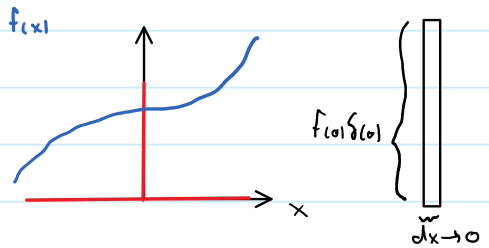
# Dirac's Spike (the $\delta$-function)

The $\delta$-function is not actually a function, but rather a distribution whose existence is only well defined under integration.

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0) \;\begin{cases} \Rightarrow \int_{-\infty}^{\infty} \delta(x)\,dx = 1 \quad \text{since } f(x)=1 \\ \Rightarrow \int_{-\infty}^{\infty} f(x')\,\delta(x'-x)\,dx' = f(x) \end{cases}$$

You could imagine that $\delta(x) = 0$ everywhere except at $x=0$ where it is undefined since




We want $f(0)\,\delta(0)\,dx \to f(0) \Rightarrow \delta(0) \to \frac{1}{0}$

The integral of any "function" that is zero everywhere except a point must be zero.
Hence $\delta$-functions are not functions!

But they can be related to the limits of functions.

(i) $\lim_{n\to 0,\infty} \delta_n(x) = 0$ for all $x \neq 0$

(ii) $\lim_{n\to 0,\infty} \int_{-\infty}^{\infty} f(x)\,\delta_n(x)\,dx = f(0)$

Now if we call $\lim_{n\to 0,\infty} \delta_n(x) = \delta(x)$ and then interchange $\lim_{n\to 0,\infty}$ w/ $\int$ (as Lesbegue let's us do) then the second line becomes $\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$ which is the defining relation.

Okay, so it turns out there are numerous sets of functions which limit to $\delta(x)$.
We will focus on one, which will play a critical role in our next topic.

Consider: $\delta_n(x) = \begin{cases} c_n(1-x^2)^n & \text{for } 0 \le |x| \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$ $\quad n = 1, 2, \ldots$

where $c_n$ are s.t. $\int_{-1}^{1} \delta_n(x)\,dx = 1$

Let's find the $c_n$: $\dfrac{1}{c_n} = \int_{-1}^{1} (1-x^2)^n\,dx = 2\int_0^1 (1-x^2)^n\,dx$

let $x = \sin\theta$, $dx = \cos\theta\,d\theta$ $\quad = 2\int_0^{\pi/2} \cos^{2n+1}\theta\,d\theta$

$\underline{n=1:}$

$$2\int_0^{\pi/2} \cos^3\theta\,d\theta = 2\int_0^{\pi/2}[\cos\theta - \cos\theta\sin^2\theta]\,d\theta$$
$$= 2[\sin\theta]_0^{\pi/2} - 2\tfrac{1}{3}[\sin^3\theta]_0^{\pi/2}$$
$$= 2 - \tfrac{2}{3} = \tfrac{4}{3} = 2^2\,1!/1\cdot 3$$

$\underline{n=2:}$

$$2\int_0^{\pi/2} \cos^5\theta\,d\theta = 2\int_0^{\pi/2}[\cos^3\theta - \cos^3\theta\sin^2\theta]\,d\theta$$
$$= \tfrac{4}{3} - 2\int_0^{\pi/2} \cos^3\theta\sin^2\theta\,d\theta$$
$$= \tfrac{4}{3} - 2\int_0^{\pi/2} \cos\theta(1-\sin^2\theta)\sin^2\theta\,d\theta$$
$$= \tfrac{4}{3} - \tfrac{2}{3}[\sin^3\theta]_0^{\pi/2} + \tfrac{2}{5}[\sin^5\theta]_0^{\pi/2}$$
$$= \tfrac{4}{3} - \tfrac{2}{3} + \tfrac{2}{5} = \tfrac{16}{15} = 2^3\,2!/1\cdot 3\cdot 5$$

