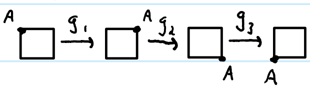
A group is a system $\{G, \bullet\}$ that consists of a set G w/ a single operation $\bullet$ that satisfies:

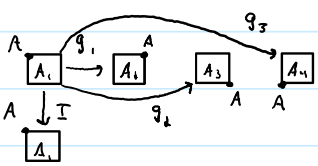
1. $\bullet$ is closed, i.e. for $a, b \in G$, $a \bullet b = c \in G$
2. $\bullet$ is associative, i.e. for $a, b, c \in G$, $a \bullet (b \bullet c) = (a \bullet b) \bullet c$
3. There exists an identity $e \in G$ s.t. for all $a \in G$, $a \bullet e = e \bullet a = a$
4. For every $a \in G$ there exists $a^{-1} \in G$ s.t. $a \bullet a^{-1} = a^{-1} \bullet a = e$

A finite and discrete case

Consider the set of transformations (group elements) in 2D that carry the corners of a square back to where there was previously a corner, i.e. $\square \to \square$.

Let's count the number of elements. To do so it will help to label at least one corner ${}^{A}\square$.

Then we have:  right? Nope!

Instead: 

don't forget

In matrix language this can be done in

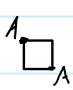
$$
\begin{array}{lccccccccc}
 & A_1 & A_2 & A_3 & A_4 & & I & g_1 & g_2 & g_3 \\
4D: & \begin{pmatrix}1\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\1\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\1\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\1\end{pmatrix} & \text{w/} & \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} & \begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix} & \begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix} & \begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix} \\
2D: & \begin{pmatrix}1\\0\end{pmatrix} & \begin{pmatrix}0\\1\end{pmatrix} & \begin{pmatrix}-1\\0\end{pmatrix} & \begin{pmatrix}0\\-1\end{pmatrix} & \text{w/} & \begin{pmatrix}1&0\\0&1\end{pmatrix} & \begin{pmatrix}0&-1\\1&0\end{pmatrix} & \begin{pmatrix}-1&0\\0&-1\end{pmatrix} & \begin{pmatrix}0&1\\-1&0\end{pmatrix} \\
\text{or even } 1D: & 1 & i & -1 & -i & \text{w/} & 1 & i & -1 & -i
\end{array}
$$

	I	g_1	g_2	g_3
I	I	g_1	g_2	g_3
g_1	g_1	g_2	g_3	I
g_2	g_2	g_3	I	g_1
g_3	g_3	I	g_1	g_2

$g_1^{-1} = g_3$

$g_1 = g_3^{-1}$

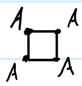
$g_2 = g_2^{-1}$

What if instead I had labelled the square by  ? In this case:

$\{I, g_1, g_2, g_3\}$

$\{I, g_1\}$ $\quad g_1 = g_1^{-1}$

$\{I\}$

What about  ?

These are called "representations" of the group. The first is called the "fundamental" and the other two are called "degenerate".

Now it turns out that for this group we have: $g_1^2 = g_2, g_1^3 = g_3, g_1^4 = I \Rightarrow g_1$ is the "generator" of the set. But so is g_3: $g_3^2 = g_2, g_3^3 = g_1, g_3^4 = I$ so we could instead use g_3 to generate the set. But we only <u>need</u> one of them. That is just having g_1 or g_3, either is enough to generate all the elements of the group. Note, I or g_2 will <u>not</u>!

But there are cases where we "need" more than one, e.g. $\{I, g, h, u, v\}$

w/ $g^2 = h, g^3 = I$ and $u^2 = v, u^3 = I \Rightarrow$

hence $h^2 = g, h^3 = I$ and $v^2 = u, v^3 = I$

So any of $\{g,u\}, \{g,v\}, \{h,u\}, \{h,v\}$ is enough to generate everything else.

	I	g	h	u	v
I	I	g	h	u	v
g	g	h	I		
h	h	I	g		
u	u			v	I
v	v			I	u

We are not always guaranteed a <u>subset</u> of generators, e.g.

	I	a	b	c
I	I	a	b	c
a	a	I	c	b
b	b	c	I	a
c	c	b	a	I

Obviously these groups are finite, but we can also have infinite discrete groups, i.e. $\{T_n\}$ $n \in \mathbb{Z}$ which is the set of translations along a line in integer steps.

Now it is time to go continuous. Obviously for a continuous group there are an infinite number of elements. But it is still useful to evaluate them as compact vs. non-compact.

