To be more general, consider: $L = \frac{d^2}{dx^2} + \underbrace{a\frac{d}{dx} + b}_{\text{constants}} \Rightarrow L\,y(x) = f(x)$ w/ $x \in (-\infty, \infty)$

Now let's assume that $f(x)$, $y(x)$ and $\frac{dy}{dx}$ all $\to 0$ for $|x| \to \infty$. Therefore we can do Fourier transforms of each:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\underbrace{\left(\frac{d^2y}{dx^2} + a\frac{dy}{dx} + b\,y(x)\right)e^{ikx}}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{ikx}\,dx$$

integrate by parts to get: $\left[\frac{dy}{dx}e^{ikx}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{dy}{dx}\frac{d}{dx}\left(e^{ikx}\right)dx = 0 - ik\int_{-\infty}^{\infty}\frac{dy}{dx}e^{ikx}\,dx$

Then we have:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(a-ik)\underbrace{\frac{dy}{dx}e^{ikx}}\,dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} b\,y(x)e^{ikx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{ikx}\,dx$$

i.b.p.s again: $\left[ye^{ikx}\right]_{-\infty}^{\infty} - ik\int_{-\infty}^{\infty} ye^{ikx}\,dx = -ik\int_{-\infty}^{\infty} y\,e^{ikx}\,dx$

So in the end we have:

$$\left[-(a-ik)ik + b\right]\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y(x)e^{ikx}\,dx}_{\hat{y}(k)} = \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{ikx}\,dx}_{\hat{f}(k)}$$

$$\underbrace{(-k^2 - iak + b)}_{\text{no zeroes on } \mathbb{R}^1}\hat{y}(k) = \hat{f}(k) \Rightarrow \hat{y}(k) = \frac{\hat{f}(k)}{-k^2 - iak + b}$$

$\Downarrow$ undoing F.T. on y

$$y(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\hat{f}(k)}{-k^2 - iak + b}e^{-ikx}\,dk$$

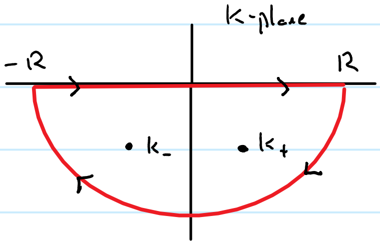
$\Downarrow$ undoing F.T. on f

$$y(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dk\,\frac{f(x')e^{ikx'}}{-k^2 - iak + b}e^{-ikx} = \int_{-\infty}^{\infty}G(x,x')f(x')\,dx'$$

Therefore:

$$G(x,x') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ik(x-x')}}{-k^2 - iak + b}\,dk = \frac{-1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ik(x-x')}}{(k-k_+)(k-k_-)}\,dk \quad \text{w/ } k_\pm = \pm\left(b - \frac{a^2}{4}\right)^{1/2} - i\frac{a}{2}$$

To do the integral we resort to the results of the last chapter. Consider the contour:



$$I = -\frac{1}{2\pi}\oint_C \frac{e^{-ik(x-x')}}{(k-k_+)(k-k_-)}\,dk$$

$$= \underbrace{-\frac{1}{2\pi}\int_{-R}^{R}\frac{e^{-ik(x-x')}}{(k-k_+)(k-k_-)}\,dk}_{\text{Note that the limit as } R\to\infty \text{ of this is what we want.}} \overbrace{- \frac{1}{2\pi}\int_0^{-\pi}\frac{e^{-iR(x-x')e^{i\phi}}\, iRe^{i\phi}}{(Re^{i\phi} - k_+)(Re^{i\phi} - k_-)}\,d\phi}^{\text{using } k = Re^{i\phi}}$$

First of all: $\oint_c \frac{e^{-ik(x-x')}}{(k-k_+)(k-k_-)}\,dk = 2\pi i\,(R_+ + R_-)$ where $R_\pm$ is the residue coming from $k_\pm$.

$$R_\pm = \frac{-1}{2\pi i}\oint_{c_\pm}\frac{e^{-ik(x-x')}/(k-k_\mp)}{(k-k_\pm)}\,dk$$ where $c_\pm$ includes $k_\pm$ but not $k_\mp$ (hence the numerator is analytic)

Using CIF : $w(z_0) = \frac{1}{2\pi i} \oint_C \frac{w(z)}{z - z_0}\, dz$

then

$$R_\pm = \frac{-e^{-ik_\pm (x-x')}}{(k_\pm - k_\mp)} = \frac{-e^{-i\left(\pm\sqrt{b-\frac{a^2}{4}} - i\frac{a}{2}\right)(x-x')}}{\pm 2\sqrt{b-\frac{a^2}{4}}} = \mp \frac{1}{2} \frac{e^{-\frac{a}{2}(x-x')}\, e^{\mp i\sqrt{b-\frac{a^2}{4}}(x-x')}}{\sqrt{b-\frac{a^2}{4}}}$$

