For Legendre we started on $[-1,1]$ w/ $(f,g) = \int_{-1}^{1} f^*(x) g(x) dx$.

Let's consider other options.

$$(-\infty, \infty) \quad w/ \quad (f,g) = \int_{-\infty}^{\infty} f^*(x) g(x) \underbrace{e^{-x^2}}_{\text{nontrivial weight}} dx$$

This "weight" renders finite what might otherwise be infinite over $(-\infty, \infty)$.

Starting w/ $\{x^0, x^1, x^2, \dots\}$ let's Gram-Schmidt it:

$$\hat{H}_0(x) = \frac{1}{(\pi)^{1/4}} \Rightarrow (\hat{H}_0, \hat{H}_0) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} dx = 1$$

$$\hat{H}_1(x) = \frac{x - \hat{H}_0(\hat{H}_0, x)}{\| x - \hat{H}_0(\hat{H}_0, x) \|} \quad \text{but} \quad (\hat{H}_0, x) = \frac{1}{(\pi)^{1/4}} \int_{-\infty}^{\infty} x e^{-x^2} dx = 0$$

$$= \frac{x}{\|x\|} = \frac{\sqrt{2}}{(\pi)^{1/4}} x \quad \text{since} \quad (x,x) = \int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \frac{1}{2}\sqrt{\pi} \Rightarrow \|x\| = \frac{(\pi)^{1/4}}{\sqrt{2}}$$

$$\hat{H}_2(x) = \frac{x^2 - \hat{H}_0(\hat{H}_0, x^2) - \hat{H}_1(\hat{H}_1, x^2)}{\| x^2 - \hat{H}_0(\hat{H}_0, x^2) - \hat{H}_1(\hat{H}_1, x^2) \|} \quad \text{but} \quad (\hat{H}_0, x^2) = \frac{(\pi)^{1/4}}{2} \quad \text{while} \quad (\hat{H}_1, x^2) = \frac{\sqrt{2}}{(\pi)^{1/4}} \int_{-\infty}^{\infty} x^3 e^{-x^2} dx = 0$$

$$= \frac{x^2 - \frac{1}{2}}{\| x^2 - \frac{1}{2} \|} = \frac{\sqrt{2}}{(\pi)^{1/4}} \left(x^2 - \frac{1}{2} \right) \quad \text{since} \quad \int_{-\infty}^{\infty} x^4 e^{-x^2} dx = \frac{3}{4}\sqrt{\pi} \quad \text{and others from above}$$

Once again we can Rodrigues it and find:

$$\underbrace{H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}}_{\text{These are the "Hermite" polynomials}} \Rightarrow \begin{cases} H_0(x) = (-1)^0 e^{x^2} e^{-x^2} = 1 & \Rightarrow N = (\pi)^{1/4} \\ H_1(x) = (-1)^1 e^{x^2} \frac{d}{dx} e^{-x^2} = -(-2x) = 2x & N = \frac{(\pi)^{1/4}}{2\sqrt{2}} \\ H_2(x) = (-1)^2 e^{x^2} \frac{d^2}{dx^2} e^{-x^2} = e^{x^2} \frac{d}{dx}\left(-2x e^{-x^2}\right) = -2 + 4x^2 & N = \frac{4(\pi)^{1/4}}{\sqrt{2}} \end{cases}$$

Are these complete? Recall that for Legendre, we utilised Weierstrass, however in this case our "interval" is $(-\infty, \infty)$ and so does not fall under the Weierstrass argument. Now since the Hermites are polynomials, we know that closure (and hence completeness) can be shown by demonstrating that $(f(x), x^n) = \int_{-\infty}^{\infty} f(x) x^n e^{-x^2} dx = 0 \Rightarrow f(x)$ is almost 0 for all n. This can be shown via techniques a bit beyond our aims.

