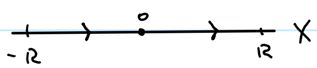
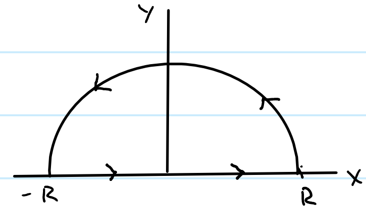
Okay, so another nice aspect of the analytic function story is using it to evaluate integrals over real variables. Recall that $x \in \mathbb{R}^1$ is just the real axis in $\mathbb{C}$.

So we might consider:



which we can bump up to



$$\int_{-R}^{R} f(x)\,dx$$

$$\oint_c f(z)\,dz$$ but if $w(z)$ is analytic this is 0!

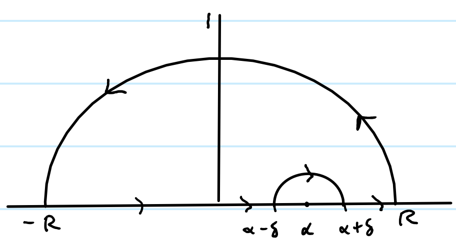
Notice the switch from $w(z) \to f(z)$ because then $f(x)$ is its baby!

This means that $\oint_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{S_R} f(z)\,dz = 0 \Rightarrow \int_{-R}^{R} f(x)\,dx = -\int_{S_R} f(z)\,dz$.

Now if $|f(z)| \to 0$ as $|z| \to \infty$ (pretty reasonable), then $\int_{-R}^{R} f(x)\,dx = 0$ (pretty special!)

In fact we can use bumping to get "around" troubling points on x for $f(x)$. Imagine $x = \alpha$ was a problem point, i.e. consider $\frac{f(z)}{z-\alpha}$ where $f(z)$ is analytic everywhere above and including the real line. Of course $\frac{1}{z-\alpha}$ is analytic everywhere except $z = \alpha$.

We can dodge the trouble point by considering:



So both are analytic on and within the given contour, and hence so is the product $\frac{f(z)}{z-\alpha}$. But this of course means $\oint_c \frac{f(z)}{z-\alpha}\,dz = 0$.

Breaking up the closed contour into 4 bits: $\int_{-R}^{\alpha-\delta} \frac{f(x)}{x-\alpha}\,dx + \underbrace{\int_{S_\delta} \frac{f(z)}{z-\alpha}\,dz + \int_{\alpha+\delta}^{R} \frac{f(x)}{x-\alpha}\,dx + \int_{S_R} \frac{f(z)}{z-\alpha}\,dz}_{\text{These are the contours in } z} = 0$

Now let's add in the assumption that $|f(z)| \to 0$ as $|z| \to \infty$.

In this case the $\int_{S_R} \to 0$ as $R \to \infty$ and we can consider the limit as $\delta \to 0$:

$$\underbrace{\lim_{R\to\infty} \lim_{\delta \to 0} \left[\int_{-R}^{\alpha-\delta} \frac{f(x)}{x-\alpha}\,dx + \int_{\alpha+\delta}^{R} \frac{f(x)}{x-\alpha} \right]}_{P\int_{-R}^{R} \frac{f(x)}{x-\alpha}\,dx} = \lim_{\delta\to 0} \left[-\int_{S_\delta} \frac{f(z)}{z-\alpha}\,dz \right]$$

just adding $0 = \int \frac{f(\alpha)-f(\alpha)}{z-\alpha}dz$

$$= \lim_{\delta\to 0} \left[\underbrace{-f(\alpha)\int_{S_\delta} \frac{dz}{z-\alpha}}_{i\pi f(\alpha)} - \underbrace{\int_{S_\delta} \frac{f(z)-f(\alpha)}{z-\alpha}\,dz}_{=0} \right]$$

$P\int_{-R}^{R} \frac{f(x)}{x-\alpha}\,dx$ ← principal-value integral

$i\pi f(\alpha)$ (see below)

$=0$ (not obvious, but we'll just accept)

