Space and Spacetime

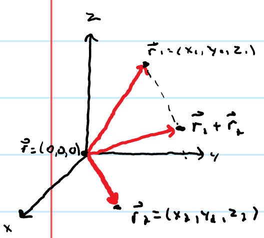
Often times it seems that students get only exposed to limited, and often wrong, perspectives by staying in the simple landscape of 3D space that is flat, i.e. $\mathbb{R}^3$. For example they might believe that: spinors are always "smaller" than vectors, that a cross-product of two vectors gives another vector, that the sum of angles in a triangle is always $180°$, the shortest distance between two points is always a straight line, a Klein bottle must always intersect itself, that you can spin around an axis, that a magnetic field is a vector field (just like $\vec{E}$), and that vectors live in space itself.

Turns out that these misconceptions can be addressed by appropriately adding time to the story, or by giving space a little curvature, or by considering spaces of higher dimension, or just by studying 3D flat space a little more carefully.

To be honest our actual physical universe includes time, is curved, and may have $D > 4$.

To begin, let's consider $\mathbb{R}^3$. Is this a space or a vector space? Turns out it can be either, but in a certain sense not both.

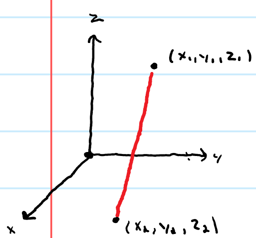
$\mathbb{R}^3$ as a vector space ($\vec{E}$): In $\mathbb{R}^3$ we pick any point and call it a origin w/ $\vec{r}=(0,0,0)$. Then any other point in the space given by (x,y,z) defines a vector w/ components $\vec{r}=(x,y,z)$.



Note that the point $\vec{r}=(0,0,0)$ is special since it serves the role of the required identity in the abelian group w/ addition as the operation.

Also note that vectors can be added (tip to tail) and vectors can have length w/ the inner product $(\vec{v}_1,\vec{v}_1)=v_{1x}^2+v_{1y}^2+v_{1z}^2$.

$\mathbb{R}^3$ as Euclidean Space (E): hat we mean by this is that the set of points in $\mathbb{R}^3$ actually designate locations in the space. Now this implies that there is no preferred point or origin.



Moreover, what is the "length" of a point?

Furthermore we cannot "add" two points in the space together.

However the relative position of two points given by their difference makes sense and this has a length associated with it.

So wait, if we consider velocity as a vector, in which does it live? Obviously zero velocity has meaning, but we can "move" the velocity vector around right?!

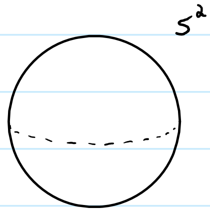
Well here is one fix: At each point A in Euclidean space, we designate that as the origin and use subtraction in the space $B-A$ to define a vector space $\vec{E}$. Since no point is special, we should do the same at every other point in the space.

The result is an "affine" space over the reals.

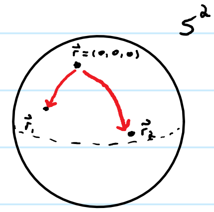
Velocity lives in the vector spaces defined at each point.

While this works in this case, it fails in general.

To understand the failure, and for a hint of what we should do, let's curve the space.

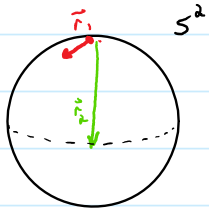


Is this a vector space?



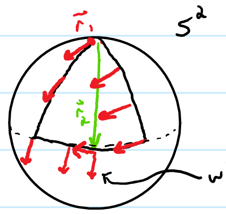
Can't we just add these as before? Then this would form a vector space no?

Consider:



Let's move the tail of $\vec{r}_1$ to the tip of $\vec{r}_2$ along different paths.

