Various properties

Eigenvalues: $LQ_n = \lambda_n Q_n \Rightarrow (\alpha_0 x^2 + \alpha_1 x + \alpha_2)\frac{d^2}{dx^2}Q_n + (\beta_0 x + \beta_1)\frac{d}{dx}Q_n = \lambda_n Q_n$

All x^n terms must have coefficients that satisfy this so $\alpha_0 n(n-1) + \beta_0 n = \lambda_n = n(\alpha_0 n + \beta_0 - \alpha_0)$

Thus λ_ns are spaced linear in n for $\alpha_0 = 0$ cases (Hermite and Laguerre) and quadratically for $\alpha_0 \neq 0$ or Jacobi.

Rodrigues: Recall

$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$ Legendre w/ $w(x)=1$

$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$ Hermite w/ $w(x)=e^{-x^2}$

$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^n)$ Laguerre w/ $w(x)=e^{-x}$

These can all be generated from: $Q_n(x) = \underbrace{k_n}_{\text{constant}}\frac{1}{w(x)}\frac{d^n}{dx^n}(\alpha(x)^n w(x))$ $\alpha(x)$ raised to n power

Completeness:

[The orthonormal set of Sturm-Liouville polynomials $\{Q_n(x)\}$ is complete in Hilbert space.

To prove it we can show they are closed, hence complete.

To prove closed we need to show that if $(f, Q_n) = 0$ for all $n \Rightarrow f$ is almost 0.

Since Q_n are polynomials in x, then x^n can be written as lin. comb. of $Q_n(x)$.

That is if $(f, Q_n) = 0 \Rightarrow (f, x^n) = 0$ for all n.

Consider: $g(k) = (e^{ikx}, f) = \int_{-\infty}^{\infty} f(x)w(x)e^{-ikx}\,dx$ w/ real k

Expanding: $e^{-ikx} = \sum_{n=0}^{\infty}\frac{(-ik)^n}{n!}x^n$ and using $(x^n, f) = 0 \Rightarrow g(k) = 0$ for all k.

But an argument to come later implies that the above implies $f(x)w(x) = 0$ almost everywhere.

$\Rightarrow$ whenever $w(x) \neq 0$, $f(x) = 0$ almost everywhere $\Rightarrow$ S.L. polys are closed, hence complete.

Okay, so let's do a quick review:

For continuous $f(x)$ on $[a,b]$ there is a uniformly converging sequence of polynomials $P_N(x)$,
$f(x) = \lim_{N\to\infty} P_N(x) = \lim_{N\to\infty} \sum_{n=0}^{N} C_N x^n$

Any square integrable function $g(x)$ on $[a,b]$ can be approximated in the mean by a continuous function, hence in the mean by $P_N(x)$ as well, $g(x) \doteq \lim_{N\to\infty} P_N(x)$.

Note that in the convergent sequence $P_N(x)$, the coefficients generally change as we increase N. This is in contrast to a convergent series, where the coefficients of each degree are unchanged as we calculate higher degree contributions.

Therefore, if starting w/ $\{x^0, x^1, \ldots, x^{\infty}\}$ we can Gram-Schmidt to obtain a complete orthonormal basis $Q_i(x)$ which provides a series which converges to $g(x)$ in the mean,
$g(x) \doteq \sum_{i=1}^{\infty} a_i Q_i(x)$

We had two methods for looking for sets of orthogonal polynomials:

1. Specify the interval $[a,b]$ and weight $w(x)$ of inner product $(f,g) = \int_a^b f^*(x) g(x) w(x) dx$
Then just G.S. to find them. Note: G.S. uses the inner-product and hence $[a,b]$ and $w(x)$, G.S. gives a unique result if we demand that the degree of each matches the label on it.
From this, we can find the differential equation satisfied by the functions.

2. Specify the differential equation $\Rightarrow$ $[a,b]$ and $w(x)$! For $n \leq 2$ it's S.L.

Having "completed" good old polynomials, it is now time to turn to more sophisticated basis functions. Well, we'll use the polynomial path to sneak up on them!