For: $-f(\alpha)\int_{S_\delta} \frac{dz}{z-\alpha}$ let's take $z-\alpha = \delta e^{i\theta}$ since  contour in z, then $dz = i\delta e^{i\theta}d\theta$ and we have $-f(\alpha)\int_{S_\delta} \frac{dz}{z-\alpha} = -if(\alpha)\int_{\pi}^{0} d\theta = i\pi f(\alpha)$

Thus:

$$\lim_{R\to\infty} P\int_{-R}^{R} \frac{f(x)}{x-\alpha}\,dx = i\pi f(\alpha)$$

Breaking up $f(x) = f_R(x) + i f_I(x)$ this becomes:

$$\left.\begin{aligned} f_R(\alpha) &= \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{f_I(x)}{x-\alpha}\,dx \\ f_I(\alpha) &= -\frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{f_R(x)}{x-\alpha}\,dx \end{aligned}\right\} \text{Hilbert transform pair}$$

This can be handy in several different ways:

1. Consider $\int_{-R}^{R}\frac{dx}{x}$ which we know is 0 since $\frac{1}{x}$ is odd, but $\frac{1}{x}$ diverges as $x\to 0$ so we can't formally do it.

But: $$P\int_{-R}^{R}\frac{dx}{x}=\lim_{\delta\to 0}\Big[\underbrace{\int_{-R}^{-\delta}\frac{dx}{x}}_{y=-x,\ dy=-dx}+\int_{\delta}^{R}\frac{dx}{x}\Big]=\lim_{\delta\to0}\Big[\int_{R}^{\delta}\frac{dy}{y}+\int_{\delta}^{R}\frac{dx}{x}\Big]=\lim_{\delta\to0}\Big[\int_{R}^{\delta}\frac{dy}{y}-\int_{R}^{\delta}\frac{dx}{x}\Big]=0$$

2. Consider $\int_{-R}^{R}\frac{dx}{x-\alpha}dx$ which obviously has problems at $x=\alpha$, so:

$$P\int_{-R}^{R}\frac{dx}{x-\alpha}dx=\lim_{\delta\to0}\Big[\int_{-R}^{\alpha-\delta}\frac{dx}{x-\alpha}+\int_{\alpha+\delta}^{R}\frac{dx}{x-\alpha}\Big]$$

setting $x=-y$ in

$$P\int_{-R}^{R}\frac{dx}{x-\alpha}dx=\lim_{\delta\to0}\Big[\int_{R}^{\delta-\alpha}\frac{dy}{y+\alpha}+\ln(R-\alpha)-\ln\delta\Big]$$

$$=\lim_{\delta\to0}\Big[\ln\delta-\ln(R+\alpha)+\ln(R-\alpha)-\ln\delta\Big]$$

$$=\ln\left(\frac{R-\alpha}{R+\alpha}\right)$$ note: if $\alpha=0$ $\ln\left(\frac{R}{R}\right)=0$ as in 1.

3. Lastly consider $$P\int_{-R}^{R}\frac{f(x)}{x-\alpha}=\underbrace{P\int_{-R}^{R}\frac{f(\alpha)}{x-\alpha}dx}_{\underbrace{f(\alpha)\ln\left(\frac{R-\alpha}{R+\alpha}\right)}_{\text{from above}}}+P\int_{-R}^{R}\frac{f(x)-f(\alpha)}{x-\alpha}dx$$

as $x\to\alpha$ this becomes $\frac{0}{0}$ and so as long as $f'(x)$ exists, then using L'Hopital's $\to\frac{f'}{1}$. So we can drop P.

Thus: $$P\int_{-R}^{R}\frac{f(x)}{x-\alpha}=f(\alpha)\ln\left(\frac{R-\alpha}{R+\alpha}\right)+\int_{-R}^{R}\frac{f(x)-f(\alpha)}{x-\alpha}dx$$

and in the limit $R\to\infty$ this becomes:

$$i\pi f(\alpha)=0+\int_{-\infty}^{\infty}\frac{f(x)-f(\alpha)}{x-\alpha}dx\Rightarrow f(\alpha)=-\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{f(x)-f(\alpha)}{x-\alpha}dx$$

$$f_R(\alpha)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f_I(x)-f_I(\alpha)}{x-\alpha}dx$$

$$f_I(\alpha)=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f_R(x)-f_R(\alpha)}{x-\alpha}dx$$

