More Fourier Fun... Opening our Minds Beyond Periodicity or Intervals

We have so far found that the Fourier series uniformly converges for continuous functions w/ piecewise continuous derivatives either over a finite interval $[a,b]$ w/ $f(a)=f(b)$ or everywhere for periodic functions $f(x+L)=f(x)$.

But there are certainly many functions that are not periodic that we would like to approximate over the whole real line $x \in (-\infty,\infty)$.

Clearly we can adjust the interval from $[-\pi,\pi]$ to $[-L,L]$ w/ $f(x)=\sum_{n=-\infty}^{\infty} c_n \frac{e^{in\pi x/L}}{(2L)^{1/2}}$

$$c_n = \frac{1}{(2L)^{1/2}}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx$$

Okay, so let's just try taking $L\to\infty$. To get ready define $\underbrace{\left(\frac{\pi}{L}\right)^{1/2}x \equiv y,\ n\left(\frac{\pi}{L}\right)^{1/2} \equiv k_n}$

$n\left(\frac{\pi}{L}\right)x = k_n y$

and

$\left(\frac{\pi}{L}\right)^{1/2} = k_{n+1}-k_n = \Delta k_n$

and

$dx = \left(\frac{L}{\pi}\right)^{1/2}dy$

then our $[-L,L]$ set becomes:

$$f(y) = \frac{1}{(2\pi)^{1/2}}\sum_{k_n=-\infty}^{\infty} \underbrace{g_{k_n}}_{n\to\Delta k_n} e^{ik_n y}\Delta k_n$$

and

$$g_{k_n} = \frac{1}{(2\pi)^{1/2}}\int_{-\sqrt{\pi L}}^{\sqrt{\pi L}} f(y)e^{-ik_n y}dy$$

And now we take the limit as $L\to\infty$, in which case the discrete steps Δk_n become smooth since $\Delta k_n \to 0$. In this case $\sum_{\Delta k_n} \Delta k_n \Rightarrow \int dk$.

$f(y)$ is the Fourier transform of $g(k)$, and vice versa

$$\left\{\begin{aligned} f(y) &= \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} g(k)e^{iky}dk \\ &\text{and} \\ g(k) &= \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} f(y)e^{-iky}dy \end{aligned}\right.$$

Recall that in proving completeness in Sturm-Liouville we used that if $g(k)=\int_{-\infty}^{\infty} f(x)w(x)dx\, e^{-ikx}dx = 0$
$\Rightarrow f(x)w(x)=0$ (as is now obvious)

Fun facts about F. I.s

Shoving $g(k)$ into $f(y)$: $f(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(y')e^{-iky'}dy'\right]e^{iky}dk$

and then switcherooing $\int dy' \leftrightarrow \int dk$

$$f(y) = \int_{-\infty}^{\infty} f(y')\underbrace{\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(y-y')}dk\right]}_{\Rightarrow\ \delta(y-y')}dy'$$

One more time : $\int_{-\infty}^{\infty}|f(y)|^2dy = \int_{-\infty}^{\infty}\left[\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} g^*(k)e^{-iky}dk\,\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} g(k')e^{ik'y}dk'\right]dy$

switcherooing

$$= \int_{-\infty}^{\infty}\left[g^*(k)\int_{-\infty}^{\infty}dk'\,g(k')\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(k'-k)y}dy\right]\right]dk$$

$$= \int_{-\infty}^{\infty}\left[g^*(k)\int_{-\infty}^{\infty}dk'\,g(k')\,\delta(k'-k)\right]dk$$

$$= \int_{-\infty}^{\infty} g^*(k)g(k)dk = \int_{-\infty}^{\infty}|g(k)|^2dk$$

An interesting question to ask is "Is the Fourier transform of a product of functions equal to the product of the Fourier transforms of the individual functions?" The answer of course is no, in part because the transforms involve calculus where we know $\frac{d}{dx}(uv) \neq \frac{du}{dx}\frac{dv}{dx}$, $\int(uv) \neq \int u\int v$. But what does it give?

$$G(k) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} f_1(y)f_2(y)e^{-iky}dy$$

$$= \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\left[\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} g_1(k')e^{ik'y}dk'\right]f_2(y)e^{-iky}dy$$

switcherooing $dk' \leftrightarrow dy$

$$= \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} g_1(k')\left[\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} f_2(y)e^{ik'y-iky}dy\right]dk'$$

$$= \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} g_1(k')g_2(k-k')dk'$$ which is called the "convolution" of g_1 and g_2.

