So the space of s.i. functions w/ our inner-product is complete. It's called Hilbert space.

We define orthogonality, normalization and orthonormality w/ the inner product as usual:

$(f_i, f_j) \equiv \int_a^b f_i^*(x) f_j(x) dx = \delta_{ij} \Rightarrow \{f_i\}$ is an orthonormal set

Recall we can generalize this (and soon will) to use a weighted inner-product:

$(f_i, f_j)_w \equiv \int_a^b f_n^*(x) f_m(x) \underbrace{w(x)}_{\text{same function for any } f_i \text{ and } f_j} dx = \delta_{ij}$

As an example of an orthonormal set (though not necessarily a complete basis) consider the Fourier functions:

$f_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$ $n = 0, \pm 1, \pm 2, \ldots$ over the interval from $-\pi$ to π

Then:

$$(f_m, f_n) = \int_{-\pi}^{\pi} f_n^* f_m dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-n)x} dx = \frac{1}{2\pi i(m-n)} e^{i(m-n)x} \Big|_{-\pi}^{\pi} = 0 \text{ if } m \neq n$$

$$(f_n, f_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^0 dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx = 1$$

We talked about the completeness of a space, and learned that s.i. functions w/ our inner-product formed a complete space, basically since no sequence of them led to functions outside of the space.

Now we would like to look for a complete orthonormal set of functions, i.e. a basis. In this case we could appeal to:

? [An orthonormal set of functions $\{f_i(x)\}$ is complete if any function in Hilbert space can be expressed as a linear combination $f(x) = \sum_{i=1}^{\infty} c_i f_i(x)$ s.t. the sum converges at every x.

It turns out that this too strong of a statement and will in fact rule out any set as being complete in Hilbert space. The problem is the pointwise convergence (at every x) which, if you think about the Dirichlet function (which is part of Hilbert space), would lead to problems. In fact anything involving a set of measure zero would do so.

So let's consider three different manners of convergence.

(i) Pointwise:

A sequence of functions $h_n(x)$ converges pointwise to $h(x)$ on $[a,b]$ if for every $x \in [a,b]$ and every $\epsilon > 0$ there exists an integer $N(x,\epsilon)$ such that for $n > N$: $|h(x) - h_n(x)| < \epsilon$.

(ii) Uniform:

A sequence of functions $h_n(x)$ converges uniformly to $h(x)$ on $[a,b]$ if for every $\epsilon > 0$ there exists an integer $N(\epsilon)$ such that for $n > N$: $|h(x) - h_n(x)| < \epsilon$ for all $x \in [a,b]$.

(iii) Convergence in the Mean:

A sequence of functions $h_n(x)$ converges in the mean to $h(x)$ on $[a,b]$ if for every ϵ there exists an $N(\epsilon)$ s.t. for $n > N$: $\int_a^b |h(x) - h_n(x)|^2 dx < \epsilon$.

Clearly
$$\begin{array}{c} (ii) \Rightarrow (i) \\ \Downarrow \\ (iii) \end{array} \quad \text{however } (i) \not\Rightarrow (iii)$$

So (ii) is the most extreme, while (i) and (iii) are weaker.

Note that (iii) uses the Lesbegues measure meaning that $h_n(x)$ could converge in the mean to the Dirichlet function, whereas they would not converge pointwise to it.

Also note that for pointwise or uniform convergence, $h(x) = \lim_{n\to\infty} h_n(x)$, but not so for in the mean.

Now what does this have to do with completeness of a basis?

Well, let the set $h_n(x)$ be the partial sums $h_n(x) = \sum_{i=1}^{n} k_i(x)$.

Then for pointwise or uniform convergence, $h(x) = \sum_{i=1}^{\infty} k_i(x)$. This is starting to look like $k_i(x)$ is a complete set. But we want to relax the definition to use "mean". Here we go.

Let $g(x)$ be any s.i. function in Hilbert space and let $\{f_i(x)\}$ be an orthonormal set of functions. If there exist constant coefficients $\{a_i\}$ s.t. $g_n(x) \equiv \sum_{i=1}^{n} a_i f_i(x)$ converges in the mean to $g(x)$, then the set $\{f_i(x)\}$ is complete.

We call this equivalence $\underbrace{g(x) \doteq \sum_{i=1}^{\infty} a_i f_i(x)}$ which does not imply $\underbrace{g(x) = \sum_{i=1}^{\infty} a_i f_i(x)}_{\text{only if pointwise or uniform}}$.

for pointwise, uniform, or mean

$$\underbrace{\hphantom{xxxxxxxxxxxxxxxxxxxx}}$$

$$\lim_{n\to\infty} \int_a^b \left| g - \sum_{i=1}^{n} a_i f_i \right|^2 dx = 0$$

Note that since uniform convergence implies mean convergence, then a set of orthonormal functions satisfying the former is obviously complete by definition. However, pointwise does not necessarily imply mean, so these are not necessarily complete.

Okay, so when we talk about $g(x) = \sum_{i=1}^{\infty} a_i f_i(x)$ or $g(x) \doteq \sum_{i=1}^{\infty} a_i f_i(x)$, how do we get the coefficients a_i?

If $g(x) = \sum_{i=1}^{\infty} a_i f_i(x)$, then obviously $a_i = \langle f_i, g(x) \rangle$ since $\langle f_i, f_j \rangle = \delta_{ij}$.