Let's use this: $f(z)=e^{iz}$ and w/ $z=Re^{i\theta}\Rightarrow|f(z)|\to0$ as $R\to0$ for $0<\theta<\pi$.

$$\left.\begin{aligned}f_R(x)&=\cos x\\ f_I(x)&=\sin x\end{aligned}\right\}\quad \cos\alpha=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(x)-\sin(\alpha)}{x-\alpha}dx$$

If in fact we choose $\alpha=0$ this gives the inobvious but useful: $\int_{-\infty}^{\infty}\frac{\sin(x)}{x}dx=\pi$

We are going to start with the complex generalization of Taylor's theorem.
As you might expect, the generalization is simple after we specify a few features to be satisfied.

If $f(z)$ is analytic everywhere inside a circle C centered about z_0 then in any closed region inside of C, $w(z) = \sum_{n=0}^{\infty} \frac{1}{n!} w^{(n)}(z_0)(z-z_0)^n$ uniformly converges.

Examples:

1. $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ for $|z| < \infty$
2. $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$ for $|z| < \infty$
3. $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ for $|z| < 1$

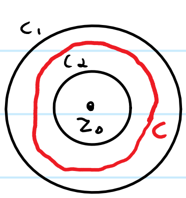
$z_0 = 0$ for all (McLauren type)

So far so good. But now let's try to take one of these and do something interesting.

Consider e^z but w/ $z \rightarrow \frac{1}{z}$ $\Rightarrow$ $e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n! z^n}$ for $|z| > 0$.

But wait, this can't be a Taylor series because the powers of z are negative.
Moreover, $\frac{1}{z}$ is singualar at $z = z_0 = 0$, so none of the derivatives actually exist!
But the expression itself makes sense because if $e^{\frac{1}{z}}$ is not analytic at $z=0$, the neither should the polynomials in its expansion.

We need something else that handles this.
Laurent's theorem:



Let $w(z)$ be analytic in the region between C_1 and C_2 as shown.
Then $w(z) = \sum_{n=-\infty}^{\infty} A_n (z-z_0)^n$ w/ $A_n = \frac{1}{2\pi i} \oint_C \frac{w(z')}{(z'-z_0)^{n+1}} dz'$ where C is between C_1 and C_2 and encloses z_0.
The series converges uniformly in any closed region lying between C_1 and C_2.

Let's first of all find Taylor in this. If $w(z)$ is analytic at all points inside of C (we are getting rid of the C_2 shield of singularities), then recall from CIF that $w(z_0) = \frac{1}{2\pi i} \oint_C \frac{w(z)}{z - z_0} dz$.

Now we can take this and calculate derivatives w.r.t. z_0 which is a continuous parameter, i.e. variable within C.

$$\frac{dw}{dz_0} = \frac{1}{2\pi i} \oint_C \frac{w(z)}{(z-z_0)^2} dz$$

$$\frac{d^2w}{dz_0^2} = \frac{2}{2\pi i} \oint_C \frac{w(z)}{(z-z_0)^3} dz$$

$$\vdots$$

$$\frac{d^n w}{dz_0^n} = \frac{n!}{2\pi i} \oint_C \frac{w(z)}{(z-z_0)^{n+1}} dz \Rightarrow \frac{1}{n!} \frac{d^n w}{dz_0^n} = \frac{1}{2\pi i} \oint_C \frac{w(z)}{(z-z_0)^{n+1}} dz$$

And into Laurent this goes giving: $w(z) = \sum_{n=-\infty}^{\infty} A_n (z-z_0)^n$ w/ $A_n = \frac{1}{n!} w^{(n)}(z_0)$

This is almost Taylor's result, except for the $n<0$ values. What the hell is $w^{-1}(z_0)$?!

Well recall that $w(z)$ is analytic within C, and in fact so too is $\frac{1}{(z-z_0)^{n+1}}$ for $n<0$, i.e. $(z-z_0)^k$ where $k = -(n+1) \geq 0$ for $n<0$ (and integer of course). But now we have the product of two functions which are analytic within C, thus $\oint_C \frac{w(z)}{(z-z_0)^{n+1}} dz = 0$ for $n<0$.

