Problem 2.10

(a) Using Eqs. 2.47 and 2.59,

$$a_+\psi_0 = \frac{1}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{d}{dx} + m\omega x\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

$$= \frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[-\hbar\left(-\frac{m\omega}{2\hbar}\right)2x + m\omega x\right]e^{-\frac{m\omega}{2\hbar}x^2} = \frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} 2m\omega x e^{-\frac{m\omega}{2\hbar}x^2}.$$

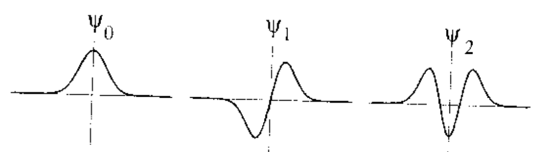
$$(a_+)^2\psi_0 = \frac{1}{2\hbar m\omega}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} 2m\omega\left(-\hbar\frac{d}{dx} + m\omega x\right) xe^{-\frac{m\omega}{2\hbar}x^2}$$

$$= \frac{1}{\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[-\hbar\left(1 - x\frac{m\omega}{2\hbar}2x\right) + m\omega x^2\right]e^{-\frac{m\omega}{2\hbar}x^2} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)e^{-\frac{m\omega}{2\hbar}x^2}.$$

Therefore, from Eq. 2.67,

$$\psi_2 = \frac{1}{\sqrt{2}}(a_+)^2\psi_0 = \boxed{\frac{1}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)e^{-\frac{m\omega}{2\hbar}x^2}.}$$

(b)



(c) Since ψ_0 and ψ_2 are even, whereas ψ_1 is odd, $\int \psi_0^*\psi_1 dx$ and $\int \psi_2^*\psi_1 dx$ vanish automatically. The only one we need to check is $\int \psi_2^*\psi_0\, dx$:

$$\int \psi_2^*\psi_0\, dx = \frac{1}{\sqrt{2}}\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)e^{-\frac{m\omega}{\hbar}x^2}dx$$

$$= -\sqrt{\frac{m\omega}{2\pi\hbar}}\left(\int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}x^2}dx - \frac{2m\omega}{\hbar}\int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2}dx\right)$$

$$= -\sqrt{\frac{m\omega}{2\pi\hbar}}\left(\sqrt{\frac{\pi\hbar}{m\omega}} - \frac{2m\omega}{\hbar}\frac{\hbar}{2m\omega}\sqrt{\frac{\pi\hbar}{m\omega}}\right) = 0. \checkmark$$

Problem 2.11

(a) Note that ψ_0 is even, and ψ_1 is odd. In either case $|\psi|^2$ is even, so $\langle x\rangle = \int x|\psi|^2 dx = \boxed{0.}$ Therefore $\langle p\rangle = md\langle x\rangle/dt = \boxed{0.}$ (These results hold for *any* stationary state of the harmonic oscillator.)

From Eqs. 2.59 and 2.62, $\psi_0 = \alpha e^{-\xi^2/2}$, $\psi_1 = \sqrt{2}\alpha\xi e^{-\xi^2/2}$. So

<u>$n = 0$</u>:

$$\langle x^2\rangle = \alpha^2\int_{-\infty}^{\infty} x^2 e^{-\xi^2/2}dx = \alpha^2\left(\frac{\hbar}{m\omega}\right)^{3/2}\int_{-\infty}^{\infty}\xi^2 e^{-\xi^2}d\xi = \frac{1}{\sqrt{\pi}}\left(\frac{\hbar}{m\omega}\right)\frac{\sqrt{\pi}}{2} = \boxed{\frac{\hbar}{2m\omega}.}$$

$$\langle p^2\rangle = \int \psi_0\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2\psi_0\,dx = -\hbar^2\alpha^2\sqrt{\frac{m\omega}{\hbar}}\int_{-\infty}^{\infty} e^{-\xi^2/2}\left(\frac{d^2}{d\xi^2}e^{-\xi^2/2}\right)d\xi$$
$$= -\frac{m\hbar\omega}{\sqrt{\pi}}\int_{-\infty}^{\infty}(\xi^2-1)e^{-\xi^2/2}d\xi = -\frac{m\hbar\omega}{\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{2}-\sqrt{\pi}\right) = \boxed{\frac{m\hbar\omega}{2}.}$$