Going back to Weierstrass:

If $f(x)$ is continuous on $[a,b]$ then there exists a sequence of polynomials $P_N(x)$ s.t. $\lim_{N\to\infty} P_N(x) = \lim_{N\to\infty} \sum_{n=0}^{N} c_n x^n = f(x)$ uniformly on $[a,b]$.

one may ask about multivariate extensions, to which one finds:

If $f(x_1, x_2, \ldots, x_k)$ is continuous in each variable over $x_i \in [a_i, b_i]$, then there exists a sequence of polynomials $P_N(x_1, x_2, \ldots, x_k)$ s.t. $\lim_{N\to\infty} P_N(x_1, x_2, \ldots, x_k) = \lim_{N\to\infty} \sum_{n,m,\ldots,p=0}^{N} A^{(N)}_{nm\ldots} x_1^n x_2^m \cdots x_k^p = f(x_1, x_2, \ldots, x_k)$ uniformly on $[a_i, b_i]$.

Now suppose we take $k=2$: $P_N(x_1, x_2) = \sum_{n,m=0}^{N} A^{(N)}_{nm} x_1^n x_2^m$ (recall $A^{(N)}_{nm}$ depends on N, so not a power series)

Let's restrict x_1 and x_2 to lie on the unit circle and use this to define $x_1 = \cos\theta$ and $x_2 = \sin\theta$.

Then: $f(\cos\theta, \sin\theta) \equiv f(\theta) = \lim_{N\to\infty} \sum_{n,m=0}^{N} A^{(N)}_{nm} \cos^n\theta \sin^m\theta \Rightarrow f(\theta + 2\pi) = f(\theta)$.

Rewriting w/ Euler and replacing $\theta \to x$ and inserting $(2\pi)^{1/2}$ for convenience gives:

Note, not powers!

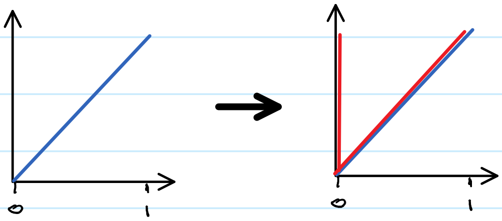
$$f(x) = \lim_{N\to\infty} f_N(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{c_n^{(N)}}{(2\pi)^{1/2}} e^{inx} = \lim_{N\to\infty} \left[\frac{a_0^{(N)}}{2} + \sum_{n=1}^{N} \left(a_n^{(N)} \cos nx + b_n^{(N)} \sin nx \right) \right]$$

so a uniform approximation to $f(x)$ is given by the <u>trigonometric polynomials</u>:

$$f_N(x) = \sum_{n=-N}^{N} \frac{c_n^{(N)}}{(2\pi)^{1/2}} e^{inx} = \frac{a_0^{(N)}}{2} + \sum_{n=1}^{N} \left(a_n^{(N)} \cos nx + b_n^{(N)} \sin nx \right)$$ (again a sequence not a series)

Now so far everything has been restricted to periodic functions, $f(x+2\pi) = f(x)$. Well it turns out that we can extend it to non-periodic functions as long as we are willing to trade uniform convergence for convergence in the mean.

Consider the very non-periodic function (in blue) over $[0,1]$ and its periodic approximation in red:



Clearly $|R(x) - B(x)| \not< \epsilon$ for arbitrary ϵ! So no uniform (or pointwise) convergence. However $\int_0^1 |R(x) - B(x)|^2 dx < \epsilon$ for any ϵ so $R(x)$ converges in the mean to $B(x)$, i.e. $R(x) \doteq B(x)$.

So we may say that any function $g(x)$ over an interval $[a,b]$ can be approximated in the mean by a periodic function $f(x)$ over the same interval.

Therefore the trigonometric sequence that converges uniformly to the periodic $f(x)$ will converge in the mean to non-periodic $g(x)$.

Also recall that any function in Hilbert space (of s.i. functions) can be approximated in the mean by a continuous function, therefore any function in Hilbert space can be approximated in the mean by the trigonometric sequence given.

Okay, so we have an approximate <u>sequence</u>, converges uniformly "$=$" to periodic functions and in the mean "$\doteq$" to arbitrary functions in Hilbert space.

