Problem 4.12

(a)

$$L_0 = e^x e^{-x} = \boxed{1.} \quad L_1 = e^x \frac{d}{dx}\left(e^{-x}x\right) = e^x\left[e^{-x} - e^{-x}x\right] = \boxed{1 - x.}$$

$$\begin{aligned} L_2 &= e^x \left(\frac{d}{dx}\right)^2 \left(e^{-x}x^2\right) = e^x \frac{d}{dx}\left(2xe^{-x} - e^{-x}x^2\right) \\ &= e^x\left(2e^{-x} - 2xe^{-x} + e^{-x}x^2 - 2xe^{-x}\right) = \boxed{2 - 4x + x^2.} \end{aligned}$$

$$\begin{aligned} L_3 &= e^x \left(\frac{d}{dx}\right)^3 \left(e^{-x}x^3\right) = e^x \left(\frac{d}{dx}\right)^2 \left(-e^{-x}x^3 + 3x^2e^{-x}\right) \\ &= e^x \frac{d}{dx}\left(e^{-x}x^3 - 3x^2e^{-x} - 3x^2e^{-x} + 6xe^{-x}\right) \\ &= e^x\left(-e^{-x}x^3 + 3x^2e^{-x} + 6x^2e^{-x} - 12xe^{-x} - 6xe^{-x} + 6e^{-x}\right) \\ &= \boxed{6 - 18x + 9x^2 - x^3.} \end{aligned}$$

(b)

$$v(\rho) = L_2^5(2\rho); \quad L_2^5(x) = L_{7-5}^5(x) = (-1)^5\left(\frac{d}{dx}\right)^5 L_7(x).$$

$$\begin{aligned} L_7(x) &= e^x\left(\frac{d}{dx}\right)^7\left(x^7e^{-x}\right) = e^x\left(\frac{d}{dx}\right)^6\left(7x^6e^{-x} - x^7e^{-x}\right) \\ &= e^x\left(\frac{d}{dx}\right)^5\left(42x^5e^{-x} - 7x^6e^{-x} - 7x^6e^{-x} + x^7e^{-x}\right) \\ &= e^x\left(\frac{d}{dx}\right)^4\left(210x^4e^{-x} - 42x^5e^{-x} - 84x^5e^{-x} + 14x^6e^{-x} + 7x^6e^{-x} - x^7e^{-x}\right) \\ &= e^x\left(\frac{d}{dx}\right)^3\Big[840x^3e^{-x} - (210 + 630)x^4e^{-x} \\ &\quad + (126 + 126)x^5e^{-x} - (21 + 7)x^6e^{-x} + x^7e^{-x}\Big] \\ &= e^x\left(\frac{d}{dx}\right)^2\Big(2520x^2e^{-x} - (840 + 3360)x^3e^{-x} \\ &\quad + (840 + 1260)x^4e^{-x} - (252 + 168)x^5e^{-x} + (28 + 7)x^6e^{-x} - x^7e^{-x}\Big) \\ &= e^x\left(\frac{d}{dx}\right)\Big[5040xe^{-x} - (2520 + 12600)x^2e^{-x} + (4200 + 8400)x^3e^{-x} \\ &\quad - (2100 + 2100)x^4e^{-x} + (420 + 210)x^5e^{-x} - (35 + 7)x^6e^{-x} + x^7e^{-x}\Big] \\ &= e^x\Big[5040e^{-x} - (5040 + 30240)xe^{-x} + (15120 + 37800)x^2e^{-x} \\ &\quad - (12600 + 8400 + 8400)x^3e^{-x} + (2100 + 2100 + 3150)x^4e^{-x} \\ &\quad - (630 + 252)x^5e^{-x} + (42 + 7)x^6e^{-x} - x^7e^{-x}\Big] \\ &= 5040 - 35280x + 52920x^2 - 29400x^3 + 7350x^4 - 882x^5 + 49x^6 - x^7. \end{aligned}$$

$$L_2^5 = -\left(\frac{d}{dx}\right)^5 \left(-882x^5 + 49x^6 - x^7\right)$$
$$= -\left[-882(5\cdot4\cdot3\cdot2) + 49(6\cdot5\cdot4\cdot3\cdot2)x - 7\cdot6\cdot5\cdot4\cdot3x^2\right]$$
$$= 60\left[(882\times2) - (49\times12)x + 42x^2\right] = 2520(42 - 14x + x^2).$$

$$v(\rho) = 2520(42 - 28\rho + 4\rho^2) = \boxed{5040\left(21 - 14\rho + 2\rho^2\right).}$$