R_θ $\theta \in [0, 2\pi)$ compact
T_x $x \in \mathbb{R}^1$ non-compact

A side note: Compact vs. non-compact determines quantization in QM. For example, T_x w/ $x \in \mathbb{R}^1 \Rightarrow p_x$ cont.
T_x w/ $x \in [0, L) \Rightarrow p_x$ quant.
$R_\theta \Rightarrow L_x$ quant.

In discussing continuous groups it is first of all impossible to "list" all of the elements, and you can forget a multiplication table. However, if a continuous group has a finite # of generators, then perhaps focussing on them makes the story easier.

It turns out that a rather large category of continuous groups plays important roles in physics, i.e. the Lie groups. These groups are continuous collections of elements that actually form a manifold. Remember those?! But in doing so, we can bring in what we know about manifolds, for example they are locally isomorphic to $\mathbb{R}^n$, and so the set of elements of a Lie group must also be isomorphic to $\mathbb{R}^n$. What is n for a Lie group? Let's find out in an example.

Consider the group $SU(3)$ which is defined as the set of complex valued 3×3 matrices which are unitary (U), i.e. $A^\dagger A = I$, and special (S), i.e. $\det A = +1$.

First let's confirm that these form a group:

1. Closure - if $A \in G$, $B \in G$ then $(AB)^\dagger AB = B^\dagger A^\dagger A B = B^\dagger B = I$
   and $\det(AB) = \det A \det B = 1$
   so $AB \in G$.
2. Associative - square matrix mult. guarantees this
3. Identity - obviously $I = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ and $IA = A$.
4. Inverse - since $A^\dagger A = I \Rightarrow A^{-1} = A^\dagger$.

Now one immediate question to be asked is how many generators does $SU(3)$ have?

Well $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ where each entry is complex, so there are 18 parameters in general. Note $i \neq \sqrt{-1}$

Now apply that $A^\dagger A = \mathbb{I} \Rightarrow \begin{pmatrix} a^* & d^* & g^* \\ b^* & e^* & h^* \\ c^* & f^* & i^* \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \Rightarrow$

$$\begin{cases} a^*a + d^*d + g^*g = 1 & \text{1 real eq.} \\ a^*b + d^*e + g^*h = 0 & \text{2 real eq. } \binom{Re}{Im} \\ a^*c + d^*f + g^*i = 0 & \text{2 real eq. } \binom{Re}{Im} \\ b^*a + e^*d + h^*g = 0 & \\ b^*b + e^*e + h^*h = 1 & \text{1 real eq.} \\ b^*c + e^*f + h^*f = 0 & \text{2 real eq. } \binom{Re}{Im} \\ c^*a + f^*d + i^*g = 0 & \\ c^*b + f^*e + i^*h = 0 & \\ c^*c + f^*f + i^*i = 1 & \text{1 real eq.} \end{cases}$$

same

Same

same

9 real eq.

So we start w/ 18 parameters, but they must satisfy 9 independent eqs. $\Rightarrow$ $18 - 9 = 9$ free parameters

But they must also satisfy $\det A = +1$, however $\det \mathbb{I} = +1 = \det(A^\dagger A) = \det A^\dagger \det A$

$$= (\det A^T)^* \det A$$

$$= (\det A)^* \det A \Rightarrow \det A = e^{i\theta}$$

Now if $\det = e^{i\theta} \Rightarrow \begin{cases} Re(\det A) = \cos\theta \\ Im(\det A) = \sin\theta \end{cases}$ so if we choose elements enough to find $Re(\det A)$, then identifying this as $\cos\theta$, we need to find the rest s.t. $Im(\det A) = \sin\theta$.

This provides one more constraint and hence leaves 8 free parameters.

In fact: $U(N)$ has N^2

$SU(N)$ has $N^2 - 1$

Now we come full circle and answer the question, what is the dimension of the manifold that the elements of $SU(3)$ constitute? $n = 8$

As we discussed in the finite case, if we had the 8 generators of $SU(3)$ in hand, then we can build elements of $SU(3)$ by combining many copies of these. How do we do that?

It turns out that an exponential will do the trick.

Let's call the 8 generators $\{T_j\}$ $j=1,\dots,8$.