$\underline{n=1}$:

$$\langle x^2\rangle = 2\alpha^2\int_{-\infty}^{\infty}x^2\xi^2e^{-\xi^2}dx = 2\alpha^2\left(\frac{\hbar}{m\omega}\right)^{3/2}\int_{-\infty}^{\infty}\xi^4e^{-\xi^2}d\xi = \frac{2\hbar}{\sqrt{\pi}m\omega}\frac{3\sqrt{\pi}}{4} = \boxed{\frac{3\hbar}{2m\omega}.}$$

$$\langle p^2\rangle = -\hbar^2 2\alpha^2\sqrt{\frac{m\omega}{\hbar}}\int_{-\infty}^{\infty}\xi e^{-\xi^2/2}\left[\frac{d^2}{d\xi^2}(\xi e^{-\xi^2/2})\right]d\xi$$
$$= -\frac{2m\omega\hbar}{\sqrt{\pi}}\int_{-\infty}^{\infty}(\xi^4-3\xi^2)e^{-\xi^2}d\xi = -\frac{2m\omega\hbar}{\sqrt{\pi}}\left(\frac{3}{4}\sqrt{\pi}-3\frac{\sqrt{\pi}}{2}\right) = \boxed{\frac{3m\hbar\omega}{2}.}$$

(b) $\underline{n=0}$:

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{\hbar}{2m\omega}};\ \sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{m\hbar\omega}{2}};$$

$$\sigma_x\sigma_p = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{m\omega\hbar}{2}} = \frac{\hbar}{2}.\quad \text{(Right } at \text{ the uncertainty limit.)}\checkmark$$

$\underline{n=1}$:

$$\sigma_x = \sqrt{\frac{3\hbar}{2m\omega}};\quad \sigma_p = \sqrt{\frac{3m\hbar\omega}{2}};\quad \sigma_x\sigma_p = 3\frac{\hbar}{2} > \frac{\hbar}{2}.\ \checkmark$$

(c)

$$\langle T\rangle = \frac{1}{2m}\langle p^2\rangle = \boxed{\left\{\begin{array}{l}\frac{1}{4}\hbar\omega\ (n=0)\\ \frac{3}{4}\hbar\omega\ (n=1)\end{array}\right\};}\quad \langle V\rangle = \frac{1}{2}m\omega^2\langle x^2\rangle = \boxed{\left\{\begin{array}{l}\frac{1}{4}\hbar\omega\ (n=0)\\ \frac{3}{4}\hbar\omega\ (n=1)\end{array}\right\}.}$$

$$\langle T\rangle + \langle V\rangle = \langle H\rangle = \left\{\begin{array}{l}\frac{1}{2}\hbar\omega\ (n=0) = E_0\\ \frac{3}{2}\hbar\omega\ (n=1) = E_1\end{array}\right\},\ \text{as expected.}$$

Problem 2.12

From Eq. 2.69,

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-),\quad p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-),$$

so

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\int\psi_n^*(a_+ + a_-)\psi_n\,dx.$$

But (Eq. 2.66)

$$a_+\psi_n = \sqrt{n+1}\,\psi_{n+1}, \quad a_-\psi_n = \sqrt{n}\,\psi_{n-1}.$$

So

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\int \psi_n^*\psi_{n+1}\,dx + \sqrt{n}\int \psi_n^*\psi_{n-1}\,dx\right] = \boxed{0} \text{ (by orthogonality).}$$

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = \boxed{0.} \quad x^2 = \frac{\hbar}{2m\omega}(a_+ + a_-)^2 = \frac{\hbar}{2m\omega}(a_+^2 + a_+a_- + a_-a_+ + a_-^2).$$