(c)

$$\text{Eq. 4.62} \Rightarrow v(\rho) = \sum_{j=0}^{\infty} c_j\rho^j. \quad \text{Eq. 4.76} \Rightarrow c_1 = \frac{2(3-5)}{(1)(6)}c_0 = -\frac{2}{3}c_0.$$

$$c_2 = \frac{2(4-5)}{(2)(7)}c_1 = -\frac{1}{7}c_1 = \frac{2}{21}c_0; \quad c_3 = \frac{2(5-5)}{(3)(8)}c_2 = 0.$$

$$v(\rho) = c_0 - \frac{2}{3}c_0\rho + \frac{2}{21}c_0\rho^2 = \boxed{\frac{c_0}{21}\left(21 - 14\rho + 2\rho^2\right).} \quad \checkmark$$

Problem 4.13

(a)

$$\psi = \frac{1}{\sqrt{\pi a^3}}e^{-r/a}, \quad \text{so } \langle r^n\rangle = \frac{1}{\pi a^3}\int r^n e^{-2r/a}\left(r^2\sin\theta\, dr\, d\theta\, d\phi\right) = \frac{4\pi}{\pi a^3}\int_0^\infty r^{n+2}e^{-2r/a}\,dr.$$

$$\langle r\rangle = \frac{4}{a^3}\int_0^\infty r^3 e^{-2r/a}\,dr = \frac{4}{a^3}3!\left(\frac{a}{2}\right)^4 = \boxed{\frac{3}{2}a;} \quad \langle r^2\rangle = \frac{4}{a^3}\int_0^\infty r^4 e^{-2r/a}\,dr = \frac{4}{a^3}4!\left(\frac{a}{2}\right)^5 = \boxed{3a^2.}$$

(b)

$$\boxed{\langle x\rangle = 0;} \quad \langle x^2\rangle = \frac{1}{3}\langle r^2\rangle = \boxed{a^2.}$$

(c)

$$\psi_{211} = R_{21}Y_1^1 = -\frac{1}{\sqrt{\pi a}}\frac{1}{8a^2}re^{-r/2a}\sin\theta e^{i\phi} \quad \text{(Problem 4.11(b)).}$$

$$\langle x^2\rangle = \frac{1}{\pi a}\frac{1}{(8a^2)^2}\int\left(r^2e^{-r/a}\sin^2\theta\right)\left(r^2\sin^2\theta\cos^2\phi\right)r^2\sin\theta\, dr\, d\theta\, d\phi$$
$$= \frac{1}{64\pi a^5}\int_0^\infty r^6 e^{-r/a}\,dr\int_0^\pi \sin^5\theta\, d\theta\int_0^{2\pi}\cos^2\phi\, d\phi$$
$$= \frac{1}{64\pi a^5}\left(6!a^7\right)\left(2\frac{2\cdot4}{1\cdot3\cdot5}\right)\left(\frac{1}{2}\cdot2\pi\right) = \boxed{12a^2.}$$

Problem 4.18

$$\langle f|L_\pm g\rangle = \langle f|L_x g\rangle \pm i\langle f|L_y g\rangle = \langle L_x f|g\rangle \pm i\langle L_y f|g\rangle = \langle (L_x \mp iL_y)f|g\rangle = \langle L_\mp f|g\rangle, \text{ so } (L_\pm)^\dagger = L_\mp.$$

