Now recall that for our "rotations of a square" group, we found a variety of matrix realizations of the group. Does any of these define the group? No. Some are more faithful representations than others.

So while $SU(3)$ is defined as a group of particular matrix operators, there should be a sense in which we can provide a more abstract definition of the group.
At least for a definition of a Lie group in the vicinity of the identity, we can take a set of generators (any set will do) and determine their "Lie algebra".

To get the Lie algebra we first clean things up by defining $g_j = \frac{1}{2} T_j$. Then we consider commutators of pairs of elements. We find:

$[g_i, g_j] = i f^{ijk} g_k$ where $f^{123} = 1$, $f^{147} = f^{165} = f^{246} = f^{257} = f^{345} = f^{376} = \frac{1}{2}$, $f^{458} = f^{678} = \frac{\sqrt{3}}{2}$

and f^{ijk} is antisymmetric under index exchange, e.g. $f^{123} = 1 = -f^{213}$, hence $f^{iij} = 0$.

and if any f^{ijk} is missing from the list it is zero, e.g. $f^{137} = 0$.

As an example we find: $[g_1, g_2] = i f^{12k} g_k = i f^{123} g_3 = i g_3$ which $g_1 = \frac{1}{2}\begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&0 \end{pmatrix}$ $g_2 = \frac{1}{2}\begin{pmatrix} 0&-i&0\\ i&0&0\\ 0&0&0 \end{pmatrix}$ $g_3 = \frac{1}{2}\begin{pmatrix} 1&0&0\\ 0&-1&0\\ 0&0&0 \end{pmatrix}$ satisfy!

Now where the power of using the Lie algebra to define a group is that it is not written with any aforementioned constraints on the matrix form of generators. So any set that satisfies it is the root of a representation of the more abstractly defined group.

As an interesting example, consider $SO(3)$ which has 3 generators $\{g_j\}$ which satisfy $[g_i, g_j] = i\epsilon^{ijk} g_k$ where ϵ^{ijk} is the anti-symmetric Levi-Civita. Three examples of generators are $g_1 = \begin{pmatrix} 0&0&0\\ 0&0&-i\\ 0&i&0 \end{pmatrix}$ $g_2 = \begin{pmatrix} 0&0&i\\ 0&0&0\\ -i&0&0 \end{pmatrix}$ $g_3 = \begin{pmatrix} 0&-i&0\\ i&0&0\\ 0&0&0 \end{pmatrix}$
Clearly from the exponential map, these 3×3 generators will create 3×3 matrix elements of $SO(3)$.
However, there is at least a second solution to $[g_i, g_j] = i\epsilon^{ijk} g_k$ $\Rightarrow$ $g_1 = \begin{pmatrix} 0&1/2\\ 1/2&0 \end{pmatrix}$ $g_2 = \begin{pmatrix} 0&-i/2\\ i/2&0 \end{pmatrix}$ $g_3 = \begin{pmatrix} 1/2&0\\ 0&-1/2 \end{pmatrix}$
But obviously the exponential map will bring these to 2×2 matrices. One big difference is that while elements of $SO(3)$ are real, the elements of the 2×2's are complex. In fact one can flesh them out to discover that the corresponding matrix group in this case is $SU(2)$.

So what we have found is that $SO(3) \cong SU(2)$, at least in the vicinity of the identity. This equivalence is the basis of spinor representations of the "rotation group". For which we are most familiar with its vector representation, i.e. 3×3 matrices.

$SO(3)$ as it turns plays the important role of being a subgroup of the "isometries" of $\mathbb{R}^3$. Isometries are transformations that leave the form of the metric on a space unchanged. Note that we introduced isometries on real or complex vector spaces as matrices that satisfy $U^T U = I$ or $U^\dagger U = I$. But now we are talking about the isometries of a space.

In $\mathbb{R}^3$ w/ (x, y, z) $ds^2 = dx^2 + dy^2 + dz^2 = (dx\ dy\ dz)\left(g \right)\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \Rightarrow g = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$.