What about a series (where the coefficients do <u>not</u> change as we go to larger n)?
As an example of an orthonormal set (though not necessarily a complete basis) consider the Fourier functions:
$f_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$ $n = 0, \pm 1, \pm 2, \cdots$ over the interval from $-\pi$ to π

$$(f_m, f_n) = \int_{-\pi}^{\pi} f_n^* f_m \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-n)x} dx = \frac{1}{2\pi i(m-n)} e^{i(m-n)x} \Big|_{-\pi}^{\pi} = 0 \text{ if } m \neq n$$

$$(f_n, f_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^0 dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx = 1$$

Let's first prove its completeness. We proceed as usual by showing it is closed, hence complete.
To show it is closed we demonstrate that if $(f, T_n) = 0$ for all $n \Rightarrow f = 0$ almost everywhere.
So start w/ $(f, T_n) = 0$ for all n. Now since trigonometric f_N from earlier can be written as lin. combs. of the orthonormal T_n, this implies $(f, f_N) = 0$ for all N.
However any function can be approximated in the mean by f_N, that is:

$$\| f - f_N \| = (\|f\|^2 - 2\underbrace{(f, f_N)}_{=0} + \|f_N\|^2)^{1/2} < \epsilon \Rightarrow \|f\| \text{ and } \|f_N\| = 0 \text{ almost everywhere.}$$

Therefore $\{T_n\}$ is a complete orthonormal set. This tells us that we can approximate in the mean an arbitrary $f(x)$ by an infinite series in T_n, i.e.

The Fourier Series

$$f(x) \doteq \sum_{n=-\infty}^{\infty} c_n T_n(x) = \sum_{n=-\infty}^{\infty} \frac{c_n}{(2\pi)^{1/2}} e^{inx} = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

and in this case the coefficients are given by:

$$c_n = (T_n, f) = \frac{1}{(2\pi)^{1/2}} \int_{-\pi}^{\pi} f(x) e^{-inx} dx, \quad \overset{n=0,1,\cdots}{a_n} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad \overset{n=1,2,\cdots}{b_n} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

Since complete we also know: $(f, f) = \int_{-\pi}^{\pi} |f|^2 dx = \sum_{-\infty}^{\infty} |c_n|^2 = \pi \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right]$

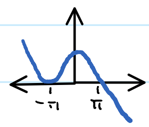
So what do we have so far? Both use $[-\pi, \pi]$

1. We have a trigonometric sequence which converges uniformly to continuous $f(x)$ w/ $f(-\pi) = f(\pi)$.
2. We have the Fourier series which converges in the mean to any function on Hilbert space.

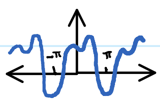
What we would like to do is promote the convergence of Fourier series from mean $\to$ uniform.

The convergence of the Fourier series to $f(x)$ is uniform in $[-\pi, \pi]$ if $f(x)$ is continuous and its derivative is piecewise continuous and $f(-\pi) = f(\pi)$. If, in addition $f(x+2\pi) = f(x)$, the convergence will be uniform everywhere.

Note for the first case we can approximate uniformly on $[-\pi, \pi]$ something like



While for the second we need:



Preliminaries:

Let's start w/ the derivative as a series: $f'(x) = \frac{a_0'}{2} + \sum_{n=1}^{\infty} (a_n' \cos nx + b_n' \sin x)$

Note that a_0', a_n' and b_n' are just labels, **not** derivatives of (constant) coefficients.

Now recall that coefficients can be obtained via:

$a_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx\,dx = \frac{1}{\pi}\left[f\cos nx\right]_{-\pi}^{\pi} + \frac{n}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{n}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = n b_n$

$b_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx = -\frac{n}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx = -n a_n$

$a_0' = \int_{-\pi}^{\pi} f'(x)\,dx = f(\pi) - f(-\pi) = 0$

Since f' is piecewise continuous on $[-\pi, \pi]$ (hence s.i.) and since the functions form a complete set then: $(f', f') = \pi\left[\frac{a_0'^2}{2} + \sum_{n=1}^{\infty}(a_n'^2 + b_n'^2)\right] = \pi\sum_{n=1}^{\infty} n^2(b_n^2 + a_n^2)$

The Meat of the Matter:

We want to use the Cauchy criterion for uniform convergence of the Fourier series.

Reminder:

A sequence of functions $h_n(x)$ converges $\left\{\begin{matrix}\text{pointwise}\\ \text{uniformly}\\ \text{in the mean}\end{matrix}\right\}$ to $h(x)$ on $[a,b]$ if for every ϵ there exists $\left\{\begin{matrix}N(x,\epsilon)\\ N(\epsilon)\\ N(\epsilon)\end{matrix}\right\}$ s.t. for $n > N$: $\left\{\begin{matrix}|h_n(x) - h(x)|\\ |h_n(x) - h(x)|\\ \int_a^b |h_n(x) - h(x)|\end{matrix}\right\} < \epsilon$.

All of these levels of convergence require one to know what it is converging to, i.e. $h(x)$.