So the sum $\sum_{n=-\infty}^{\infty} \Rightarrow \sum_{n=0}^{\infty}$ counting only nonzero contributions. This of course makes sense since any negative power of $(z-z_0)$ would make $z=z_0$ a singularity which screws up the assumption of analyticity within C.

Okay, so to be honest, calculating Laurent expansions sucks because we have to do integrals instead of derivatives. For functions that are analytic within C, I would much rather use Taylor cause derivatives are much easier.

But of course if the point z_0 is where the analyticity breaks down we must use Laurent. Note that if a singularity exists, but away from it the function is analytic, then we can use Taylor for any other point $z_0' \neq z_0$.

Just to see Laurent in action (and doable) consider:

$$w(z) = \frac{z^2+3z+1}{z^2} \quad w/\ z_0 = 0$$

$$A_n = \frac{1}{2\pi i}\oint_c \frac{z^2+3z+1}{z^2 z^{n+1}}\,dz = \frac{1}{2\pi i}\oint_c \frac{z^2+3z+1}{z^{n+3}}\,dz$$

Note that for $n \leq -3$ this becomes the integral of an analytic function $\Rightarrow 0$.

So starting w/ $n=-2$: $A_{-2} = \frac{1}{2\pi i}\oint_c \frac{z^2+3z+1}{z}\,dz = \frac{1}{2\pi i}\oint_c \left(z+3+\frac{1}{z}\right)dz$

$= \frac{1}{2\pi i}\oint_c \frac{1}{z}\,dz$ since z and 3 are analytic

let $C: z = e^{i\theta}$ w/ $r=1 \Rightarrow dz = ie^{i\theta}d\theta$ $\quad = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\theta}e^{i\theta}\,d\theta = 1$

$$A_{-1} = \frac{1}{2\pi i}\oint_c \frac{z^2+3z+1}{z^2}\,dz = \frac{1}{2\pi i}\oint_c \Big(\underbrace{1}_{=0} + \underbrace{\frac{3}{z}}_{=3} + \frac{1}{z^2}\Big)\,dz$$

$$= 3 + \frac{1}{2\pi}\underbrace{\int_{-\pi}^{\pi} e^{-2i\theta}e^{i\theta}\,d\theta}_{=0}$$

$$= 3$$

$$A_0 = \frac{1}{2\pi i}\oint_c \frac{z^2+3z+1}{z^3}\,dz = \frac{1}{2\pi i}\oint_c \Big(\underbrace{\frac{1}{z}}_{=1} + \underbrace{\frac{3}{z^2}}_{=0} + \frac{1}{z^3}\Big)dz$$

$$= 1 + \frac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{e^{-3i\theta}e^{i\theta}}_{=0}\,d\theta$$

$$= 1$$

$$A_{n>0} = \frac{1}{2\pi i}\oint_c \left(z^{-1-n} + 3z^{-2-n} + z^{-3-n}\right)dz = 0$$

Then: $w(z) = \sum_{n=-\infty}^{\infty} A_n z^n = z^{-2} + 3z^{-1} + 1 = \frac{1}{z^2} + \frac{3}{z} + 1$ which $= \frac{z^2+3z+1}{z^2}$

Although most integrals in Laurent coefficients are hard to impossible, we can use what we know to get some results.

Recall: $e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!z^n}$ (from Taylor w/ $z \to \frac{1}{z}$)

Comparing to Laurent we learn: $\oint_c e^{\frac{1}{z}} z^{n-1}\,dz = \frac{2\pi i}{n!}$ for $n \geq 0$

$\oint_c e^{\frac{1}{z}} z^{n-1}\,dz = 0$ for $n < 0$

Breaking up the Laurent series: $w(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z-z_0)^n}$

Clearly the first part is analytic everywhere, while the second part encodes the singularities. If $b_n = 0$ for $n = N+1, N+2, \cdots, \infty$ then we say the function has a "pole" at N. Some do, e.g. $\frac{1}{z} + \frac{1}{z^2}$, while some don't, e.g. $e^{\frac{1}{z}}$.