What about $g(x) \doteq \sum_{i=1}^{\infty} a_i f_i(x) \Rightarrow \lim_{n\to\infty} \int_a^b |g - \sum_{i=1}^{n} a_i f_i|^2 dx = 0 \Rightarrow a_i = ?$

Consider $M_n = \int_a^b |g - \sum_{i=1}^{n} a_i f_i|^2 dx$ for a given $g(x)$ and the set $\{f_i(x)\}_{1,\dots,n}$.
Since this is positive definite, $M_n \geq 0$, we can look for the coefficients a_i which minimize the value of M_n.

$$M_n = \int_a^b \left(g^* g - \sum_{i=1}^{n} a_i g^* f_i - \sum_{i=1}^{n} a_i^* g f_i^* + \sum_{i,j=1}^{n} a_i^* a_j f_i^* f_j \right) dx$$

$$= \langle g, g \rangle - \sum_{i=1}^{n} a_i c_i^* - \sum_{i=1}^{n} a_i^* c_i + \sum_{i=1}^{n} a_i^* a_j \delta_{ij} \quad w/\ c_i = \langle f_i, g \rangle$$

Consider $M_n + \sum_{i=1}^{n} |c_i|^2 - \sum_{i=1}^{n} |c_i|^2 = \langle g, g \rangle + \underbrace{\sum_{i=1}^{n} |a_i - c_i|^2}_{\text{only place } a_i \text{ appears!}} - \sum_{i=1}^{n} |c_i|^2 \geq 0$

M_n is obviously minimized by $a_i = c_i$, then we are left with:

$$\langle g, g \rangle \geq \sum_{i=1}^{n} |c_i|^2 = \sum_{i=1}^{n} |\langle f_i, g \rangle|^2$$

If we take $n \to \infty$, the l.h.s. doesn't change, so we then have:

$\langle g, g \rangle \geq \sum_{i=1}^{\infty} |c_i|^2$ which is just the ∞-version of Bessel's inequality: $\|x\|^2 \geq \sum_i |\alpha_i|^2$

Just like in the finite case we know that if f_i is a complete orthonormal basis, the $=$ holds for any g.

We can similarly extend Parseval's equation to ∞-dimensions: If $\{f_i\}$ is complete then for any g and h on the space $\langle g, h \rangle = \sum_{i=1}^{\infty} \langle g, f_i \rangle \langle f_i, h \rangle$.

We may also argue:

[The orthonormal set $\{f_i\}$ is closed iff $(g, f_i) = 0$ for all $i \Rightarrow g = 0$.

and

[The orthonormal set $\{f_i\}$ in Hilbert space is complete iff it is closed.

Proof:

if complete then closed (if not closed then not complete):

i.e. not closed

Assume normalized $f(x) \neq 0$ s.t. $(f_i, f) \equiv c_i = \int_a^b f_i^*(x) f(x) dx = 0$ for all i.

Then $\lim_{n\to\infty} \int_a^b | f - \sum_{i=1}^{n} \underset{\overset{\shortparallel}{0}}{c_i} f_i |^2 dx = \int_a^b |f|^2 dx = 1 \neq 0 \Rightarrow \{f_i\}$ is not complete.

if closed then complete (if not complete then not closed):

Assume $f(x)$ exists s.t. $\underbrace{\|f\|^2 > \sum_{n=1}^{\infty} |c_n|^2}_{\text{a convergent series}}$ w/ $c_n = (f_n, f) \Rightarrow \{f_n\}$ is not complete.

With a convergent series, we can make a convergent sequence $\{g_n(x)\}$ w/ $g_n(x) = \sum_{n=1}^{n} c_n f_n(x)$

Now $g_n(x)$ is Cauchy in Hilbert space, and since Hilbert space is complete, the sequence must converge in the mean to a limit in the space. Call it $g(x)$ w/ $c_n = (f_n, g)$.

Now $(f_n, g) = (f_n, f) \Rightarrow (f_n, f-g) = 0$ so $f-g$ is $\perp$ to all f_n.

If $\|f-g\| \neq 0$, then this means that the set $\{f_n\}$ is not closed.

Well, we have $\|f-g\| = \|f - g_n - (g - g_n)\| \geq \big| \|f - g_n\| - \|g - g_n\| \big|$ for all n.

As $n \to \infty$ $\|g - g_n\| \to 0$, but
$$\begin{aligned}
\|f - g_n\|^2 &= \|f - \sum_{n=1}^{n} c_n f_n\|^2 \\
&= (f - \sum_{n=1}^{n} c_n f_n, f - \sum_{k=1}^{n} c_k f_k) \\
&= \|f\|^2 - (f, \sum_{k=1}^{n} c_k f_k) - (\sum_{n=1}^{n} c_n f_n, f) \\
&\quad + (\sum_{n=1}^{n} c_n f_n, \sum_{k=1}^{n} c_k f_k) \\
&= \|f\|^2 - \sum_{k=1}^{n} c_k (f, f_k) - \sum_{n=1}^{n} c_n^* (f_n, f) \\
&\quad + \sum_{n=1}^{n} \sum_{k=1}^{n} c_n^* c_k \underbrace{(f_n, f_k)}_{\delta_{nk}} \\
&= \|f\|^2 - \sum_{k=1}^{n} |c_k|^2 - \sum_{n=1}^{n} |c_n|^2 + \sum_{n=1}^{n} |c_n|^2 \\
&= \|f\|^2 - \sum_{k=1}^{n} |c_k|^2
\end{aligned}$$

But $\|f\|^2 - \sum_{k=1}^{n} |c_k|^2 > 0$ for all n, so $\|f - g\| > 0$ for all n as well, so $\{f_n\}$ is not closed.

Lastly, here are two important results proved in the book:

(a) If two functions f and g in Hilbert space have the same expansion coefficients w.r.t. some basis $\{f_i\}$, i.e. $(f_i, f) = (f_i, g) = c_i$, then $f = g$.

(b) Any function in Hilbert space has a unique set of expansion coefficients w.r.t. to a given basis.