Note: $1\cdot 3\cdot 5\cdots(2n+1) = \dfrac{(2n+1)!}{n!\,2^n}$

$\Rightarrow c_n = (2n+1)!/2^{2n+1}(n!)^2$

As $n\to\infty$ how does $c_n$ behave? Go back to $\dfrac{1}{c_n} = 2\int_0^1 (1-x^2)^n\,dx \ge 2\int_0^{1/\sqrt{n}} (1-x^2)^n\,dx$

since $\frac{1}{\sqrt{n}} \le 1$ for all $n$ and the integrand is $\ge 0$ throughout $[0,1]$

Consider $g(x) \equiv (1-x^2)^n - (1-nx^2)$ w/ $g(0) = 0$

$g'(x) = 2nx[1-(1-x^2)^{n-1}] > 0$ for $0 < x \le 1$

then $g(x) \ge 0$ or $(1-x^2)^n \ge (1-nx^2)$ for $0 < x \le 1$

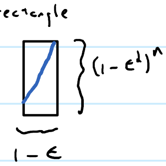
and $\dfrac{1}{c_n} \ge 2\int_0^{1/\sqrt{n}} (1-nx^2)\,dx = \dfrac{4}{3n^{1/2}} > \dfrac{1}{n^{1/2}}$

$\Rightarrow c_n < n^{1/2}$

Finally, consider that since $\delta_n(x)$ is even: $\int_\epsilon^1 \delta_n(x)\,dx = \int_{-1}^{-\epsilon} \delta_n(x)\,dx$ $\quad$ but this is $< 1$ so

but: $\int_\epsilon^1 \delta_n(x)\,dx < n^{1/2}\underbrace{(1-\epsilon^2)^n}_{\text{max value of } \delta_n(x) \text{ in interval}}\underbrace{(1-\epsilon)}_{\text{base of rectangle}} < n^{1/2}(1-\epsilon^2)^n$

Now it turns out that $x^n$ w/ $x<1$ always beats $n^k$ in terms of getting smaller faster than $n^k$ grows.

bound of integral since 


So $\lim_{n\to\infty} \int_\epsilon^1 \delta_n(x)\,dx = 0$

which means that the nonzero contribution comes increasingly from $x \simeq 0$.

Since $\delta_n(x)$ is $\ge 0$ and continuous, $\Rightarrow \lim_{n\to\infty} \delta_n(x) = 0$ for $0 < x \le 1$, but since $\int_{-1}^1 \delta_n(x)\,dx = 1$

$\Rightarrow \lim_{n\to\infty} \int_{-1}^1 f(x)\delta_n(x)\,dx = f(0)$ which is what we expect for $\delta(x)$!

Our δ-function construction is useful in the following (due to Weierstrauss):

If $f(x)$ is continuous on $[a,b]$ then there exists a sequence of polynomials $P_n(x)$ s.t. $\lim_{n\to\infty} P_n(x) = f(x)$ uniformly on $[a,b]$

We will skip the rigorous proof and do heuristic.

For $f(x)$ on $[a,b]$, consider instead $h(\underbrace{\tfrac{x-a}{b-a}}_{z}) = f(x) \Rightarrow f(a) = h(0),\ f(b) = h(1)$

Thus $x \in [a,b] \Rightarrow z \in [0,1]$.

Now if $h(z)$ can be approximated by polynomials in $z \Rightarrow$ so too can $f(x)$ be approximated by polynomials in $x$. So we might as well work w/ $[0,1]$.

Furthermore, consider $g(z) = h(z) - h(0) - z[h(1) - h(0)] \Rightarrow g(0) = g(1) = 0$ but if $h(z)$ is polynomial, so too is $g(z)$. So we can work w/ $g(z)$ on $[0,1]$ s.t. $g(0) = g(1) = 0$.

Now we just set $P_n(z) = \int_{-1}^{1} g(z+t)\underbrace{\delta_n(t)}_{\text{our sequence that limits to } \delta(t)}dt \Rightarrow \lim_{n\to\infty} P_n(z) = g(z)$ Presto!!

While this expression clearly indicates the limit in terms of the δ-function, it isn't very useful in determining the polynomial form. This can be fixed.