Now, using Eq. 4.112, in the form $L_\mp L_\pm = L^2 - L_z^2 \mp \hbar L_z$:

$$\begin{aligned}\langle f_l^m|L_\mp L_\pm f_l^m\rangle &= \langle f_l^m|(L^2 - L_z^2 \mp \hbar L_z)f_l^m\rangle = \langle f_l^m|\left[\hbar^2 l(l+1) - \hbar^2 m^2 \mp \hbar^2 m\right] f_l^m\rangle \\ &= \hbar^2\left[l(l+1) - m(m\pm 1)\right]\langle f_l^m|f_l^m\rangle = \hbar^2\left[l(l+1) - m(m+1)\right] \\ &= \langle L_\pm f_l^m|L_\pm f_l^m\rangle = \langle A_l^m f_l^{m\pm 1}|A_l^m f_l^{m\pm 1}\rangle = |A_l^m|^2\langle f_l^{m\pm 1}|f_l^{m\pm 1}\rangle = |A_l^m|^2.\end{aligned}$$

Conclusion: $\boxed{A_l^m = \hbar\sqrt{l(l+1) - m(m \pm 1)}.}$

Problem 4.19

(a)

$$[L_z, x] = [xp_y - yp_x, x] = [xp_y, x] - [yp_x, x] = 0 - y[p_x, x] = i\hbar y. \checkmark$$

$$[L_z, y] = [xp_y - yp_x, y] = [xp_y, y] - [yp_x, y] = x[p_y, y] - 0 = -i\hbar x. \checkmark$$

$$[L_z, z] = [xp_y - yp_x, z] = [xp_y, z] - [yp_x, z] = 0 - 0 = 0. \checkmark$$

$$[L_z, p_x] = [xp_y - yp_x, p_x] = [xp_y, p_x] - [yp_x, p_x] = p_y[x, p_x] - 0 = i\hbar p_y. \checkmark$$

$$[L_z, p_y] = [xp_y - yp_x, p_y] = [xp_y, p_y] - [yp_x, p_y] = 0 - p_x[y, p_y] = -i\hbar p_x. \checkmark$$

$$[L_z, p_z] = [xp_y - yp_x, p_z] = [xp_y, p_z] - [yp_x, p_z] = 0 - 0 = 0. \checkmark$$

(b)

$$[L_z, L_x] = [L_z, yp_z - zp_y] = [L_z, yp_z] - [L_z, zp_y] = [L_z, y]p_z - [L_z, p_y]z$$
$$= -i\hbar x p_z + i\hbar p_x z = i\hbar(zp_x - xp_z) = i\hbar L_y.$$

(So, by cyclic permutation of the indices, $[L_x, L_y] = i\hbar L_z$.)

(c)

$$[L_z, r^2] = [L_z, x^2] + [L_z, y^2] + [L_z, z^2] = [L_z, x]x + x[L_z, x] + [L_z, y]y + y[L_z, y] + 0$$
$$= i\hbar yx + xi\hbar y + (-i\hbar x)y + y(-i\hbar x) = \boxed{0.}$$

$$[L_z, p^2] = [L_z, p_x^2] + [L_z, p_y^2] + [L_z, p_z^2] = [L_z, p_x]p_x + p_x[L_z, p_x] + [L_z, p_y]p_y + p_y[L_z, p_y] + 0$$
$$= i\hbar p_y p_x + p_x i\hbar p_y + (-i\hbar p_x)p_y + p_y(-i\hbar p_x) = \boxed{0.}$$

(d) It follows from (c) that all three components of **L** commute with r^2 and p^2, and hence with the whole Hamiltonian, since $H = p^2/2m + V(\sqrt{r^2})$. QED