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\int \psi_n^*(a_+^2 + a_+a_- + a_-a_+ + a_-^2)\psi_n. \quad \text{But}$$

$$\begin{cases} a_+^2\psi_n &= a_+\left(\sqrt{n+1}\,\psi_{n+1}\right) = \sqrt{n+1}\sqrt{n+2}\,\psi_{n+2} = \sqrt{(n+1)(n+2)}\,\psi_{n+2}. \\ a_+a_-\psi_n &= a_+\left(\sqrt{n}\,\psi_{n-1}\right) \quad = \sqrt{n}\sqrt{n}\,\psi_n \qquad = n\psi_n. \\ a_-a_+\psi_n &= a_-\left(\sqrt{n+1}\,\psi_{n+1}\right) = \sqrt{n+1}\sqrt{n+1}\,\psi_n \quad = (n+1)\psi_n. \\ a_-^2\psi_n &= a_-\left(\sqrt{n}\,\psi_{n-1}\right) \quad = \sqrt{n}\sqrt{n-1}\,\psi_{n-2} \quad = \sqrt{(n-1)n}\,\psi_{n-2}. \end{cases}$$

So

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\left[0 + n\int|\psi_n|^2dx + (n+1)\int|\psi_n|^2\,dx + 0\right] = \frac{\hbar}{2m\omega}(2n+1) = \boxed{\left(n+\frac{1}{2}\right)\frac{\hbar}{m\omega}.}$$

$$p^2 = -\frac{\hbar m\omega}{2}(a_+ - a_-)^2 = -\frac{\hbar m\omega}{2}(a_+^2 - a_+a_- - a_-a_+ + a_-^2) \Rightarrow$$

$$\langle p^2\rangle = -\frac{\hbar m\omega}{2}[0 - n - (n+1) + 0] = \frac{\hbar m\omega}{2}(2n+1) = \boxed{\left(n+\frac{1}{2}\right)m\hbar\omega.}$$

$$\langle T\rangle = \langle p^2/2m\rangle = \boxed{\frac{1}{2}\left(n+\frac{1}{2}\right)\hbar\omega}.$$

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{n+\frac{1}{2}}\sqrt{\frac{\hbar}{m\omega}}; \quad \sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{n+\frac{1}{2}}\sqrt{m\hbar\omega}; \quad \sigma_x\sigma_p = \left(n+\frac{1}{2}\right)\hbar \geq \frac{\hbar}{2}. \checkmark$$

Problem 2.22

(a)

$$1 = |A|^2 \int_{-\infty}^{\infty} e^{-2ax^2} dx = |A|^2 \sqrt{\frac{\pi}{2a}}; \quad \boxed{A = \left(\frac{2a}{\pi}\right)^{1/4}.}$$

(b)

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx)} dx = \int_{-\infty}^{\infty} e^{-y^2+(b^2/4a)} \frac{1}{\sqrt{a}}\, dy = \frac{1}{\sqrt{a}} e^{b^2/4a} \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\frac{\pi}{a}} e^{b^2/4a}.$$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} A \int_{-\infty}^{\infty} e^{-ax^2} e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{1/4} \sqrt{\frac{\pi}{a}} e^{-k^2/4a} = \frac{1}{(2\pi a)^{1/4}} e^{-k^2/4a}.$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \frac{1}{(2\pi a)^{1/4}} \int_{-\infty}^{\infty} \underbrace{e^{-k^2/4a} e^{i(kx - \hbar k^2 t/2m)}}_{e^{-[(\frac{1}{4a}) + (i\hbar t/2m)]k^2 - ixk}} dk$$

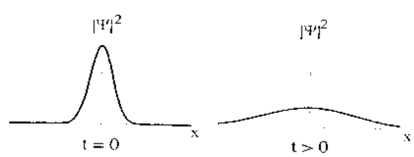
$$= \frac{1}{\sqrt{2\pi}(2\pi a)^{1/4}} \frac{\sqrt{\pi}}{\sqrt{\frac{1}{4a} + i\hbar t/2m}} e^{-x^2/4(\frac{1}{4a} + i\hbar t/2m)} = \boxed{\left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-ax^2/(1+2i\hbar at/m)}}{\sqrt{1+2i\hbar at/m}}.}$$

(c)