Since $g(z) = 0$ for $z > 1$ or $z < 0$ we can write $P_n(z) = \int_{-z}^{1-z} g(z+t)\,\delta_n(t)\,dt$

w/ $g(z+t) = 0$ for $t \leq -z,\ t \geq 1-z$

Then we just replace $t \to t - z$ to obtain:

$$P_n(z) = \int_0^1 g(t)\,\delta_n(t-z)\,dt$$
$$= \int_0^1 g(t)\,c_n\left[1-(t-z)^2\right]^n dt$$

Note that this is clearly a polynomial in $z$. In fact it is of degree $2n$.

Now you might think "Isn't this just gonna re-create the Taylor series for $g(t)$?" The answer is no, but we need an example to see why.

An example:

Suppose $g(z) = \sin(\pi z)$ then $P_n(z) = \int_0^1 g(t)\, c_n \left[1-(t-z)^2\right]^n dt$

$g(0) = g(1) = 0$ $\qquad = c_n \int_0^1 \sin(\pi t) \left[1-(t-z)^2\right]^n dt$

Then: $P_1(z) = \left(\frac{3}{4\pi} + \frac{3}{\pi^3}\right) - \frac{3}{2\pi} z + \frac{3}{2\pi} z^2$

$P_2(z) = \left(\frac{15}{16\pi} - \frac{15}{4\pi^3} + \frac{45}{\pi^5}\right) + \frac{45}{2\pi^3} z + \left(\frac{15}{8\pi} - \frac{45}{2\pi^3}\right) z^2 - \frac{15}{4\pi} z^3 + \frac{5}{8\pi} z^4$

Now to compare w/ a Taylor series, we just take the series and form a sequence:

$$g(z) = \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{d^n}{dz^n} g(z)\right]_{z=0} z^n \Rightarrow P'_K(z) = \sum_{n=0}^{K} \frac{1}{n!}\left[\frac{d^n}{dz^n} g(z)\right]_{z=0} z^n$$

For our example we have:

$P_0'(z) = 0$

$P_1'(z) = 0 + \pi z$

$P_2'(z) = 0 + \pi z + 0$

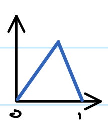
$P_3'(z) = 0 + \pi z + 0 - \frac{\pi^3}{6} z^3$

$P_4'(z) = 0 + \pi z + 0 - \frac{\pi^3}{6} z^3 + 0$

So for Taylor series the coefficients in front of each power of z do not change as we ascend the n-ladder. This ultimately leads to uniform convergence of the series as $n \to \infty$. For the Weierstrass construction the coefficients of everything depend on what step on the n-ladder we are evaluating. This leads to uniform convergence of the sequence as $n \to \infty$.

But in a sense 

since Taylor series require all derivatives of the function to exist, i.e. $g(z)$ must be analytic. Weierstrass only requires continuity.

Therefore: 

$\Rightarrow$ Weierstrass but no Taylor

In fact for Taylor we need to know the function and its derivatives at a point ($z=0$ in example above) whereas for Weierstrass we work w/ an interval and need just the function over the interval.

This leads to the following powerful result regarding Hilbert space (space of s.i. functions):
If $P_n(z)$ uniformly converges to $g(z)$, then it also converges in the mean, i.e. for any $\epsilon > 0$ there exists $n$ s.t. $\| g - P_n \| < \epsilon$.

However any element of Hilbert space (continuous or not) can be approximated in the mean by a continuous function. [This is not obvious, but true] But continuous functions can be approximated in the mean by $P_n$, therefore elements of Hilbert space may be approximated in the mean by $P_n$.

That is, if $\phi$ is a function in Hilbert space and $f$ is a continuous function, then:

$$\phi - P_n = (\phi - f) + (f - P_n)$$

by triangle inequality

$\| \phi - P_n \| \leq \| \phi - f \| + \| f - P_n \|$ but $\phi$ can be approximated by $f$ making $\| \phi - f \|$ as small as we want, and $f$ can be approximated by $P_n$ making $\| f - P_n \|$ as small as we want, therefore $\| \phi - P_n \|$ can be as small as we want $\Rightarrow P_n$ approximates $\phi$.

We are now in a position to show that there exists a complete orthonormal set of polynomials on $[a,b]$, i.e. an orthonormal basis.

Due to Weierstrass, we know that any continuous function $f(x)$ can be approximated uniformly by the sequence $P_n(x) = \sum_{m=0}^{2n} a_{nm} x^m$.