$T_1 = \begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&0 \end{pmatrix}$ $T_2 = \begin{pmatrix} 0&-i&0\\ i&0&0\\ 0&0&0 \end{pmatrix}$ $T_3 = \begin{pmatrix} 1&0&0\\ 0&-1&0\\ 0&0&0 \end{pmatrix}$ $T_4 = \begin{pmatrix} 0&0&1\\ 0&0&0\\ 1&0&0 \end{pmatrix}$ $T_5 = \begin{pmatrix} 0&0&-i\\ 0&0&0\\ i&0&0 \end{pmatrix}$ $T_6 = \begin{pmatrix} 0&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix}$ $T_7 = \begin{pmatrix} 0&0&0\\ 0&0&-i\\ 0&i&0 \end{pmatrix}$
and $T_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1&0&0\\ 0&1&0\\ 0&0&-2 \end{pmatrix}$.

This is one set. There are others which share the same critical properties, i.e. they are all traceless $Tr(T_j)=0$ and Hermitian $T_j^\dagger = T_j$. In a sense these are infinitesimal versions of the 8 degrees of freedom defining any element of $SU(3)$. An arbitrary combination of these is essentially a vector in the tangent space to the manifold, i.e. $\vec{\alpha} = \alpha_j T_j$ where the T_j basis would be fixed and how much of each is determined by the 8 parameters $\{\alpha_j\}$. But where in the vector bundle are we choosing the vector space? Is the manifold flat, i.e. globally $\mathbb{R}^8$ where all vector spaces can be parallel, or is it curved where the vector spaces at each point are different?

While for general manifolds there may not be any special points, i.e. $\mathbb{R}^2$ or S^2_1 in this case there is. Remember that the collection of points is the set of group elements, and for this set there is obviously a special one, i.e. the identity.

So our infinitesimal basis is over the identity, and with this choice in hand, to build an element of $SU(3)$ given by $U(\vec{\alpha})$ we can use:
$$U(\vec{\alpha}) = \exp(i\vec{\alpha}) = \exp\left(i\sum_{j=1}^{8}\alpha_j T_j\right)$$

Let's start by proving that this construction satisfies the definition of $SU(3)$.

1. Starting w/ (S):
$$\det[U(\vec{\alpha})] = \det\left[\exp\left(i\sum_{j=1}^{8}\alpha_j T_j\right)\right] \overset{\text{Jacobi's Theorem}}{=} \exp\left[Tr\left(i\sum_{j=1}^{8}\alpha_j T_j\right)\right] \overset{\text{since all } T_j \text{ are traceless}}{=} \exp[0] = 1$$

2. Then w/ (U):
$$U^\dagger(\vec{\alpha})U(\vec{\alpha}) = \exp\left(i\sum_{j=1}^{8}\alpha_j T_j\right)^\dagger U(\vec{\alpha}) = \exp\left(-i\sum_{j=1}^{8}\alpha_j T_j^\dagger\right)U(\vec{\alpha}) = \exp\left(-i\sum_{j=1}^{8}\alpha_j T_j\right)U(\vec{\alpha}) = U(-\vec{\alpha})U(\vec{\alpha}) \overset{\text{using Baker Campbell Hausdorff}}{=} \exp(\vec{0}) = I$$

Let's build an example. Start w/ $\alpha_1 \neq 0$, $\alpha_{j\neq 1} = 0$ $\Rightarrow U(\vec{\alpha}) = \exp(i\alpha_1 T_1) = \sum_{n=0}^{\infty}\frac{1}{n!}(i\alpha_1 T_1)^n$ but notice the quite useful property of T_1: $T_1 = \begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&0 \end{pmatrix}$, $T_1^2 = \underbrace{\begin{pmatrix} 1&0&0\\ 0&1&0\\ 0&0&0 \end{pmatrix}}_{\text{"}I_2\text{"}}$, $T_1^3 = T_1$, $T_1^4 = I_2$ (similar for other T_j's)

"I_2" $\Rightarrow I_2 I_2 = I_2$, $I_2 T_1 = T_1$

then:
$$U(\vec{\alpha}) = I + \sum_{n=\text{odd}}\frac{1}{n!}(i\alpha_1)^n T_1 + \sum_{n=\text{even}}\frac{1}{n!}(i\alpha_1)^n I_2 = \begin{pmatrix} 1+\sum_{n=\text{even}}\frac{1}{n!}(i\alpha_1)^n & \sum_{n=\text{odd}}\frac{1}{n!}(i\alpha_1)^n & 0 \\ \sum_{n=\text{odd}}\frac{1}{n!}(i\alpha_1)^n & 1+\sum_{n=\text{even}}\frac{1}{n!}(i\alpha_1)^n & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha_1 & i\sin\alpha_1 & 0 \\ i\sin\alpha_1 & \cos\alpha_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$