Problem 4.21

(a)

$$L_+L_-f = -\hbar^2 e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\left[e^{-i\phi}\left(\frac{\partial f}{\partial\theta} - i\cot\theta\frac{\partial f}{\partial\phi}\right)\right]$$

$$= -\hbar^2 e^{i\phi}\left\{e^{-i\phi}\left[\frac{\partial^2 f}{\partial\theta^2} - i\left(-\csc^2\theta\frac{\partial f}{\partial\phi} + \cot\theta\frac{\partial^2 f}{\partial\theta\,\partial\phi}\right)\right]\right.$$

$$\left.+ i\cot\theta\left[-ie^{-i\phi}\left(\frac{\partial f}{\partial\theta} - i\cot\theta\frac{\partial f}{\partial\phi}\right) + e^{-i\phi}\left(\frac{\partial^2 f}{\partial\phi\,\partial\theta} - i\cot\theta\frac{\partial^2 f}{\partial\phi^2}\right)\right]\right\}$$

$$= -\hbar^2\left(\frac{\partial^2 f}{\partial\theta^2} + i\csc^2\theta\frac{\partial f}{\partial\phi} - i\cot\theta\frac{\partial^2 f}{\partial\theta\,\partial\phi} + \cot\theta\frac{\partial f}{\partial\theta} - i\cot^2\theta\frac{\partial f}{\partial\phi} + i\cot\theta\frac{\partial^2 f}{\partial\phi\,\partial\theta} + \cot^2\theta\frac{\partial^2 f}{\partial\phi^2}\right)$$

$$= -\hbar^2\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i(\csc^2\theta - \cot^2\theta)\frac{\partial}{\partial\phi}\right]f, \quad \text{so}$$

$$L_+L_- = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right). \quad \text{QED}$$

(b) Equation 4.129 $\Rightarrow L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$, Eq. 4.112 $\Rightarrow L^2 = L_+L_- + L_z^2 - \hbar L_z$, so, using (a):

$$L^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right) - \hbar^2\frac{\partial^2}{\partial\phi^2} - \hbar\left(\frac{\hbar}{i}\right)\frac{\partial}{\partial\phi}$$

$$= -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + (\cot^2\theta + 1)\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi} - i\frac{\partial}{\partial\phi}\right) = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]. \quad \text{QED}$$

Problem 4.22

(a) $\boxed{L_+ Y_l^l = 0}$ (top of the ladder).

(b)

$$L_z Y_l^l = \hbar l Y_l^l \Rightarrow \frac{\hbar}{i}\frac{\partial}{\partial\phi}Y_l^l = \hbar l Y_l^l, \quad \text{so } \frac{\partial Y_l^l}{\partial\phi} = il Y_l^l, \text{ and hence } Y_l^l = f(\theta)e^{il\phi}.$$

[*Note:* $f(\theta)$ is the "constant" here — it's constant with respect to ϕ ... but still can depend on θ.]

$$L_+ Y_l^l = 0 \Rightarrow \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\left[f(\theta)e^{il\phi}\right] = 0, \quad \text{or } \frac{df}{d\theta}e^{il\phi} + if\cot\theta\, il\, e^{il\phi} = 0, \text{ so}$$

$$\frac{df}{d\theta} = l\cot\theta f \Rightarrow \frac{df}{f} = l\cot\theta\, d\theta \Rightarrow \int\frac{df}{f} = l\int\frac{\cos\theta}{\sin\theta}\,d\theta \Rightarrow \ln f = l\ln(\sin\theta) + constant.$$

$$\ln f = \ln(\sin^l\theta) + K \Rightarrow \ln\left(\frac{f}{\sin^l\theta}\right) = K \Rightarrow \frac{f}{\sin^l\theta} = constant \Rightarrow f(\theta) = A\sin^l\theta.$$

$$\boxed{Y_l^l(\theta,\phi) = A(e^{i\phi}\sin\theta)^l.}$$

(c)

$$1 = A^2\int \sin^{2l}\theta\sin\theta\,d\theta\,d\phi = 2\pi A^2\int_0^\pi \sin^{(2l+1)}\theta\,d\theta = 2\pi A^2\, 2\frac{(2\cdot 4\cdot 6\cdot\dots\cdot(2l))}{1\cdot 3\cdot 5\cdot\dots\cdot(2l+1)}$$

$$= 4\pi A^2\frac{(2\cdot 4\cdot 6\cdot\dots\cdot 2l)^2}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot\dots\cdot(2l+1)} = 4\pi A^2\frac{(2^l l!)^2}{(2l+1)!}, \quad \text{so} \quad \boxed{A = \frac{1}{2^{l+1}l!}\sqrt{\frac{(2l+1)!}{\pi}},}$$

the same as Problem 4.5, except for an overall factor of $(-1)^l$, which is arbitrary anyway.