$\Downarrow$

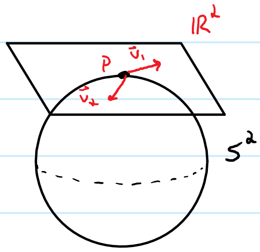


which one is it?

The addition of two vectors by tip to tail obviously fails because we need to move one from the origin to the tip of the other vector, but what we get is clearly path dependent.

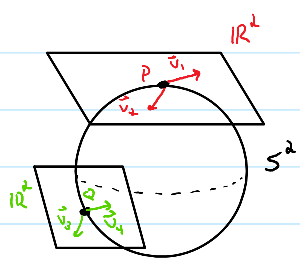
So S^2 is not a vector space. But it clearly is a space of points, upon which something can move, and hence have a velocity. Where does the velocity vector live?

It turns out that we need to adjoin an $\mathbb{R}^2$ (since S^2) to a point in S^2 to allow vectors to live.



In what orientation should we affix $\mathbb{R}^2$? Tangent of course. This is called the tangent space at point $p \in S^2$.

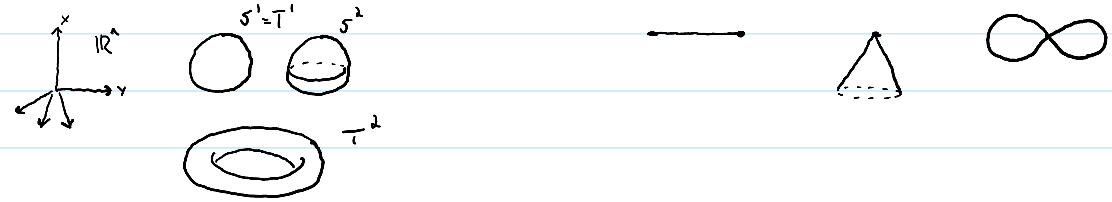
But we are gonna need these everywhere:



This is called the tangent bundle over S^2. Note that unlike Euclidean space, the tangent spaces at different points are not // to each other!

The fact that we can do this with S^2 is due to the fact that is an example of a "manifold". In essence, a manifold is a space which locally resembles Euclidean space near each point. Yes the sphere is curved, but if we take "locally" to mean a length scale $d \ll R$, then it will appear flat. Think of Earth.

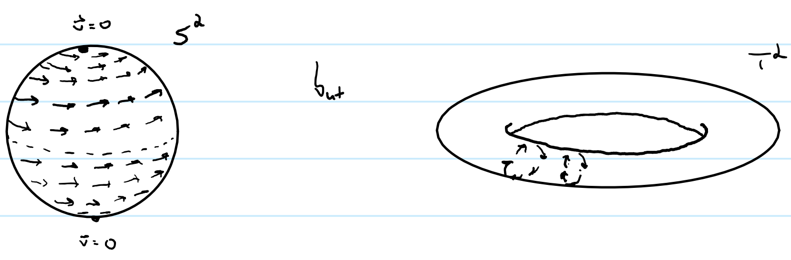
Examples: Manifolds Not Manifolds



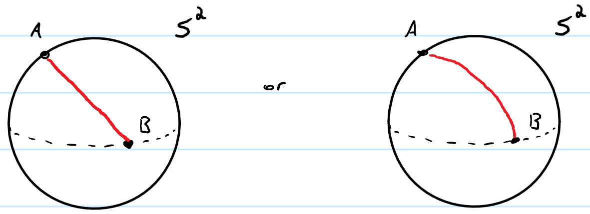
You can think of manifolds as smooth and boundaryless. The play an incredibly prominent role in physics. One of the huge advantages is that by locally looking like E^n near each point, we can afix a copy of $\mathbb{R}^n$ tangent to E^n to allow vectors to be defined.

Now the manifolds as defined so far are topological spaces. They have no measure of distance. But they are still useful.