To get the equation satisfied by these consider the following generating function:

$\phi(x,t) \equiv e^{x^2} e^{-(t-x)^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$ (from which we can derive Rodrigues)

Note: $\frac{\partial \phi}{\partial x} = 2x e^{x^2} e^{-(t-x)^2} + 2(t-x) e^{x^2} e^{-(t-x)^2} = 2t\phi \Rightarrow \sum_{n=0}^{\infty} \frac{H_n'(x)}{n!} t^n = \sum_{n=0}^{\infty} \frac{2H_n(x)}{n!} t^{n+1}$

Equating powers of $t^n \Rightarrow H_n'(x) = \frac{n!}{(n-1)!} 2 H_{n-1}(x) = 2n H_{n-1}(x)$ RR#1 (recursion relation)

Also note: $\frac{\partial \phi}{\partial t} = -2(t-x)\phi \Rightarrow \sum_{n=0}^{\infty} \frac{H_n(x)}{(n-1)!} t^{n-1} = -2 \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^{n+1} + 2x \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$

Equating powers of $t^n \Rightarrow H_{n+1}(x) = n!\left[-2 \frac{H_{n-1}(x)}{(n-1)!} + 2x \frac{H_n(x)}{n!}\right] = -2n H_{n-1}(x) + 2x H_n(x)$

RR#2

What we would like is an equation involving only $H_n(x)$. Combining RR#1 and RR#2 we have:

$H_{n+1}(x) - 2x H_n(x) + 2n H_{n-1}(x) = 0$ RR#2

$$H_n(x) - 2x \underbrace{H_{n-1}(x)}_{\frac{H_n'(x)}{2n}} + 2(n-1) \underbrace{H_{n-2}(x)}_{\frac{H_{n-1}'}{2(n-1)} = \frac{H_n''(x)}{2n\,2(n-1)}} = 0$$

$\frac{H_n'(x)}{2n} \leftarrow$ RR#1 $\rightarrow \frac{H_{n-1}'}{2(n-1)} = \frac{H_n''(x)}{2n\,2(n-1)}$ using RR#1 again

$H_n(x) - \frac{2x}{2n} H_n'(x) + \frac{1}{2n} H_n''(x) = 0$

$2n H_n(x) - 2x H_n'(x) + H_n''(x) = 0$ for $n \geq 0$ Hermite's equation

You get to do Laguerre in your HW!

We have arrived at the following alternative means of defining Legendre, Hermite and Laguerre polynomials, they are solutions of

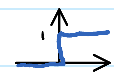
$(x^2-1)P_n''(x) + 2x P_n'(x) - n(n+1)P_n(x) = 0 \quad x\in[-1,1] \quad (f,g) = \int_{-1}^{1} f^* g \, dx$ Legendre

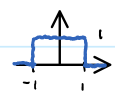
$H_n''(x) - 2x H_n'(x) + 2n H_n(x) = 0 \quad x\in(-\infty,\infty) \quad (f,g) = \int_{-\infty}^{\infty} f^* g e^{-x^2} dx$ Hermite

$x L_n''(x) + (1-x)L_n'(x) + n L_n(x) = 0 \quad \underbrace{x\in[0,\infty) \quad (f,g) = \int_0^{\infty} f^* g e^{-x} dx}$ Laguerre

All of these can be promoted to $(-\infty,\infty)$ intervals using:

In general $w \geq 0$ $\begin{cases} \text{i) Nothing on Hermite (or rather } w(x) = e^{-x^2}) \\ \text{ii) Heaviside for Laguerre} \quad w(x) = H(x) e^{-x} \\ \text{iii) Boxcar for Legendre} \quad w(x) = \text{boxcar}(x) = H(x+1) - H(x-1) \end{cases}$





All of these can be cast as:

$Lu = \lambda u$ w/ $L = \alpha(x)\frac{d^2}{dx^2} + \beta(x)\frac{d}{dx} + \gamma(x)$ and λ a constant (α, β, γ are real).

An important feature of this story is that the L's are Hermitian <u>w.r.t. the inner-products.</u>

Let's explore this in general. First of all, our definition of Hermitian:

L is Hermitian if: $(Lf,g) = \int_{-\infty}^{\infty} (Lf)^* g w \, dx \overset{?}{=} \int_{-\infty}^{\infty} f^* (Lg) w \, dx = (f, Lg)$

So for L to be Hermitian we need the r.h.s. $-$ l.h.s. $= 0$.

r.h.s. $(f, Lg) = \int_{-\infty}^{\infty} f^* [\alpha g'' + \beta g' + \gamma g] w \, dx = \left[f^* \alpha g' w\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (f^* \alpha w)' g' \, dx$
$\quad + \left[f^* \beta g w\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (f^* \beta w)' g \, dx$ } I.B.P.
$\quad + \int_{-\infty}^{\infty} f^* \gamma g w \, dx$

l.h.s. $(Lf, g) =$ same w/ $f^* \leftrightarrow g$ switched

Then: r.h.s. $-$ l.h.s. $= 0 \Rightarrow \left[w\alpha(f^* g' - f^{*\prime} g)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \left[(w\alpha)' - (w\beta)\right](f^* g' - g f^{*\prime}) \, dx = 0$