Problem 4.27

(a)

$$\chi^\dagger\chi = |A|^2(9+16) = 25|A|^2 = 1 \Rightarrow \boxed{A = 1/5.}$$

(b)

$$\langle S_x\rangle = \chi^\dagger \mathsf{S}_x \chi = \frac{1}{25}\frac{\hbar}{2}\,(-3i\ 4)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}3i\\ 4\end{pmatrix} = \frac{\hbar}{50}\,(-3i\ 4)\begin{pmatrix}4\\ 3i\end{pmatrix} = \frac{\hbar}{50}(12i + 12i) = \boxed{0.}$$

$$\langle S_y\rangle = \chi^\dagger \mathsf{S}_y \chi = \frac{1}{25}\frac{\hbar}{2}\,(-3i\ 4)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}3i\\ 4\end{pmatrix} = \frac{\hbar}{50}\,(-3i\ 4)\begin{pmatrix}-4i\\ -3\end{pmatrix} = \frac{\hbar}{50}(-12 - 12) = \boxed{-\frac{12}{25}\hbar.}$$

$$\langle S_z\rangle = \chi^\dagger \mathsf{S}_z \chi = \frac{1}{25}\frac{\hbar}{2}\,(-3i\ 4)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}3i\\ 4\end{pmatrix} = \frac{\hbar}{50}\,(-3i\ 4)\begin{pmatrix}3i\\ -4\end{pmatrix} = \frac{\hbar}{50}(9 - 16) = \boxed{-\frac{7}{50}\hbar.}$$

(c)

$$\langle S_x^2\rangle = \langle S_y^2\rangle = \langle S_z^2\rangle = \frac{\hbar^2}{4} \text{ (always, for spin 1/2), so } \sigma_{S_x}^2 = \langle S_x^2\rangle - \langle S_x\rangle^2 = \frac{\hbar^2}{4} - 0, \ \boxed{\sigma_{S_x} = \frac{\hbar}{2}.}$$

$$\sigma_{S_y}^2 = \langle S_y^2\rangle - \langle S_y\rangle^2 = \frac{\hbar^2}{4} - \left(\frac{12}{25}\right)^2\hbar^2 = \frac{\hbar^2}{2500}(625 - 576) = \frac{49}{2500}\hbar^2, \ \boxed{\sigma_{S_y} = \frac{7}{50}\hbar.}$$

$$\sigma_{S_z}^2 = \langle S_z^2\rangle - \langle S_z\rangle^2 = \frac{\hbar^2}{4} - \left(\frac{7}{50}\right)^2\hbar^2 = \frac{\hbar^2}{2500}(625 - 49) = \frac{576}{2500}\hbar^2, \ \boxed{\sigma_{S_z} = \frac{12}{25}\hbar.}$$

(d)

$$\sigma_{S_x}\sigma_{S_y} = \frac{\hbar}{2}\cdot\frac{7}{50}\hbar \stackrel{?}{\geq} \frac{\hbar}{2}|\langle S_z\rangle| = \frac{\hbar}{2}\cdot\frac{7}{50}\hbar \quad \text{(right \textit{at} the uncertainty limit).} \checkmark$$

$$\sigma_{S_y}\sigma_{S_z} = \frac{7}{50}\hbar\cdot\frac{12}{25}\hbar \stackrel{?}{\geq} \frac{\hbar}{2}|\langle S_x\rangle| = 0 \quad \text{(trivial).} \checkmark$$

$$\sigma_{S_z}\sigma_{S_x} = \frac{12}{25}\hbar\cdot\frac{\hbar}{2} \stackrel{?}{\geq} \frac{\hbar}{2}|\langle S_y\rangle| = \frac{\hbar}{2}\cdot\frac{12}{25}\hbar \quad \text{(right \textit{at} the uncertainty limit).} \checkmark$$