Since under a linear transformation U, the metric would transform as $g \to g' = U^T g U$, invariance under isometries implies $U^T g U = g$.

So given g, we can use this condition to find isometries (all except translations that is).

So for $g = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ the isometries satisfy $U^T I U = I \Rightarrow U^T U = I$, hence the isometries are orthogonal, hence $O(3)$. To preserve handedness, we insist that $\det U = +1$, hence $SO(3)$. Of course this is the same as for the $\mathbb{R}^3$ vector space. But wait...

What about spacetime, i.e. $\mathbb{M}^4$ w/ $g = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$? Well once again, solutions to $U^T g U = g$ are isometries of the space. What do these look like?

Well first of all we could imagine transformations which only hit the lower-right 3×3 block $\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$. But these are $SO(3)$ transformations, or rotations. Nothing can be done to the single upper-left element alone. But we can mix elements from the first row or column w/ elements from the lower-right 3×3 block, that is we could mix x-t, y-t or z-t. These turn out to be boosts, i.e. transformations that take us to a new frame that has a constant velocity along x, y or z.

The resulting group is called $SO(1,3)$, and the number of generators is $\frac{1}{2} 4(4-1) = 6$ just like $SO(4)$ (as you prove in your HW).

An immediate question you might ask is: Spinors? Well yes, but it is a bit more complicated.

First of all, the subset of $SO(3)$ like transformations have "4D" extensions that are trivial:

$J_1 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & -i \\ & & i & 0 \end{pmatrix}$ $J_2 = \begin{pmatrix} 0 & & & \\ & 0 & & i \\ & & 0 & \\ & -i & & 0 \end{pmatrix}$ $J_3 = \begin{pmatrix} 0 & & & \\ & 0 & -i & \\ & i & 0 & \\ & & & 0 \end{pmatrix}$

the boost on the other hand are generated by:

$K_1 = \begin{pmatrix} 0 & -i & & \\ -i & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$ $K_2 = \begin{pmatrix} 0 & & -i & \\ & 0 & & \\ -i & & 0 & \\ & & & 0 \end{pmatrix}$ $K_3 = \begin{pmatrix} 0 & & & -i \\ & 0 & & \\ & & 0 & \\ -i & & & 0 \end{pmatrix}$

Forming the Lie algebra we find:

$[J_i, J_j] = i\epsilon^{ijk} J_k$ as expected for $SO(3)$

$[K_i, K_j] = -i\epsilon^{ijk} J_k$ 2 boosts → rotation

$[J_i, K_j] = i\epsilon^{ijk} K_k$ rotation and boost → boost

So while the J_k generators form the elements of a subgroup of $SO(1,3)$, the K_j generators do not!

But, consider instead: $J_{\pm i} = \frac{1}{2}(J_i \pm i K_i) \Rightarrow$

$[J_{+i}, J_{+j}] = i\epsilon^{ijk} J_{+k} \Rightarrow SO(3)$

$[J_{-i}, J_{-j}] = i\epsilon^{ijk} J_{-k} \Rightarrow SO(3)$

$[J_{+i}, J_{-j}] = 0$ the $SO(3)$'s don't mix

So we have basically found that with this rewriting $SO(1,3) \simeq SO(3) \times SO(3)$, but now we can play the $SO(3) \simeq SU(2)$ game, hence $SO(1,3) \simeq SU(2) \times SU(2)$. Each factor of $SU(2)$ gives rise to 2 complex component spinors, so in total this representation of the isometries of $\mathbb{M}^4$ are 4-component spinors. Note, even though vectors in this spacetime have 4 components as well, the spinors are quite different. For example you can consider the components of a vector to be along t, x, y, z. Not so for spinors!

So in going from 3D to 4D, vectors go from 3 to 4 components, whereas spinors go from 2 to 4. What next? Well it turns out that to count, you just ask for the number of independent planes in a space(time). Each one gives a rotation which can be encoded in an $SU(2)$ w/ a 2-component spinor leading to $2^{d/2}$ or $2^{(d-1)/2}$ states if d is even or odd.