For L to be Hermitian, this must be true for arbitrary f and g (which means $f^* g' - g f^{*\prime} \neq 0$) so:

1. $\left[w\alpha(f^* g' - f^{*\prime} g)\right]_{-\infty}^{\infty} = 0 \Rightarrow$ either $w\alpha \to 0$ at $x \to \pm\infty$, or the functions $f, g \to 0$ as $x \to \pm\infty$

2. $(w\alpha)' = w\beta \Rightarrow (w\alpha)' = \underbrace{\frac{\beta}{\alpha}}_{\frac{\beta(x)}{\alpha(x)}} w\alpha \Rightarrow \underbrace{w\alpha}_{>0} = C \underbrace{e^{\int \frac{\beta}{\alpha} dx}}_{>0} \Rightarrow \begin{cases} C = 0 \text{ is a solution} \\ \text{If } \alpha > 0 \Rightarrow C > 0, \ \alpha < 0 \Rightarrow C < 0 \end{cases}$

Putting this requirement of Hermiticity w/ the original eqn. we can rewrite:

$Lu = \alpha u'' + \beta u' + \gamma u = \lambda u$

as

$\frac{d}{dx}\left[w\alpha \frac{du}{dx}\right] + (\gamma - \lambda) w u = 0$ $\qquad (w\alpha)'u' + w\alpha u'' + \gamma w u = \cancel{w}\beta u' + \cancel{w}\alpha u'' + \gamma \cancel{w} u = \lambda \cancel{w} u$

which together w/ $\left[w\alpha(f^* g' - f^{*\prime} g)\right]_{-\infty}^{\infty} = 0$ define a Sturm-Liouville system

Now recall that in the finite-dimensional case we had that:

[If A is normal, then the eigenvectors belonging to distinct eigenvalues are orthogonal.

Well it turns out that at least for Hermitian A, this result extends to infinite-dimensional at least for S.L. systems.

Proof: λ_m and λ_n w/ $m \neq n$ $\qquad \frac{d}{dx}\left[(w\alpha)\frac{du_m}{dx}\right] + (\gamma - \lambda_m) w u_m = 0$ $\qquad$ Recall that for

just * the other: $\frac{d}{dx}\left[(w\alpha)\frac{du_n^*}{dx}\right] + (\gamma - \lambda_n) w u_n^* = 0$ $\qquad$ Hermitian $\Rightarrow \lambda = $ real

Then: $u_n^* \frac{d}{dx}\left[(w\alpha)\frac{du_m}{dx}\right] - \lambda_m w u_n^* u_m - u_m \frac{d}{dx}\left[(w\alpha)\frac{du_n^*}{dx}\right] + \lambda_n w u_n^* u_m = 0$

or

$$\frac{d}{dx}\left[u_n^*(w\alpha)\frac{du_m}{dx} - u_m(w\alpha)\frac{du_n^*}{dx}\right] = -(\lambda_n - \lambda_m) w u_n^* u_m$$

Since $\frac{d}{dx}\left[u_n^*(w\alpha)\frac{du_m}{dx} - u_m(w\alpha)\frac{du_n^*}{dx}\right] = \frac{du_n^*}{dx}\cancel{(w\alpha)}\frac{du_m}{dx} - \frac{du_m}{dx}\cancel{(w\alpha)\frac{du_n^*}{dx}} + u_n^* \frac{d}{dx}\left[(w\alpha)\frac{du_m}{dx}\right] - u_m \frac{d}{dx}\left[(w\alpha)\frac{du_n^*}{dx}\right]$

then $\int_{-\infty}^{\infty} dx$ both sides

$$\underbrace{\left[u_n^*(w\alpha)\frac{du_m}{dx} - u_m(w\alpha)\frac{du_n^*}{dx}\right]_{-\infty}^{\infty}} = -(\lambda_n - \lambda_m)\int_{-\infty}^{\infty} w u_n^* u_m dx = -(\lambda_n - \lambda_m)(u_n, u_m)$$

$\left[w\alpha(u_n^* u_m' - u_m u_n^{*\prime})\right]_{-\infty}^{\infty} = 0$ by condition for S.L.