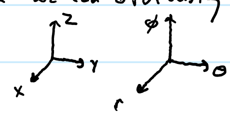
For example, suppose you were living on a 2D surface which was a manifold. It was so large that you couldn't make out if it was S^2 or T^2. You would like to figure out by making local measurements as you walked all around the surface. An option is to turn on a vector field (electric for example). After walking around the surface observing the field, you would have found that if it was nonzero everywhere, you were not on an S^2, and you could be on a T^2.



Now we might interested in the distance between two points in a space. Now the old notion of "distance" was basically the length of the straightline that connects the points. But this obviously fails in the presence of curvature.



What makes more sense is to just pick a path, and calculate the length of the path. Considering all paths and extremizing them might pick out the best candidate for the distance between the ends.
To calculate the length of a path s, we can break it up into sufficiently small intervals that have infinitesimal length ds, then $s = \int ds$.
So what is ds=? Well we will need to label the points in the space via coordinates. To do so, we take a part of the manifold U_α (need not all of it and sometimes can't do it all) and map it via a one-to-one map $\phi : U_\alpha \longrightarrow \mathbb{R}^n$ (where n is the dimension of the manifold). We call this a chart if the image in $\mathbb{R}^n$ is open (w/out boundary). Now in $\mathbb{R}^n$ we can obviously define coordinates, e.g. $\{x, y, z\}$, $\{r, \theta, \phi\}$, etc., which label points in $\mathbb{R}^n$, i.e.



Now obviously we could take $ds = \sqrt{dx^2 + dy^2 + dz^2}$ but not $ds = \sqrt{dr^2 + d\theta^2 + d\phi^2}$.

What we do instead is take the coordinate differentials, which are actually the components of infinitesimal coordinate displacement vectors in the local $\mathbb{R}^n$, and we combine two copies with a piece of machinery that has all the distance structure built in, the metric.

So $ds = \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$ where $dx^\mu = (dx, dy, dz)$ or $dx^\mu = (dr, d\theta, d\phi)$
With the index notation we should understand this as: $dx^\mu g_{\mu\nu} dx^\nu \simeq dx^T g\, dx = (\quad)(\quad)(\quad) = \#$
The form of $g_{\mu\nu}$ will depend on the space in question as well as the coordinates chosen. Examples: $\mathbb{R}^3$
S^2
To calculate the length of the path, we just coordinatize it w/ $x^\mu(\lambda)$, then

$$s = \int_{s_i}^{s_f} ds = \int \sqrt{g_{\mu\nu} dx^\mu dx^\nu} = \int_{\lambda_i}^{\lambda_f} \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda$$

Clearly the distance should not depend on coordinates, so ds^2 should be invariant. What we can do to preserve it we will see eventually.

One way to understand $g_{\mu\nu}dx^\mu dx^\nu$ is in terms of $dx_\nu dx^\nu$ where if dx^ν is an element of a vector space, then dx_ν is the corresponding element of the dual vector space. What is nice about this is that if we have any vector and a dual vector V^μ and W_μ, their "invariant" inner product is that of $\mathbb{R}^n$, i.e. $V^\mu W_\mu = V^x W_x + V^y W_y + V^z W_z = V^r W_r + V^\theta W_\theta + V^\phi W_\phi$.

To get the corresponding element of the dual vector space we just apply the metric, i.e. $dx_\nu = g_{\nu\mu}dx^\mu$.

To go back we apply the inverse of the metric notated by $g^{\mu\nu}$, i.e. $dx^\mu = g^{\mu\nu}dx_\nu$.

Obviously: $dx^\nu = g^{\nu\mu}dx_\mu = g^{\nu\mu}g_{\mu\alpha}dx^\alpha \Rightarrow g^{\nu\mu}g_{\mu\alpha} = \delta^\nu{}_\alpha \Rightarrow g^{\mu\nu} = (g_{\mu\nu})^{-1}$

Where do these live? In the cotangent bundle to the space.

Okay, so now that we know a bit about geometry and vectors, what we can do to transform them is the next step. So...