

Problem 3.2

(a)

$$\langle f|f \rangle = \int_0^1 x^{2\nu} dx = \frac{1}{2\nu+1} x^{2\nu+1} \Big|_0^1 = \frac{1}{2\nu+1} (1 - 0^{2\nu+1}).$$

Now $0^{2\nu+1}$ is finite (in fact, zero) provided $(2\nu+1) > 0$, which is to say, $\nu > -\frac{1}{2}$. If $(2\nu+1) < 0$ the integral definitely blows up. As for the critical case $\nu = -\frac{1}{2}$, this must be handled separately:

$$\langle f|f \rangle = \int_0^1 x^{-1} dx = \ln x \Big|_0^1 = \ln 1 - \ln 0 = 0 + \infty.$$

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So $f(x)$ is in Hilbert space only for ν strictly greater than $-1/2$.

(b) For $\nu = 1/2$, we know from (a) that $f(x)$ is in Hilbert space: yes.

Since $xf = x^{3/2}$, we know from (a) that it is in Hilbert space: yes.

For $df/dx = \frac{1}{2}x^{-1/2}$, we know from (a) that it is *not* in Hilbert space: no.

[*Moral:* Simple operations, such as differentiating (or multiplying by $1/x$), can carry a function *out* of Hilbert space.]

Problem 3.3

Suppose $\langle h|\hat{Q}h \rangle = \langle \hat{Q}h|h \rangle$ for all functions $h(x)$. Let $h(x) = f(x) + cg(x)$ for some arbitrary constant c . Then

$$\langle h|\hat{Q}h \rangle = \langle (f + cg)|\hat{Q}(f + cg) \rangle = \langle f|\hat{Q}f \rangle + c\langle f|\hat{Q}g \rangle + c^*\langle g|\hat{Q}f \rangle + |c|^2\langle g|\hat{Q}g \rangle;$$

$$\langle \hat{Q}h|h \rangle = \langle \hat{Q}(f + cg)|(f + cg) \rangle = \langle \hat{Q}f|f \rangle + c\langle \hat{Q}f|g \rangle + c^*\langle \hat{Q}g|f \rangle + |c|^2\langle \hat{Q}g|g \rangle.$$

Equating the two and noting that $\langle f|\hat{Q}f \rangle = \langle \hat{Q}f|f \rangle$ and $\langle g|\hat{Q}g \rangle = \langle \hat{Q}g|g \rangle$ leaves

$$c\langle f|\hat{Q}g \rangle + c^*\langle g|\hat{Q}f \rangle = c\langle \hat{Q}f|g \rangle + c^*\langle \hat{Q}g|f \rangle.$$

In particular, choosing $c = 1$:

$$\langle f|\hat{Q}g \rangle + \langle g|\hat{Q}f \rangle = \langle \hat{Q}f|g \rangle + \langle \hat{Q}g|f \rangle,$$

whereas if $c = i$:

$$\langle f|\hat{Q}g \rangle - \langle g|\hat{Q}f \rangle = \langle \hat{Q}f|g \rangle - \langle \hat{Q}g|f \rangle.$$

Adding the last two equations:

$$\langle f|\hat{Q}g \rangle = \langle \hat{Q}f|g \rangle. \quad \text{QED}$$