So we have $(\lambda_n - \lambda_m)(u_n, u_m) = 0$ but $\lambda_n \neq \lambda_m \Rightarrow (u_n, u_m) = 0$

Now believe it or not, everything we have said about S.L. systems so far has <u>not</u> been restricted to polynomial functions.

Now let's explore making this restriction.

$$L \underbrace{\underbrace{Q_n}_{} = \lambda_n \underbrace{Q_n}_{}}_{}$$ w/ $L = \alpha \frac{d^2}{dx^2} + \beta \frac{d}{dx} + \gamma$ and $(Q_n, Q_m) = 0$ for $n \neq m$

polynomial polynomial

L must then also be polynomial in α, β, γ. In order not change the degree of Q_n we need:

$\alpha(x) = \alpha_0 x^2 + \alpha_1 x + \alpha_2$

$\beta(x) = \beta_0 x + \beta_1$

$\gamma(x) = \gamma_0$

An interesting observation is that $\beta(x)$ can never be zero, because if it were then $(w\alpha)' = w\beta = 0 \Rightarrow w\alpha = \text{constant}$, but then $[w\alpha(f^* g' - g f^{*'})]_{-\infty}^{\infty} \neq 0$ since f and g being polynomial means they do not vanish at $\pm\infty$.

In fact, $w\alpha$ must $\to 0$ faster than any inverse power of x in order to beat all f and g.

Now it seems that six parameters $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \gamma_0$ are the freedom in defining a S.L. system.

It's actually less than that due to:

1. L may be scaled by $C_1 = \text{constant} \Rightarrow LC_1 \Rightarrow Q_n$ unchanged, $\lambda \to L\lambda$.
2. x may be shifted by $C_2 = \text{constant} \Rightarrow Q_n(x) = Q_n(x + C_2)$ and $\lambda \to \lambda$.
3. x may be scaled by $C_3 = \text{constant} \Rightarrow Q_n(x) = Q_n(C_3 x)$ and $\lambda \to \lambda$.
4. L may be shifted by $C_4 = \text{constant} \Rightarrow L + C_4 \Rightarrow Q_n$ unchanged and $\lambda \to \lambda + C_4$.

The last one means that we can in general set $\gamma_0 = 0$. That leaves five parameters, but owing to 1-3, we can expect there to be only two independent parameters.

So we will organize and analyze things by the order of α.

1. $\alpha(x)$ is quadratic $(\alpha_0 \neq 0)$
2. $\alpha(x)$ is linear $(\alpha_0 = 0, \alpha_1 \neq 0)$
3. $\alpha(x)$ is constant $(\alpha_0 = \alpha_1 = 0, \alpha_2 \neq 0)$

If not specified, coefficient can be zero or not!

$\underline{\underline{\alpha(x) \text{ is quadratic}}}$:

First let's choose C_1 s.t. $C_1 L \Rightarrow \alpha_0 = 1$ then $w\alpha = Ce^{\int \frac{\beta}{\alpha} dx} = C \exp\left[\int \frac{\beta_0 x + \beta_1}{x^2 + \alpha_1 x + \alpha_2} dx\right]$

Now $\alpha(x) = x^2 + \alpha_1 x + \alpha_2$ is composed of real coefficients, but the roots of $\alpha(x)$ could be complex.

For complex roots: $\alpha(x) = (x-k)(x-k^*)$ and $w\alpha = C\,[\alpha(x)]^{\beta_0/2} \exp\Big[\underbrace{\frac{\beta_1 + \beta_0 \operatorname{Re} k}{|\operatorname{Im} k|} \tan^{-1}\left(\frac{x - \operatorname{Re} k}{|\operatorname{Im} k|}\right)}_{\text{Stuff}}\Big]$

Since $\alpha(x)$ is a polynomial and we need $[w\alpha(f^* g' - g f^{*\prime})]_{-\infty}^{\infty} = 0$, there are obviously polynomials for g w/ powers such that $w\alpha$ doesn't vanish fast enough.
Also $w\alpha$ is never zero (since $\exp[\text{stuff}] \neq 0$ and the roots are complex so only $= 0$ for values of x off the real line).
Thus complex roots won't work.