Let $\theta \equiv 2\hbar at/m$. Then $|\Psi|^2 = \sqrt{\frac{2a}{\pi}} \frac{e^{-ax^2/(1+i\theta)} e^{-ax^2/(1-i\theta)}}{\sqrt{(1+i\theta)(1-i\theta)}}$. The exponent is

$$-\frac{ax^2}{(1+i\theta)} - \frac{ax^2}{(1-i\theta)} = -ax^2 \frac{(1 - i\theta + 1 + i\theta)}{(1+i\theta)(1-i\theta)} = \frac{-2ax^2}{1+\theta^2}; \quad |\Psi|^2 = \sqrt{\frac{2a}{\pi}} \frac{e^{-2ax^2/(1+\theta^2)}}{\sqrt{1+\theta^2}}.$$

Or, with $w \equiv \sqrt{\frac{a}{1+\theta^2}}$, $\boxed{|\Psi|^2 = \sqrt{\frac{2}{\pi}} w e^{-2w^2x^2}.}$ As t increases, the graph of $|\Psi|^2$ flattens out and broadens.



(d)

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\Psi|^2 dx = \boxed{0} \text{ (odd integrand)}; \quad \langle p \rangle = m\frac{d\langle x \rangle}{dt} = \boxed{0.}$$

$$\langle x^2 \rangle = \sqrt{\frac{2}{\pi}} w \int_{-\infty}^{\infty} x^2 e^{-2w^2x^2} dx = \sqrt{\frac{2}{\pi}} w \frac{1}{4w^2} \sqrt{\frac{\pi}{2w^2}} = \boxed{\frac{1}{4w^2}.} \quad \langle p^2 \rangle = -\hbar^2 \int_{-\infty}^{\infty} \Psi^* \frac{d^2\Psi}{dx^2} dx.$$

Write $\Psi = Be^{-bx^2}$, where $B \equiv \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\sqrt{1+i\theta}}$ and $b \equiv \frac{a}{1+i\theta}$.

$$\frac{d^2\Psi}{dx^2} = B\frac{d}{dx}\left(-2bxe^{-bx^2}\right) = -2bB(1-2bx^2)e^{-bx^2}.$$

$$\Psi^*\frac{d^2\Psi}{dx^2} = -2b|B|^2(1-2bx^2)e^{-(b+b^*)x^2}; \; b+b^* = \frac{a}{1+i\theta} + \frac{a}{1-i\theta} = \frac{2a}{1+\theta^2} = 2w^2.$$

$$|B|^2 = \sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\theta^2}} = \sqrt{\frac{2}{\pi}}w. \quad \text{So } \Psi^*\frac{d^2\Psi}{dx^2} = -2b\sqrt{\frac{2}{\pi}}w(1-2bx^2)e^{-2w^2x^2}.$$

$$\langle p^2\rangle = 2b\hbar^2\sqrt{\frac{2}{\pi}}w\int_{-\infty}^{\infty}(1-2bx^2)e^{-2w^2x^2}dx$$
$$= 2b\hbar^2\sqrt{\frac{2}{\pi}}w\left(\sqrt{\frac{\pi}{2w^2}} - 2b\frac{1}{4w^2}\sqrt{\frac{\pi}{2w^2}}\right) = 2b\hbar^2\left(1-\frac{b}{2w^2}\right).$$

But $1-\frac{b}{2w^2} = 1-\left(\frac{a}{1+i\theta}\right)\left(\frac{1+\theta^2}{2a}\right) = 1-\frac{(1-i\theta)}{2} = \frac{1+i\theta}{2} = \frac{a}{2b}$, so

$$\langle p^2\rangle = 2b\hbar^2\frac{a}{2b} = \boxed{\hbar^2 a.} \qquad \boxed{\sigma_x = \frac{1}{2w};} \qquad \boxed{\sigma_p = \hbar\sqrt{a}.}$$

(e)

$$\sigma_x\sigma_p = \frac{1}{2w}\hbar\sqrt{a} = \frac{\hbar}{2}\sqrt{1+\theta^2} = \frac{\hbar}{2}\sqrt{1+(2\hbar a t/m)^2} \geq \frac{\hbar}{2}. \checkmark$$

Closest at $\boxed{t=0,}$ at which time it is right *at* the uncertainty limit.