Problem 3.13

(a) $[AB, C] = ABC - CAB = ABC - ACB + ACB - CAB = A[B, C] + [A, C]B.$ ✓

(b) Introducing a test function $g(x)$, as in Eq. 2.50:

$$[x^n, p]g = x^n \frac{\hbar}{i}\frac{dg}{dx} - \frac{\hbar}{i}\frac{d}{dx}(x^n g) = x^n \frac{\hbar}{i}\frac{dg}{dx} - \frac{\hbar}{i}\left(nx^{n-1}g + x^n \frac{dg}{dx}\right) = i\hbar n x^{n-1} g.$$

So, dropping the test function, $[x^n, p] = i\hbar n x^{n-1}$. ✓

(c) $[f, p]g = f\frac{\hbar}{i}\frac{dg}{dx} - \frac{\hbar}{i}\frac{d}{dx}(fg) = f\frac{\hbar}{i}\frac{dg}{dx} - \frac{\hbar}{i}\left(\frac{df}{dx}g + f\frac{dg}{dx}\right) = i\hbar\frac{df}{dx}g \Rightarrow [f, p] = i\hbar\frac{df}{dx}.$ ✓

Problem 3.14

$$\left[x, \frac{p^2}{2m} + V\right] = \frac{1}{2m}[x, p^2] + [x, V]; \quad [x, p^2] = xp^2 - p^2x = xp^2 - pxp + pxp - p^2x = [x,p]p + p[x,p].$$

Using Eq. 2.51: $[x, p^2] = i\hbar p + p i\hbar = 2i\hbar p$. And $[x, V] = 0$, so $\left[x, \frac{p^2}{2m} + V\right] = \frac{1}{2m}2i\hbar p = \frac{i\hbar p}{m}$.

The generalized uncertainty principle (Eq. 3.62) says, in this case,

$$\sigma_x^2\sigma_H^2 \geq \left(\frac{1}{2i}\frac{i\hbar}{m}\langle p\rangle\right)^2 = \left(\frac{\hbar}{2m}\langle p\rangle\right)^2 \Rightarrow \sigma_x\sigma_H \geq \frac{\hbar}{2m}|\langle p\rangle|. \quad \text{QED}$$

For stationary states $\sigma_H = 0$ and $\langle p\rangle = 0$, so it just says $0 \geq 0$.

Problem 3.17

(a) 1 commutes with everything, so $\boxed{\frac{d}{dt}\langle\Psi|\Psi\rangle = 0}$ (this is the conservation of normalization, which we originally proved in Eq. 1.27).

(b) Anything commutes with itself, so $[H, H] = 0$, and hence $\boxed{\frac{d}{dt}\langle H\rangle = 0}$ (assuming H has no explicit time dependence); this is conservation of energy, in the sense of the comment following Eq. 2.40.

(c) $[H, x] = -\frac{i\hbar p}{m}$ (see Problem 3.14). So $\frac{d\langle x\rangle}{dt} = \frac{i}{\hbar}\left(-\frac{i\hbar\langle p\rangle}{m}\right) = \boxed{\frac{\langle p\rangle}{m}}$ (Eq. 1.33).

(d) $[H, p] = \left[\frac{p^2}{2m} + V, p\right] = [V, p] = i\hbar\frac{dV}{dx}$ (Problem 3.13(c)). So $\frac{d\langle p\rangle}{dt} = \frac{i}{\hbar}\left(i\hbar\left\langle\frac{\partial V}{\partial x}\right\rangle\right) = \boxed{-\left\langle\frac{\partial V}{\partial x}\right\rangle}.$

This is Ehrenfest's theorem (Eq. 1.38).

Problem 3.31

Equation 3.71 $\Rightarrow \frac{d}{dt}\langle xp\rangle = \frac{i}{\hbar}\langle[H,xp]\rangle$; Eq. 3.64 $\Rightarrow [H,xp] = [H,x]p + x[H,p]$; Problem 3.14 $\Rightarrow [H,x] = -\frac{i\hbar p}{m}$; Problem 3.17(d) $\Rightarrow [H,p] = i\hbar\frac{dV}{dx}$. So

$$\frac{d}{dt}\langle xp\rangle = \frac{i}{\hbar}\left[-\frac{i\hbar}{m}\langle p^2\rangle + i\hbar\langle x\frac{dV}{dx}\rangle\right] = 2\langle\frac{p^2}{2m}\rangle - \langle x\frac{dV}{dx}\rangle = 2\langle T\rangle - \langle x\frac{dV}{dx}\rangle. \quad \text{QED}$$

In a stationary state all expectation values (at least, for operators that do not depend explicitly on t) are time-independent (see item 1 on p. 26), so $d\langle xp\rangle/dt = 0$, and we are left with Eq. 3.97.

For the harmonic oscillator:

$$V = \frac{1}{2}m\omega^2x^2 \Rightarrow \frac{dV}{dx} = m\omega^2 x \Rightarrow x\frac{dV}{dx} = m\omega^2x^2 = 2V \Rightarrow 2\langle T\rangle = 2\langle V\rangle \Rightarrow \langle T\rangle = \langle V\rangle. \quad \text{QED}$$

In Problem 2.11(c) we found that $\langle T\rangle = \langle V\rangle = \frac{1}{4}\hbar\omega$ (for $n=0$); $\langle T\rangle = \langle V\rangle = \frac{3}{4}\hbar\omega$ (for $n=1$). ✓
In Problem 2.12 we found that $\langle T\rangle = \frac{1}{2}(n+\frac{1}{2})\hbar\omega$, while $\langle x^2\rangle = (n+\frac{1}{2})\hbar/m\omega$, so $\langle V\rangle = \frac{1}{2}m\omega^2\langle x^2\rangle = \frac{1}{2}(n+\frac{1}{2})\hbar\omega$, and hence $\langle T\rangle = \langle V\rangle$ for *all* stationary states. ✓