We need: $R_+ + R_- = \frac{e^{-\frac{a}{2}(x-x')}}{2\sqrt{b-\frac{a^2}{4}}} \left[-\cos(\ \ ) + i\sin(\ \ ) + \cos(\ \ ) + i\sin(\ \ ) \right]$

$$= \frac{i\, e^{-\frac{a}{2}(x-x')}}{\sqrt{b-\frac{a^2}{4}}} \sin\left[\sqrt{b-\frac{a^2}{4}}\,(x-x')\right]$$

Thus: $I = \frac{e^{-\frac{a}{2}(x-x')}}{\sqrt{b-\frac{a^2}{4}}} \sin\left[\sqrt{b-\frac{a^2}{4}}\,(x-x')\right]$ which is the same if we take $R \to \infty$

which also equals:

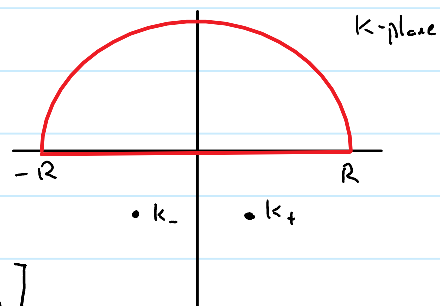
$$I = G(x,x') - \underbrace{\lim_{R\to\infty} \frac{1}{2\pi} \int_0^{-\pi} \frac{e^{-iR(x-x')e^{i\phi}}\, iRe^{i\phi}}{(Re^{i\phi} - k_+)(Re^{i\phi} - k_-)}\, d\phi}_{}$$

$= 0$ by Jordan's Lemma, but only for $x - x' > 0$

So we finally have:

$$G(x,x') = \frac{e^{-\frac{a}{2}(x-x')}}{\sqrt{b-\frac{a^2}{4}}} \sin\left[\sqrt{b-\frac{a^2}{4}}\,(x-x')\right] \quad \text{for } x > x'$$

On the other hand, for $x < x'$ we alternatively use:



Jordan's lemma to argue that $\int_0^{\pi} d\phi = 0$

But $I = 0$ since C contains no singularities $\Rightarrow G(x,x') = 0$.

Putting these together:

$$G(x,x') = \Theta(x-x') \frac{e^{-\frac{a}{2}(x-x')}}{\sqrt{b-\frac{a^2}{4}}} \sin\left[\sqrt{b-\frac{a^2}{4}}\,(x-x')\right]$$

Now of course one can show that $LG(x,x') = \delta(x-x')$, which you will do.

To get the most general solution, let's add solutions of the homogenous $\underbrace{Ly(x) = 0}_{\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0}$, i.e.

$y_1(x) = e^{-\frac{a}{2}x} \sin\left[\sqrt{b-\frac{a^2}{4}}\, x\right]$

$y_2(x) = e^{-\frac{a}{2}x} \cos\left[\sqrt{b-\frac{a^2}{4}}\, x\right]$

So in total we have, as the general solution to $Ly(x) = \frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x)$ w/ $f(x) = 0$ $x < x_0$

$$y(x) = Ay_1(x) + By_2(x) + \int_{x_0}^{x} \frac{e^{-\frac{a}{2}(x-x')}}{\sqrt{b-\frac{a^2}{4}}} \sin\left[\sqrt{b-\frac{a^2}{4}}\,(x-x')\right] f(x')\,dx'$$

Now let's translate: $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + b\,y(x) = f(x) \Rightarrow y(x) \to x(t),\ a = 2\gamma,\ b = \omega_0^2$

$$y(x) = A y_1(x) + B y_2(x) + \int_{x_0}^{x} \frac{e^{-\frac{a}{2}(x-x')}}{\sqrt{b - \frac{a^2}{4}}} \sin\left[\sqrt{b - \frac{a^2}{4}}\,(x-x')\right] f(x')dx'$$

$\Downarrow$

$$x(t) = A x_1(t) + B x_2(t) + \int_{t_0}^{t} \frac{e^{-\gamma(t-t')}}{\sqrt{\omega_0^2-\gamma^2}} \sin\left[\sqrt{\omega_0^2-\gamma^2}\,(t-t')\right] F(t')dt'$$

Which satisfies:

$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x(t) = F(t)$ the equation for a forced and damped harmonic oscillator

Let $F(t) = \begin{cases} 0 & t<0 \\ F_0 e^{-\alpha t} & t \geq 0 \end{cases}$ and assume it is at rest in equilibrium at $t=0 \Rightarrow A = B = 0$

Then: $x(t) = \displaystyle\int_0^t \frac{e^{-\gamma(t-t')}}{\sqrt{\omega_0^2-\gamma^2}} \sin\left[\sqrt{\omega_0^2-\gamma^2}\,(t-t')\right] F_0 e^{-\alpha t'} dt'$

$$= F_0 e^{-\gamma t} \int_0^t \frac{1}{\sqrt{\omega_0^2-\gamma^2}} \sin\left[\sqrt{\omega_0^2-\gamma^2}\,(t-t')\right] e^{(\gamma-\alpha)t'} dt'$$

Integration skills to the rescue!