However for Cauchy:

[The sequence $h_n(x)$ converges uniformly on $[a,b]$ if for every $\epsilon > 0$ there exists an $N(\epsilon)$ s.t. for all $n,m > N$ and $x \in [a,b]$, $|h_n(x) - h_m(x)| < \epsilon$.

So using Cauchy will give us uniform convergence, but to what? Well remember that we already have convergence in the mean, so that answers this question.

Here we go:

To form a sequence (as opposed to a series) we just: $S_n = \frac{a_0}{2} + \sum_{p=1}^{n} a_p \cos px + \sum_{p=1}^{n} b_p \sin px$

Then:

$$|S_n - S_m| = \left|\sum_{p=n+1}^{n} (a_p \cos px + b_p \sin px)\right| \quad \text{assuming } n > m$$

$$= \underbrace{\left|\sum_{p=m+1}^{n} \frac{1}{p}(p a_p \cos px + p b_p \sin px)\right|}$$

This is like a dot product between $(\frac{1}{p}, f(p)) \leq \sqrt{(\frac{1}{p},\frac{1}{p})(f(p),f(p))}$

$$\leq \sqrt{\sum_{p=m+1}^{n} \frac{1}{p^2}}\sqrt{\sum_{p=m+1}^{n} |p a_p \cos px + p b_p \sin px|^2}$$

$$p^2|a_p \cos px + b_p \sin px|^2 = p^2\left|\sqrt{a_p^2+b_p^2}\cos\left(px - \tan^{-1}\frac{b_p}{a_p}\right)\right|^2 \leq p^2\left|\sqrt{a_p^2+b_p^2}\right|^2$$

obviously $\leq$

$$\leq \sqrt{\sum_{p=1}^{\infty} \frac{1}{p^2}}\sqrt{\sum_{p=1}^{\infty} p^2\left|\sqrt{a_p^2+b_p^2}\right|^2} = \sqrt{\sum_{p=1}^{\infty} \frac{1}{p^2}}\sqrt{\frac{1}{\pi}(f',f')} = \sqrt{\sum_{p=1}^{\infty} \frac{1}{p^2}}\underbrace{\sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi} |f'(x)|^2 dx}}_{\equiv M}$$

$|S_n - S_m| \leq \sqrt{\sum_{p=1}^{\infty} \frac{1}{p^2}}\, M$ where M is finite because f' is piecewise continuous and finite.

Now it turns out that $\sum_{p=1}^{\infty} \frac{1}{p^2}$ converges (to $\frac{\pi^2}{6}$) and so must satisfy the Cauchy condition, hence for $|k_n - k_m| = \left|\sum_{p=1}^{n} \frac{1}{p^2} - \sum_{p=1}^{m} \frac{1}{p^2}\right| = \sum_{p=m+1}^{n} \frac{1}{p^2} < \epsilon$ for $n, m > N(\epsilon)$. Instead of ϵ let's call it $\frac{\epsilon^2}{M^2}$, hence $\sum_{p=m+1}^{n} \frac{1}{p^2} < \frac{\epsilon^2}{M^2}$ and $N(\frac{\epsilon^2}{M^2})$.

But then $|S_n - S_m| \leq \frac{\epsilon}{M} M = \epsilon$ for $n, m > N(\frac{\epsilon^2}{M^2})$ so Cauchy is satisfied and the sequence converges uniformly.

Of course the sequence S_n is a sequence of series, so the series is converging uniformly as well, and adding in the target of convergence in the mean, the Fourier series is uniformly convergent to $f(x)$. As long as $f(x)$ is continuous and its derivative is piecewise continuous, and $f(-\pi) = f(\pi)$.

Clearly we can adjust the interval from $[-\pi, \pi]$ to $[-L, L]$ w/ $f(x) = \sum_{n=-\infty}^{\infty} c_n \frac{e^{in\pi x/L}}{(2L)^{1/2}}$

$$c_n = \frac{1}{(2L)^{1/2}}\int_{-L}^{L} f(x) e^{-in\pi x/L} dx$$

For $f(x)$ an $\begin{cases} \text{even function} \Rightarrow b_n = 0 & \cos(nx) \text{ series} \\ \text{odd function} \Rightarrow a_n = 0 & \sin(nx) \text{ series} \end{cases}$

And lastly, the differential (eigenvector) equation satisfied by these is: $\frac{d^2 u}{dx^2} = -n^2 u$