In d-dimensions we have : $F(\vec{r}) = \frac{1}{(2\pi)^{d/2}}\left(\int_{-\infty}^{\infty}\right)^d G(\vec{k})e^{i\vec{k}\cdot\vec{r}}d\vec{k}$ w/ $d\vec{k} = dk_1dk_2\cdots dk_d$

$G(\vec{k}) = \frac{1}{(2\pi)^{d/2}}\left(\int_{-\infty}^{\infty}\right)^d F(\vec{r})e^{-i\vec{k}\cdot\vec{r}}d\vec{r}$ w/ $d\vec{r} = dx_1dx_2\cdots dx_d$

and of course

$$\delta^d(\vec{r}-\vec{r}_0) = \delta(x_1-x_{10})\,\delta(x_2-x_{20})\cdots\delta(x_d-x_{d0}) = \frac{1}{(2\pi)^d}\int_{-\infty}^{\infty} e^{i\vec{k}\cdot(\vec{r}-\vec{r}_0)}d\vec{k}$$

Okay let's review our process:

First we use Weierstrass to argue that if $f(x)$ is continuous on $[a,b]$ there exists a sequence of polynomials $P_n(x)$ s.t. $\lim_{n\to\infty} P_n(x) = f(x)$ w/ uniform (hence also mean) convergence.

Then we take the lin. ind. set of which $P_n(x)$ is a superposition and G.S. it if needed to get an orthonormal basis $\{Q_n\}$. Then $P_n(x) = \sum_{i=0}^{n} \underbrace{c_{ni}}_{\text{depend on } n} Q_i(x)$.

We then prove the completeness of this basis via closure using if $(f, Q_n) = 0 \Rightarrow f = 0$, but if $(f, Q_n) = 0 \Rightarrow (f, P_n) = 0$ but we know that P_n converges in the mean to $f \Rightarrow \|f - P_n\| < \epsilon$ hence $\|f\|^2 + \|P_n\|^2 < \epsilon^2 \Rightarrow f = 0$ almost everywhere.

But since $\{Q_n\}$ is complete and orthonormal, then $f_n(x) = \sum_{i=0}^{n} c_i Q_i(x)$ which converges in the mean $f(x) \doteq \sum_{i=0}^{\infty} c_i Q_i(x)$.

does not depend on n

This is the story for finite intervals, but can be generalized via S.L. to more general settings.

Let's repeat the Fourier trickery of going from $x,y \to \cos\theta, \sin\theta$, but this time in 3D.

$$\text{Weierstrass} \Rightarrow F(\vec{r}) = \lim_{M\to\infty} F_M(\vec{r}) = \lim_{M\to\infty} \sum_{i,j,k=0}^{M} C_{ijk}^{(M)} \overbrace{\underbrace{x_1^i x_2^j x_3^k}}^{\text{poly of degrees } i,j,k} \quad \text{w/ uniform (mean) convergence.}$$

since z_i is a lin. comb. of x_i

In terms of: $z_1 \equiv x_1 + i x_2 = r\sin\theta\, e^{i\phi}$ ($\phi \in [0,2\pi)$)

$z_2 \equiv x_1 - i x_2 = r \sin\theta\, e^{-i\phi}$

$z_3 \equiv x_3 = r\cos\theta$ ($\theta \in [0,\pi]$)

$$F_M(\vec{r}) = \sum_{\alpha,\beta,\gamma=0}^{M} A_{\alpha\beta\gamma}^{(M)} z_1^\alpha z_2^\beta z_3^\gamma = \sum_{\ell=0}^{3M} r^\ell \sum_{\substack{\alpha,\beta,\gamma=0 \\ (\alpha+\beta+\gamma=\ell)}}^{M} A_{\alpha\beta\gamma}^{(M)} e^{i(\alpha-\beta)\phi} \sin^{(\alpha+\beta)}\theta \cos^\gamma\theta$$

Down a relabelling rabbit hole:

Now just like w/ Fourier we want to restrict to $r=1$ and relabel $\alpha-\beta = m$. We know that

$\alpha,\beta,\gamma \geq 0 \Rightarrow \alpha+\beta = m + 2\beta \geq 0$ and $\alpha+\beta \geq |\alpha-\beta| = |m| \Rightarrow \alpha+\beta-|m| = m - |m| + 2\beta \geq 0$.

Now if $m \geq 0 \Rightarrow m - |m| = 0$ and $\alpha+\beta-|m| = 2\beta$ its even

if $m < 0 \Rightarrow m - |m| = 2m$ and $\alpha+\beta-|m| = 2m + 2\beta$ its even again

$\Big\}$ important in a moment

We can now rewrite $\sin^{(\alpha+\beta)}\theta \cos^\gamma\theta = \sin^{(\alpha+\beta-|m|)}\theta \sin^{|m|}\theta \cos^\gamma\theta = (\sin^2\theta)^{\left[\frac{1}{2}(\alpha+\beta-|m|)\right]} \sin^{|m|}\theta\cos^\gamma\theta$

$= (1-\cos^2\theta)^{\frac{1}{2}(\alpha+\beta-|m|)} \sin^{|m|}\theta \cos^\gamma\theta$

$= \sin^{|m|}\theta\, f_{\ell m}(\cos\theta)$ where $f_{\ell m}(\cos\theta)$ is a poly in $\cos\theta$ of degree $\alpha+\beta+\gamma-|m| = \ell - |m|$

Now if we prefer m as our label we need summation limits. Using $\alpha,\beta,\gamma \geq 0$:

First: $\underbrace{m = \alpha - \beta \leq \ell}$ (since $\alpha+\beta+\gamma = \ell$) $\Rightarrow$ $\alpha+\beta+\gamma - |m| = \ell - |m| \geq 0$

