psi_{tot}}_{\text{antisymmetric}}$

If $a=b \Rightarrow \psi_{tot} = 0$ "Pauli Exclusion Principles"

Several effects from QM symmetrization requirements

Statistics Start w/ a system V(x) w/ energy levels



At high T particles tend to spread out so differences btwn bosons and fermions disappear.

What changes for distinguishable particles?

Dist. Maxwell-Boltzmann Statistics

Indist. Bose-Einstein "

Fermi-Dirac "

} at high T are the same

Spin-Statistics Theorem

"Bosons" – any particles w/ integer spin

e.g. (spin-0 Higgs boson , spin-1 photon, strong and weak force particles

spin-2 graviton)

"Fermions" – any particles w/ ½-integer spin

e.g. (spin-½ all matter)

All of this can be proven in QFT.

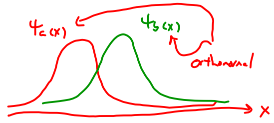
Exchange Force

Go back to 2 particles

Symmetrization creates "force"

2 particles (1,2), 2 states (a,b), 1-dimension

$$\left.\begin{aligned}\psi_{tot} &= \psi_a(x_1)\psi_b(x_2)\\ \psi_{tot} &= \frac{1}{\sqrt{2}}\left(\psi_a(x_1)\psi_b(x_2) \pm \psi_b(x_1)\psi_a(x_2)\right)\end{aligned}\right\} \psi_{tot} = f\left(\psi_a(x_1)\psi_b(x_2) + g\,\psi_b(x_1)\psi_a(x_2)\right)$$



Dist. $f=1$ $g=0$

Bos. $f=\frac{1}{\sqrt{2}}$ $g=+1$

Ferm. $f=\frac{1}{\sqrt{2}}$ $g=-1$

Compute: $\langle (x_1-x_2)^2\rangle = \langle x_1^2 + x_2^2 - 2x_1x_2\rangle = \langle x_1^2\rangle + \langle x_2^2\rangle - 2\langle x_1 x_2\rangle$

Start with:

$$\langle\ \ \rangle_{tot} = f^2\Big\{\int \cancelto{1}{\psi_a^*\psi_a}\, dx_1 \int \cancelto{1}{\psi_b^*\psi_b}\, dx_2 + g^2 \int \cancelto{1}{\psi_b^*\psi_b}\, dx_1 \int \cancelto{1}{\psi_a^*\psi_a}\, dx_2$$
$$+ g\int \cancelto{0}{\psi_a^*\psi_b}\, dx_1 \int \cancelto{0}{\psi_b^*\psi_a}\, dx_2 + g\int \cancelto{0}{\psi_b^*\psi_a}\, dx_1 \int \cancelto{0}{\psi_a^*\psi_b}\, dx_2\Big\}$$

$$\langle x_1^2\rangle_{tot} = f^2\Big\{\int x_1^2\psi_a^*\psi_a\, dx_1 \int \cancelto{1}{\psi_b^*\psi_b}\, dx_2 + g^2 \int x_1^2\psi_b^*\psi_b\, dx_1 \int \cancelto{1}{\psi_a^*\psi_a}\, dx_2$$
$$+ g\int x_1^2\psi_a^*\psi_b\, dx_1 \int \cancelto{0}{\psi_b^*\psi_a}\, dx_2 + g\int x_1^2\psi_b^*\psi_a\, dx_1 \int \cancelto{0}{\psi_a^*\psi_b}\, dx_2\Big\}$$
$$= f^2\{\langle x_1^2\rangle_a + g^2\langle x_1^2\rangle_b\}$$

$$\langle x_2^2\rangle_{tot} = f^2\{\langle x_2^2\rangle_a + g^2\langle x_2^2\rangle_b\}$$

$$\langle x_1x_2\rangle_{tot} = f^2\Big\{\int x_1\psi_a^*\psi_a\, dx_1 \int x_2\psi_b^*\psi_b\, dx_2 + g^2 \int x_1\psi_b^*\psi_b\, dx_1 \int x_2\psi_a^*\psi_a\, dx_2$$
$$+ g\int x_1\psi_a^*\psi_b\, dx_1 \int x_2\psi_b^*\psi_a\, dx_2 + g\int x_1\psi_b^*\psi_a\, dx_1 \int x_2\psi_a^*\psi_b\, dx_2\Big\}$$
$$= f^2\Big\{\langle x_1\rangle_a\langle x_2\rangle_b + g^2\langle x_1\rangle_b\langle x_2\rangle_a + g\langle x_1\rangle_{ab}\langle x_2\rangle_{ba} + g\langle x_1\rangle_{ba}\langle x_2\rangle_{ab}\Big\}$$

$$\langle x_1\rangle_{ab}\left(\langle x_2\rangle_{ab}\right)^* = |\langle x\rangle_{ab}|^2$$

Note: No reason to distinguish $\langle x_1\rangle_a = \langle x_2\rangle_a \equiv \langle x\rangle_a$

$$\langle (x_1-x_2)^2\rangle = \underbrace{f^2\{2\langle x^2\rangle_a + 2g^2\langle x^2\rangle_b - 2\langle x\rangle_a\langle x\rangle_b - 2g^2\langle x\rangle_a\langle x\rangle_b\}}_{\text{for } f=1\ g=0,\ f=\frac{1}{\sqrt{2}}\ g=\pm 1 \text{ gives same thing}}$$
$$\underbrace{-4f^2 g\,|\langle x\rangle_{ab}|^2}_{\text{the only difference}}$$

Dist. 0

Bosons $-2\left|\int x\psi_a^*\psi_b\, dx\right|^2$ attraction

Fermions $+2\left|\underbrace{\int x\psi_a^*\psi_b\, dx}_{\text{only relevant when wavefunctions overlap}}\right|^2$ repulsion