	3D	4D	5D	6D	7D	8D	9D	10D
vector	3	4	5	6	7	8	9	10
spinor	2	4	4	8	8	16	16	32

spinors win!!

Yet another representation will play an important role in what is to come.

First of all, the commutators satisfy: $\underbrace{[g_i,[g_j,g_k]]+[g_j,[g_k,g_i]]+[g_k,[g_i,g_j]]=0}$

this is trivial.. just write it out!

But using the Lie algebra for generators this becomes:

$$[g_i, if^{jkm}g_m]+[g_j, if^{kim}g_m]+[g_k, if^{ijm}g_m]=0$$

$$if^{jkm}[g_i,g_m]+if^{kim}[g_j,g_m]+if^{ijm}[g_k,g_m]=0$$

$$f^{jkm}f^{imn}g_n+f^{kim}f^{jmn}g_n+f^{ijm}f^{kmn}g_n=0$$

$$f^{jkm}f^{imn}+f^{kim}f^{jmn}+f^{ijm}f^{kmn}=0 \quad \text{Jacobi Identity}$$

or $$f^{kjm}f^{min}-f^{kim}f^{mjn}-f^{jim}f^{kmn}=0$$

by switching indices (once gets a −, and twice gets +)

Now f^{ijk} has three indices, but one way to view them is that one index (say j) labels which matrix you are considering, while the other two (i and k) label the elements of the matrix.

To clarify this we can write $(t^j)_{ik}=if^{ijk}$, but then the second version of the Jacobi identity is:

$$(t^j)_{km}(t^i)_{mn}-(t^i)_{kn}(t^j)_{mn}-if^{jim}(t^m)_{kn}=0$$

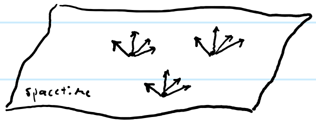
Now squint... what do you see?

$$[t^j,t^i]=if^{jim}t^m$$

That is, the structure constants f^{ijk} themselves satisfy the Lie algebra, and so form the "adjoint" representation of the associated group.

Now what sort of role does SU(3) play in physics? Well SU(3), as well as SU(2) and U(1) are the symmetry groups underlying three of the four fundamental forces in nature, i.e. the strong force (SU(3)), the weak force (SU(2)) and electromagnetism (U(1)).

To give you a quick idea of how this works, we first start out with a free field theory that contains no interactions, i.e. $\mathcal{L} = \int \partial_\mu \phi \partial^\mu \phi \, d^4x$. Then we expand our notion of spacetime by attaching to each point a copy of the Lie groups SU(3), SU(2) and U(1) which creates what is often called a fiber bundle over spacetime.



This is much like the tangent bundle, except we have freedom in defining the fibers.

Now any field which lived on spacetime, say $\phi(x^\mu)$, must now be endowed with extra degrees of freedom associated with these fibers, i.e. they must transform in some representation of SU(3), SU(2) and U(1).

To make this interesting, let's use the fundamental rep of each. So for SU(3) the field is represented by $\phi(x^\mu)_3 = \begin{pmatrix} \phi_r \\ \phi_g \\ \phi_b \end{pmatrix}$ (3 "colors"), for SU(2) by $\phi(x^\mu)_2 = \begin{pmatrix} \phi_u \\ \phi_d \end{pmatrix}$ (flavor), and for U(1) by $\phi(x^\mu)$ being complex and hence admitting a phase.

With this setup we now follow what is roughly a three step process.