Problem 4.28

$$\langle S_x\rangle = \frac{\hbar}{2}\,(a^*\ b^*)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \frac{\hbar}{2}\,(a^*\ b^*)\begin{pmatrix}b\\a\end{pmatrix} = \boxed{\frac{\hbar}{2}(a^*b+b^*a)} = \hbar\,\mathrm{Re}(ab^*).$$

$$\langle S_y\rangle = \frac{\hbar}{2}\,(a^*\ b^*)\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \frac{\hbar}{2}\,(a^*\ b^*)\begin{pmatrix}-ib\\ia\end{pmatrix}$$
$$= \frac{\hbar}{2}(-ia^*b+iab^*) = \boxed{\frac{\hbar}{2}i(ab^*-a^*b)} = -\hbar\,\mathrm{Im}(ab^*).$$

$$\langle S_z\rangle = \frac{\hbar}{2}\,(a^*\ b^*)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \frac{\hbar}{2}\,(a^*\ b^*)\begin{pmatrix}a\\-b\end{pmatrix} = \frac{\hbar}{2}(a^*a-b^*b) = \boxed{\frac{\hbar}{2}(|a|^2-|b|^2).}$$

$$S_x^2 = \frac{\hbar^2}{4}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix}1&0\\0&1\end{pmatrix} = \frac{\hbar^2}{4};\quad S_y^2 = \frac{\hbar^2}{4}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \frac{\hbar^2}{4};$$

$$S_z^2 = \frac{\hbar^2}{4}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \frac{\hbar^2}{4};\quad \text{so } \boxed{\langle S_x^2\rangle = \langle S_y^2\rangle = \langle S_z^2\rangle = \frac{\hbar^2}{4}.}$$

$$\langle S_x^2\rangle + \langle S_y^2\rangle + \langle S_z^2\rangle = \frac{3}{4}\hbar^2 \stackrel{?}{=} s(s+1)\hbar^2 = \frac{1}{2}(\frac{1}{2}+1)\hbar^2 = \frac{3}{4}\hbar^2 = \langle S^2\rangle.\ \checkmark$$

Problem 4.29

(a)

$$S_y = \frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix};\quad \begin{vmatrix}-\lambda & -i\hbar/2\\ i\hbar/2 & -\lambda\end{vmatrix} = \lambda^2 - \frac{\hbar^2}{4} \Rightarrow \boxed{\lambda = \pm\frac{\hbar}{2}}\ \text{(of course)}.$$

$$\frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix}\alpha\\\beta\end{pmatrix} \Rightarrow -i\beta = \pm\alpha;\quad |\alpha|^2+|\beta|^2 = 1 \Rightarrow |\alpha|^2+|\alpha|^2 = 1 \Rightarrow \alpha = \frac{1}{\sqrt{2}}.$$

$$\boxed{\chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix};\quad \chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}.}$$

(b)

$$c_+ = \left(\chi_+^{(y)}\right)^\dagger\chi = \frac{1}{\sqrt{2}}\,(1\ -i)\begin{pmatrix}a\\b\end{pmatrix} = \frac{1}{\sqrt{2}}(a-ib);\quad \boxed{+\frac{\hbar}{2},\ \text{with probability } \frac{1}{2}|a-ib|^2.}$$

$$c_- = \left(\chi_-^{(y)}\right)^\dagger\chi = \frac{1}{\sqrt{2}}\,(1\ i)\begin{pmatrix}a\\b\end{pmatrix} = \frac{1}{\sqrt{2}}(a+ib);\quad \boxed{-\frac{\hbar}{2},\ \text{with probability } \frac{1}{2}|a+ib|^2.}$$

$$P_+ + P_- = \frac{1}{2}\left[(a^*+ib^*)(a-ib) + (a^*-ib^*)(a+ib)\right]$$
$$= \frac{1}{2}\left[|a|^2 - ia^*b + iab^* + |b|^2 + |a|^2 + ia^*b - iab^* + |b|^2\right] = |a|^2+|b|^2 = 1.\ \checkmark$$

(c) $\boxed{\frac{\hbar^2}{4},\ \text{with probability 1.}}$

Problem 4.30