For real roots: Using C_1, C_2, C_3 we can make $\alpha(x) = 1 - x^2$

$$C_1: L \to \frac{4\alpha_0}{4\alpha_0\alpha_2 - \alpha_1^2} L \qquad C_3: x \to \frac{1}{2\alpha_0}\sqrt{\alpha_1^2 - 4\alpha_0\alpha_2}\; x$$

$$\alpha_0 x^2 + \alpha_1 x + \alpha_2 \Rightarrow \frac{4\alpha_0}{4\alpha_0\alpha_2 - \alpha_1^2}(\alpha_0 x^2 + \alpha_1 x + \alpha_2) \Rightarrow \frac{4\alpha_0}{4\alpha_0\alpha_2 - \alpha_1^2}\left[\frac{1}{4\alpha_0}(\alpha_1^2 - 4\alpha_0\alpha_2)x^2 + \frac{\alpha_1}{2\alpha_0}\sqrt{\alpha_1^2 - 4\alpha_0\alpha_2}\; x + \alpha_2\right]$$

$$C_2: x \to x - \frac{\alpha_1}{\sqrt{\alpha_1^2 - 4\alpha_0\alpha_2}}$$

$$\Rightarrow \frac{-4\alpha_0}{\alpha_1^2 - 4\alpha_0\alpha_2}\left[\frac{1}{4\alpha_0}(\alpha_1^2 - 4\alpha_0\alpha_2)\left(x^2 - \frac{2\alpha_1}{\sqrt{\alpha_1^2 - 4\alpha_0\alpha_2}} x + \frac{\alpha_1^2}{\alpha_1^2 - 4\alpha_0\alpha_2}\right) + \frac{\alpha_1}{2\alpha_0}\sqrt{\alpha_1^2 - 4\alpha_0\alpha_2}\; x - \frac{\alpha_1^2}{2\alpha_0} + \alpha_2\right]$$

$$= -\left(x^2 - \frac{2\alpha_1}{\sqrt{\alpha_1^2 - 4\alpha_0\alpha_2}} x + \frac{\alpha_1^2}{\alpha_1^2 - 4\alpha_0\alpha_2}\right) - \frac{2\alpha_1}{\sqrt{\alpha_1^2 - 4\alpha_0\alpha_2}} x + \frac{2\alpha_1^2}{\alpha_1^2 - 4\alpha_0\alpha_2} - \frac{4\alpha_0\alpha_2}{\alpha_1^2 - 4\alpha_0\alpha_2}$$

$$= -x^2 + 1$$

We can also use $\beta_1 = q - p$, $\beta_0 = -(p+q+2)$ (just a redefinition) and then things clean up nicely:

$$\frac{\beta}{\alpha} = \frac{-(p+q+2)x + q - p}{1 - x^2} = \frac{q+1}{1+x} - \frac{p+1}{1-x} \Rightarrow w\alpha = Ce^{\int \frac{\beta}{\alpha} dx} = C(1+x)^{q+1}(1-x)^{p+1}$$

$$= C(1+x)^q(1-x)^p(1-x^2)$$

$$\Rightarrow w(x) = C(1+x)^q(1-x)^p$$

Now, once again $w\alpha$ won't go to zero fast enough to beat higher powers of x in f or g.
But the roots of α at $x = \pm 1$ allow us to use the nonzero part between these and glue it to 0 for $x < -1$, $x > 1$ (the boxcar). So we are led to $[-1, 1]$ though this can easily be shifted to any $[a, b]$ by adjusting c_1, c_2, c_3.

So in the end $[-1,1]$ w/ $w(x)=(1+x)^q(1-x)^p$ works and gives rise to the Jacobi polynomials $J_n^{p,q}(x)$ where p,q refer to the values in $w(x)$. That is, you pick p and q and using $w(x)$ find an entire set of polynomials specific to these p and q.

Special cases:

If $p=q=m$ $(n>1)$ w/ $w(x)=(1-x^2)^m$ $\Rightarrow$ Gegenbauer polynomials $G_n^m(x)=J_n^{m,m}(x)$.

If $p=q=-\frac{1}{2}$ w/ $w(x)=(1-x^2)^{-1/2}$ $\Rightarrow$ Tschebycheff polynomials $T_n(x)=J_n^{-\frac{1}{2},-\frac{1}{2}}(x)$.

If $p=q=0$ w/ $w(x)=1$ $\Rightarrow$ Legendre polynomials $P_n(x)=J_n^{0,0}(x)$.

