Problem 3.2

(a)

$$\langle f|f\rangle = \int_0^1 x^{2\nu}\,dx = \frac{1}{2\nu+1}x^{2\nu+1}\Big|_0^1 = \frac{1}{2\nu+1}\left(1 - 0^{2\nu+1}\right).$$

Now $0^{2\nu+1}$ is finite (in fact, *zero*) provided $(2\nu+1) > 0$, which is to say, $\boxed{\nu > -\frac{1}{2}}$. If $(2\nu+1) < 0$ the integral definitely blows up. As for the critical case $\nu = -\frac{1}{2}$, this must be handled separately:

$$\langle f|f\rangle = \int_0^1 x^{-1}\,dx = \ln x\Big|_0^1 = \ln 1 - \ln 0 = 0 + \infty.$$

So $f(x)$ is in Hilbert space only for ν strictly *greater* than -1/2.

(b) For $\nu = 1/2$, we know from (a) that $f(x)$ *is* in Hilbert space: $\boxed{\text{yes.}}$

Since $xf = x^{3/2}$, we know from (a) that it *is* in Hilbert space: $\boxed{\text{yes.}}$

For $df/dx = \frac{1}{2}x^{-1/2}$, we know from (a) that it is *not* in Hilbert space: $\boxed{\text{no.}}$

[*Moral:* Simple operations, such as differenting (or multiplying by $1/x$), can carry a function *out* of Hilbert space.]

Problem 3.3

Suppose $\langle h|\hat{Q}h\rangle = \langle \hat{Q}h|h\rangle$ for all functions $h(x)$. Let $h(x) = f(x) + cg(x)$ for some arbitrary constant c. Then

$$\langle h|\hat{Q}h\rangle = \langle (f+cg)|\hat{Q}(f+cg)\rangle = \langle f|\hat{Q}f\rangle + c\langle f|\hat{Q}g\rangle + c^*\langle g|\hat{Q}f\rangle + |c|^2\langle g|\hat{Q}g\rangle;$$
$$\langle \hat{Q}h|h\rangle = \langle \hat{Q}(f+cg)|(f+cg)\rangle = \langle \hat{Q}f|f\rangle + c\langle \hat{Q}f|g\rangle + c^*\langle \hat{Q}g|f\rangle + |c|^2\langle \hat{Q}g|g\rangle.$$

Equating the two and noting that $\langle f|\hat{Q}f\rangle = \langle \hat{Q}f|f\rangle$ and $\langle g|\hat{Q}g\rangle = \langle \hat{Q}g|g\rangle$ leaves

$$c\langle f|\hat{Q}g\rangle + c^*\langle g|\hat{Q}f\rangle = c\langle \hat{Q}f|g\rangle + c^*\langle \hat{Q}g|f\rangle.$$

In particlar, choosing $c = 1$:

$$\langle f|\hat{Q}g\rangle + \langle g|\hat{Q}f\rangle = \langle \hat{Q}f|g\rangle + \langle \hat{Q}g|f\rangle,$$

whereas if $c = i$:

$$\langle f|\hat{Q}g\rangle - \langle g|\hat{Q}f\rangle = \langle \hat{Q}f|g\rangle - \langle \hat{Q}g|f\rangle.$$

Adding the last two equations:

$$\langle f|\hat{Q}g\rangle = \langle \hat{Q}f|g\rangle. \quad \text{QED}$$