First consider: $\underbrace{\int_a^b x^2 e^{kx}dx}$, how could you approach this? How about IBP?

$$\frac{1}{k}\int_a^b x^2 k e^{kx}dx = \frac{1}{k}\left[x^2 e^{kx}\right]_a^b - \frac{2}{k}\underbrace{\int_a^b x e^{kx}dx}$$

$$\frac{1}{k}\int_a^b x k e^{kx}dx = \frac{1}{k}\left[x e^{kx}\right]_a^b - \frac{1}{k}\int_a^b e^{kx}dx$$

$$\int_a^b x^2 e^{kx}dx = \frac{1}{k}\left[x^2 e^{kx}\right]_a^b - \frac{2}{k^2}\left[x e^{kx}\right]_a^b + \frac{2}{k^3}\left[e^{kx}\right]_a^b$$

Now what about: $\int_a^b \sin(\omega x) e^{kx}dx$, how about IBP? $\sin(\omega x)$ doesn't go away, but..

$$\frac{1}{k}\int_a^b \sin(\omega x) k e^{kx}dx = \frac{1}{k}\left[\sin(\omega x) e^{kx}\right]_a^b - \frac{\omega}{k}\underbrace{\int_a^b \cos(\omega x) e^{kx}dx}$$

$$\frac{1}{k}\int_a^b \cos(\omega x) k e^{kx}dx = \frac{1}{k}\left[\cos(\omega x) e^{kx}\right]_a^b + \frac{\omega}{k}\int_a^b \sin(\omega x) e^{kx}dx$$

Then: $\displaystyle\int_a^b \sin(\omega x) e^{kx}dx = \frac{1}{k}\left[\sin(\omega x) e^{kx}\right]_a^b - \frac{\omega}{k^2}\left[\cos(\omega x) e^{kx}\right]_a^b - \frac{\omega^2}{k^2}\int_a^b \sin(\omega x) e^{kx}dx$

$\Rightarrow \displaystyle\int_a^b \sin(\omega x) e^{kx}dx = \frac{1}{1-\frac{\omega^2}{k^2}}\left[\frac{1}{k}\sin(\omega x) e^{kx} - \frac{\omega}{k^2}\cos(\omega x) e^{kx}\right]_a^b$

Using (a slightly modified version of) this, and employing some trig, the result can be written as:

$$x(t) = \frac{F_0}{\sqrt{\omega_0^2-\gamma^2}} \frac{\sin\left[\sqrt{\omega_0^2-\gamma^2}\,t - \delta\right]}{\sqrt{\omega_0^2+\alpha^2-2\alpha\gamma}} e^{-\gamma t} + \frac{F_0}{\omega_0^2+\alpha^2-2\alpha\gamma} e^{-\alpha t}$$

Taking the dampening to zero ($\gamma \to 0$) we have: $x(t) = \frac{F_0}{\omega_0} \frac{\sin[\omega_0 t - \delta]}{\sqrt{\omega_0^2+\alpha^2}} + \frac{F_0}{\omega_0^2+\alpha^2} e^{-\alpha t}$

which

$$x(t\to\infty) = \frac{F_0}{\omega_0} \frac{\sin[\omega_0 t - \delta]}{\sqrt{\omega_0^2+\alpha^2}} \quad \text{SHM}$$

Now remember the spirit of Green's functions: They let you solve $L\,y(x) = f(x)$ by finding what is essentially the inverse of L, i.e. $L\,G(x,x') = \delta(x,x')$. The inverse does not depend on $f(x)$, so once we have it, we can apply it to the equation w/ any $f(x)$. It does depend on the boundary conditions, i.e. for 2nd order L we could use $y(0), y'(0)$ or $y(0), y(1)$.

But even more so, consider $L\,y(x) = f(x,y)$, which cannot be easily manipulated into an integral equation for $y(x)$. Or can it? $y(x) = \int G(x,x') f[x', y(x')]\,dx'$
So Green's help turn differential equations into integral equations.

Now in 1D, there is actually an approach to finding G by solving $L\,G(x,x') = \delta(x-x')$ directly for a given L. This process can be applied to operators of arbitrary order. This doesn't actually work in higher dimensions, so we won't worry about it.