1. Insist that the transformations associated w/ SU(3), SU(2) and U(1) are a global symmetry of $\mathcal{L}$.
   That is, $\mathcal{L} = \int \partial_\mu \phi \partial^\mu \phi \, d^4x \rightarrow \mathcal{L} = \int \partial_\mu \tilde{\phi} \partial^\mu \phi \, d^4x$ where if ϕ transforms as $\phi \rightarrow \phi' = k\phi$ for k being any combination of elements of SU(3), SU(2) and U(1), then $\tilde{\phi}$ transforms as $\tilde{\phi} \rightarrow \tilde{\phi}' = \tilde{\phi} k^{-1}$.
   Note that we use the same k at every fiber over the bundle, hence global.
2. Now we promote the global symmetry to a local one, that is $k \rightarrow k(x^\mu)$.
   Notice that now we can't cancel $k(x^\mu)$ w/ $k^{-1}(x^\mu)$ because we can't move these past the derivative without generating extra terms, i.e. $\partial_\mu \phi \rightarrow \partial_\mu(k\phi) = (\partial_\mu k)\phi + k(\partial_\mu \phi)$. To fix this, we redefine the derivative ∂_μ to what we call a "covariant" one by adding terms to it $\partial_\mu \rightarrow D_\mu = \partial_\mu + A^1_\mu + A^2_\mu + \cdots$
   If the additional terms transform in just the right way under SU(3), SU(2) and U(1), then $D_\mu(\phi) \rightarrow D_\mu(k\phi) = k D_\mu \phi$ and invariance is restored. What representation is this? The adjoint!
   What are the A^i_μ? They are the generators of SU(3), SU(2) and U(1).
   Behold: $\mathcal{L} = \int \partial_\mu \tilde{\phi} \partial^\mu \phi \, d^4x \rightarrow \mathcal{L} = \int D_\mu \tilde{\phi} D^\mu \phi \, d^4x = \int (\partial_\mu + A^i_\mu)\tilde{\phi}(\partial^\mu + A^{i\mu})\phi \, d^4x = \int \{ \partial_\mu \tilde{\phi} \partial^\mu \phi + \underbrace{A^i_\mu \tilde{\phi} \partial^\mu \phi + \partial_\mu \tilde{\phi} A^{i\mu} \phi + A^i_\mu \tilde{\phi} A^{i\mu} \phi}_{\text{interactions !!}} \} d^4x$
   This Lagrangian describes "charged" particles interacting w/ fixed sources A^i_μ.
3. Let these new "gauge" fields be dynamical by adding in kinetic terms for them.
   $\mathcal{L} = \int \{ D_\mu \tilde{\phi} D^\mu \phi + F^i_{\mu\nu} F^{i\mu\nu} \} d^4x$ where $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu$ $\Rightarrow$ for U(1) this gives Maxwell + Lorentz Force

So we have three of four fundamental forces coming from the localization or gauging of the global symmetries associated w/ fibers over spacetime.

But what about gravity? While the other forces are associated w/ fibers over spacetime, gravity seems to be more closely associated w/ spacetime itself. What global symmetries are there that we might try to localize? How about the isometries?

Starting in $\mathbb{M}^4$ where the isometries include translations, rotations and boosts, all of which are global, i.e. the transformation operators have constant elements, e.g. $R_{xy} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta & \sin\theta \\ & & -\sin\theta & \cos\theta \end{pmatrix}$, we can consider localizing these transformations.
But the result is really just an arbitrary coordinate transformation, an element of $GL(3, \mathbb{R})$. 1D
general linear real valued elements.
But elements of $GL(3, \mathbb{R})$ can be functions of position in spacetime. As before, these transformations can't easily be commuted w/ the derivative, i.e. $\partial_\mu (UV) \neq U \partial_\mu V$.
But we need it to, so we redefine the derivative once again, this time using a gauge field known as Christoffel symbols, i.e. $\partial_\mu \to D_\mu = \partial_\mu + \Gamma^\alpha_{\mu\beta}$.
Now we have that $D_\mu (UV) = U D_\mu V$, and the lagrangian now contains interactions between particles and a fixed, but possibly curved geometry.
We finish up by letting the $\Gamma^\alpha_{\mu\beta}$ be dynamical by adding in a kinetic term for them (built out of the Riemann curvature tensor) and the result is a theory where sources deform the geometry of spacetime, and this impacts the movement of objects on it, i.e. General Relativity.

I've given you a cursory review of my single favorite thing in physics; that symmetry underlies all of the fundamental forces in nature. In my particle physics and GR classes I dive into the details of each case. But 'nuff said for now!