$\alpha(x)$ is linear:

$\alpha(x)=x+\alpha_2$ (by C_1) $\Rightarrow$ $\alpha(x)=x$ (by C_2) $\Rightarrow$ $\alpha(x)=0$ @ $x=0$

then: $w\alpha = C\exp\left[\int \frac{\beta_0 x+\beta_1}{x}dx\right] = Cx^{\beta_1}e^{\beta_0 x}$

If $\beta_0<0$ then $w\alpha\to 0$ fast enough w/ $x\to\infty$ for any power of x in f or g, but of course as $x\to-\infty$ then $w\alpha\to\infty$.

So we use the root at $x=0$ to pair it up with 0 for all $x<0$ (Heaviside).

Using C_3 to set $\beta_0=-1$ and calling $\beta_1=s+1$ w/ $s>-1$ then $w(x)=x^s e^{-x}$ for $[0,\infty)$.

For $s=0$ these are the Laguerre polynomials $L_n(x)=L_n^0(x)$.

For $s\neq 0$ they are the associated Laguerre polynomials $L_n^s(x)$.

$\alpha(x)$ is constant:

$C_1 \Rightarrow \alpha_2 = 1 \Rightarrow \alpha(x) = 1 \Rightarrow w\alpha = C\exp\left[\int(\beta_0 x + \beta_1)\,dx\right]$

$$= C\exp\left[\frac{\beta_0}{2}x^2 + \beta_1 x\right]$$

$$= \exp\left(-\frac{\beta_1^2}{2\beta_0}\right) C \exp\left[\frac{\beta_0}{2}\left(x + \frac{\beta_1}{\beta_0}\right)^2\right]$$

If $\beta_0 < 0$ then $w\alpha = w$ falls exponentially as $|x| \to \infty$ so it works!

w/ C_2 and C_3 we make $\beta_0 = -2$, $\beta_1 = 0$ and $c' = 1 \Rightarrow w(x) = e^{-x^2}$ w/ $[-\infty, \infty]$

In this case $Q_n(x)$ are the Hermites.

Various properties

Eigenvalues: $LQ_n = \lambda_n Q_n \Rightarrow (\alpha_0 x^2 + \alpha_1 x + \alpha_2)\frac{d^2}{dx^2}Q_n + (\beta_0 x + \beta_1)\frac{d}{dx}Q_n = \lambda_n Q_n$

All x^n terms must have coefficients that satisfy this so $\alpha_0 n(n-1) + \beta_0 n = \lambda_n = n(\alpha_0 n + \beta_0 - \alpha_0)$

Thus λ_ns are spaced linear in n for $\alpha_0 = 0$ cases (Hermite and Laguerre) and quadratically for $\alpha_0 \neq 0$ or Jacobi.

Rodrigues: Recall
$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \quad \text{Legendre w/ } w(x)=1$$
$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} \quad \text{Hermite w/ } w(x)=e^{-x^2}$$
$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^n) \quad \text{Laguerre w/ } w(x)=e^{-x}$$

These can all be generated from: $Q_n(x) = \underbrace{k_n}_{\text{constant}}\frac{1}{w(x)}\frac{d^n}{dx^n}(\alpha(x)^n w(x))$ ← $\alpha(x)$ raised to n power

Completeness:

The orthonormal set of Sturm-Liouville polynomials $\{Q_n(x)\}$ is complete in Hilbert space.

To prove it we can show they are closed, hence complete.
To prove closed we need to show that if $(f, Q_n)=0$ for all $n \Rightarrow f$ is almost 0.
Since Q_n are polynomials in x, then x^n can be written as lin. comb. of $Q_n(x)$.
That is if $(f, Q_n)=0 \Rightarrow (f, x^n)=0$ for all n.

Consider: $g(k) = (e^{ikx}, f) = \int_{-\infty}^{\infty} f(x)w(x)e^{-ikx}\,dx$ w/ real k
Expanding: $e^{-ikx} = \sum_{n=0}^{\infty}\frac{(-ik)^n}{n!}x^n$ and using $(x^n, f)=0 \Rightarrow g(k)=0$ for all k.

But an argument to come later implies that the above implies $f(x)w(x)=0$ almost everywhere.
$\Rightarrow$ whenever $w(x)\neq 0$, $f(x)=0$ almost everywhere $\Rightarrow$ S.L. polys are closed, hence complete.