Obviously the set $\{x^0, x^1, x^2, \ldots, x^{2n}\}$ is linearly independent, but not orthogonal.
Who ya gonna call? $\underbrace{\text{Gram}}_{\text{corrected}}$-Schmidt

If we do then we can get $x^m = \sum_{i=0}^{m} c_{mi} Q_i(x)$ where $Q_i(x)$ are orthonormal polynomials up to degree $m$.

Then: $P_n(x) = \sum_{m=0}^{2n} a_{nm} \sum_{i=0}^{m} c_{mi} Q_i(x)$

Now we know the $Q_i(x)$ are orthonormal, but are they complete? Well we know they are complete if they are closed, and they are closed if $(f, Q_n) = 0$ for all $n$ requires $f = 0$.
Note that since $P_n(x)$ is a polynomial in $Q_i(x)$, the condition $(f, Q_n) = 0$ can be replaced w/ $(f, P_n) = 0$.

Now we know from Weierstrass that $f(x)$ can be approximated in the mean by $P_n(x)$.
This means that given any $\epsilon$ there exists an $n$ s.t. $\|f - P_n\| < \epsilon$.

Suppose that $(f, P_n) = 0$, then $\|f - P_n\| = (f - P_n, f - P_n)^{1/2} = [\|f\|^2 + \|P_n\|^2]^{1/2}$
so we then have $\epsilon^2 > \underbrace{\|f\|^2 + \|P_n\|^2}$
this is arbitrarily small therefore both of these are, i.e. $f \to 0$ almost everywhere

Therefore $P_n(x)$ and hence $Q_n(x)$ is closed, thus $Q_n(x)$ is a complete orthonormal set, a basis.

Going back to what we can do w/ a complete orthonormal set $\{Q_i\}$ for Hilbert space, recall that any function on Hilbert space $f(x)$ is converged in the mean to by $f_n(x) = \sum_{i=1}^{n} a_i Q_i(x)$ w/ constant coefficients $a_i$.

Thus $f(x)$ can be approximated in the mean by an infinite <u>series</u>, i.e. $f(x) \doteq \sum_{i=1}^{\infty} a_i Q_i(x)$ w/ $a_i = (Q_i, f)$.

Example: Consider the interval $[-1,1]$ w/ $(f,g) = \int_{-1}^{1} f^*(x) g(x) dx$. We would like to find an orthonormal basis.

Starting w/ $\{x^0, x^1, x^2, \dots\}$ we just need to Gram-Schmidt it.

$$\hat{P}_0(x) = \frac{x^0}{\sqrt{2}} = \frac{1}{\sqrt{2}} \Rightarrow (\hat{P}_0, \hat{P}_0) = \int_{-1}^{1} \frac{1}{2} dx = 1$$

$$\hat{P}_1(x) = \frac{x^1 - \hat{P}_0(\hat{P}_0, x^1)}{\| x^1 - \hat{P}_0(\hat{P}_0, x^1)\|} = \frac{x - \frac{1}{\sqrt{2}}\int_{-1}^{1} \frac{1}{\sqrt{2}} x \, dx}{\| x - \frac{1}{\sqrt{2}} \int_{-1}^{1} \frac{1}{\sqrt{2}} x \, dx \|} = \sqrt{\frac{3}{2}}\, x$$

$$\hat{P}_2(x) = \frac{x^2 - \hat{P}_0(\hat{P}_0, x^2) - \hat{P}_1(\hat{P}_1, x^2)}{\| x^2 - \hat{P}_0(\hat{P}_0, x^2) - \hat{P}_1(\hat{P}_1, x^2)\|} = \sqrt{\frac{5}{2}}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)$$

This gets big and nasty, but it turns out there is a cleaner way.

$$\hat{P}_n(x) = \underbrace{\left(\frac{2n+1}{2}\right)^{1/2}}_{\text{this is the normalization factor}} \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n \Rightarrow \text{example } \hat{P}_2(x) = \left(\frac{5}{2}\right)^{1/2} \frac{1}{8} \frac{d^2}{dx^2}(x^4 - 2x^2 + 1) = \sqrt{\frac{5}{2}}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)$$