$\alpha - \beta \leq \alpha+\beta+\gamma$

$0 \leq 2\beta + \gamma$ true since $\alpha,\beta,\gamma \geq 0$

Therefore $|m| \leq \ell$ and we have: $F_M(\vec{r}) = \sum_{\ell=0}^{3M}\sum_{m=-\ell}^{\ell} B_{\ell m}^{(M)} e^{im\phi} \sin^{|m|}\theta\, f_{\ell m}(\cos\theta) = \sum_{\ell=0}^{3M}\sum_{m=-\ell}^{\ell} B_{\ell m}^{(M)} Y_{\ell m}(\theta,\phi)$

Now we would like to G.S. the $\bar{Y}_{\ell m}(\theta,\phi)$. That is we want $Y_{\ell m}(\theta,\phi)$ s.t.

$$(Y_{\ell' m'}, Y_{\ell m}) = \int_\Omega Y^*_{\ell' m'} Y_{\ell m}\, d\Omega = \int_0^{2\pi}\int_0^\pi \sin\theta\, Y^*_{\ell' m'} Y_{\ell m}\, d\theta\, d\phi = \delta_{\ell'\ell}\delta_{m'm}$$

Starting w/ $\bar{Y}_{00}$ we first recall that $f_{\ell m}(\cos\theta)$ has degree $\ell - |m|$ and thus is constant for $\ell = 0 = m$. Then $\bar{Y}_{00} = C \Rightarrow \int_0^{2\pi}\int_0^\pi C^2 \sin\theta\, d\theta\, d\phi = 4\pi C^2 \Rightarrow Y_{00}(\theta,\phi) = \frac{1}{2\sqrt{\pi}}$

Now for $\ell = 1$ we have

$$m = 0 \quad f_{10}(\cos\theta) = A + B\cos\theta \Rightarrow \bar{Y}_{10} = (A + B\cos\theta)$$
$$m = -1 \quad f_{1-1}(\cos\theta) = \text{constant} = C \Rightarrow \bar{Y}_{1-1} = e^{-i\phi}\sin\theta\, C$$
$$m = +1 \quad f_{11}(\cos\theta) = \text{constant} = D \Rightarrow \bar{Y}_{11} = e^{i\phi}\sin\theta\, D$$

Lets do $\bar{Y}_{10}$, $Y_{10} = \frac{\bar{Y}_{10} - Y_{00}(Y_{00},\bar{Y}_{10})}{\|\bar{Y}_{10} - Y_{00}(Y_{00},\bar{Y}_{10})\|} = \left[A + B\cos\theta - \frac{1}{2\sqrt{\pi}}\int_0^{2\pi}\int_0^\pi \frac{1}{2\sqrt{\pi}}\sin\theta(A + B\cos\theta)\, d\theta\, d\phi\right] / \|\ \|$

$$= \left[A + B\cos\theta - \frac{2\pi}{4\pi}(2A + 0)\right] / \|\ \|$$

$$= B\cos\theta \Big/ \sqrt{\int_0^{2\pi}\int_0^\pi \sin\theta\, B^2\cos^2\theta\, d\theta\, d\phi}$$

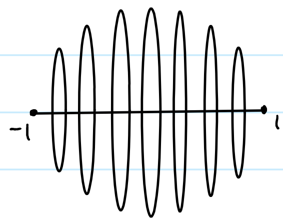
$$= \sqrt{\frac{3}{4\pi}}\cos\theta$$

Similarly $Y_{11} = -\sqrt{\frac{3}{8\pi}}\, e^{i\phi}\sin\theta$, $Y_{1-1} = \sqrt{\frac{3}{8\pi}}\, e^{-i\phi}\sin\theta$

The $Y_{\ell m}(\theta,\phi)$ are the spherical harmonics. These are obviously orthonormal. We can also prove their completeness which implies mean convergence, hence:

$$F(\vec{r}) \doteq \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \underbrace{C_{\ell m}}_{\text{does not depend on each other}} Y_{\ell m}(\theta,\phi)$$

A bit of "geometry". The unit sphere S^2 we can think of as a sequence of circles along a line that goes from $[-1,1]$ and the circles begin and end at $r=0$ and swell to $r=1$ in the middle.



There is a certain sense in which we could imagine the story that plays out here is a mix of Fourier functions for the circle d.o.f. and Legendre modes for the $[-1,1]$ d.o.f.

In fact the "Rodrigues" formula in this case is:

$$Y_{\ell m}(\theta,\phi) = (-1)^m \left[\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} P_\ell^m(\cos\theta)\, e^{im\phi} \qquad m \geq 0 \quad \text{and} \quad Y_{\ell -m} = (-1)^m Y_{\ell m}^*$$

where

$P_\ell^m(x) = (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_\ell(x)$ where $P_\ell(x)$ are the Legendre polynomials, and of course the $e^{im\phi}$ are the Fourier functions.

In fact, removing the Fourier part by setting $m=0 \Rightarrow Y_{\ell 0}(\theta,\phi) = \left(\frac{2\ell+1}{4\pi}\right)^{1/2} P_\ell(\cos\theta)$

In fact the P_ℓ^m are just the associated Legendre functions.