Thus:

$$\underbrace{P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n}_{\text{Rodrigues formula}} \quad \text{these are the "Legendre polynomials"}$$

$$\frac{n!}{m!(n-m)!}$$

We can do even better using the binomial theorem: $(x^2-1)^n = \sum_{m=0}^{n} \binom{n}{m} x^{2m} (-1)^{n-m}$

e.g.

$$(x^2-1)^2 = \sum_{m=0}^{2} \binom{2}{m} x^{2m} (-1)^{2-m}$$
$$= \binom{2}{0}(-1)^2 + \binom{2}{1} x^2 (-1)^1 + \binom{2}{2} x^4 (-1)^0$$
$$= 1 - 2x^2 + x^4$$

Then:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \sum_{m=0}^{n} \binom{n}{m} x^{2m} (-1)^{n-m}$$
$$= \frac{1}{2^n n!} \sum_{m \geq p}^{n} (-1)^{n-m} \binom{n}{m} \frac{(2m)!}{(2m-n)!} x^{2m-n}$$

$$p = \begin{cases} \frac{n}{2} & \text{for } n \text{ even} \\ \frac{n+1}{2} & \text{for } n \text{ odd} \end{cases}$$

$$\Rightarrow P_2(x) = \frac{1}{8} \sum_{m=1}^{2} (-1)^{2-m} \binom{2}{m} \frac{(2m)!}{(2m-2)!} x^{2m-2}$$
$$= \frac{1}{8}(-1)\, 2\, \frac{2}{1} x^0 + \frac{1}{8}(-1)^0\, 1\, \frac{24}{2} x^2$$
$$= -\frac{1}{2} + \frac{3}{2} x^2$$

Now recall that Gram-Schmidt can give various orthonormal bases depending on which vectors you pick, and the order that you go through the rest. However, to have the polynomial degree match the labelling $P_n$, it turns out that there is only one-route through Gram-Schmidt, hence only one basis s.t. $P_n = n^{th}$.

We could demonstrate that Rodrigues' formula satisfies the requirements of a complete orthonormal set, or we can just trust that since it gives us the first few Legendre polynomials, it probably works for all.

Are these complete? Well recall from our discussion of Weierstrass that over any finite interval $[a,b]$, the orthornal set due to Gram-Schmidting $\{x^0, x^1, \dots\}$ is complete. Thus so too are these Legendres.

Now as you are probably aware, Legendre polynomials satisfy a differential equation.

To see it start with an identity:

$$(x^2-1)\, d\, (x^2-1)^n = 2nx\,(x^2-1)^n \qquad \text{where } d = \frac{d}{dx}$$

differentiating each side $n+1$ times and using the Leibnitz rule:

$$d^n(uv) = \sum_{m=0}^{n} \binom{n}{m} d^m u\, d^{n-m} v = u\, d^n v + n\, du\, d^{n-1} v + \frac{n(n-1)}{2!} d^2 u\, d^{n-2} v + \frac{n!}{k!(n-k)!} d^k u\, d^{n-k} v + \dots + d^n u\, v$$

we have for the left (since $d^k u = 0$ for $k \geq 3$):

$$d^{n+1}\left[(x^2-1)\, d\,(x^2-1)^n\right] = (x^2-1)\, d^{n+2}(x^2-1)^n + (n+1)2x\, d^{n+1}(x^2-1)^n + n(n+1)\, d^n(x^2-1)^n$$

while for the right:

$$d^{n+1}\left[2nx(x^2-1)^n\right] = 2nx\, d^{n+1}(x^2-1)^n + (n+1)2n\, d^n(x^2-1)^n$$

and the difference of these two must be zero:

$$(x^2-1)\, d^{n+2}(x^2-1)^n + 2x\, d^{n+1}(x^2-1)^n - n(n+1)\, d^n(x^2-1)^n = 0$$

now we grab Rodrigues as $d^n(x^2-1)^n = 2^n n!\, P_n(x)$ and shove him in:

$$(x^2-1)\, 2^n n!\, d^2 P_n(x) + 2x\, 2^n n!\, d\, P_n(x) - n(n+1)\, 2^n n!\, P_n(x) = 0$$

or

$$(1-x^2)\, P_n''(x) - 2x\, P_n'(x) + n(n+1)\, P_n(x) = 0 \qquad \text{Legendre's equation}$$