Particle Physics HW4

1. Now that we have seen at least one set of the Dirac matrices and you have some hint as to their use, it is now time for you to get a little bit more familiar with what they are and some useful properties which will come in very handy later on.
   There is also a "fifth" gamma matrix that we will eventually come to use, however its index is not a spacetime value since spacetime indices run 0,1,2,3. The fifth gamma matrix is defined by $\gamma^5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3$.
   Prove the following identities satisfied by any set of Dirac matrices (that means do NOT use the explicit matrices I gave you in class, but rather the defining relationship $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I_{4x4}$, the expression above for γ^5 and what you know about matrices and the metric). Note, "Tr" means trace over the spinor indices (not spacetime). I have intentionally left off the identity in spin space I_{4x4} so you can get used to figuring out where and when it should be present which is useful since most resources leave it off anyway. Note: Several of these require that you use the results from the previous parts. It helps to remember that the trace is cyclic, e.g. $Tr(ABC) = Tr(CAB) = Tr(BCA)$ (which holds for any number of matrices).
   a) $Tr(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}$
   b) $Tr(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho) = 4(\eta^{\mu\nu}\eta^{\lambda\rho} - \eta^{\mu\lambda}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\lambda})$ **Hint:** This is the hard one. Start with $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, let both sides act on $\gamma^\lambda\gamma^\rho$, then trace and use the cyclicity of the trace, commutation properties and the result from (a).
   c) $\gamma^5\gamma^5 = 1$
   d) $\{\gamma^\mu, \gamma^5\} = 0$
   e) The trace of an odd product of gamma matrices is always zero. **Hint:** Craftily insert the identity inside of the trace written as $\gamma^5\gamma^5 = 1$ and then use the results of part (d).
   f) $Tr(\gamma^5\gamma^\mu\gamma^\nu) = -Tr(\gamma^5\gamma^\nu\gamma^\mu)$
2. To build an invariant out of spinors we had to introduce a somewhat unexpected form for the dual spinor, i.e. $\tilde{\psi} = i\gamma^0\psi$. Then showing that $\tilde{\psi}^\dagger\psi$ is invariant depended on the result that $\left(e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\right)^\dagger \gamma^0 = \gamma^0 e^{-\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}$. Prove this by expanding out the exponential for the first three terms and using the (anti)commutation relations of the gamma matrices. **Hint:** It will be useful to recall that ${\sigma^{ij}}^\dagger = \sigma^{ij}$ and ${\sigma^{0j}}^\dagger = -\sigma^{0j}$ where $i, j = 1,2,3$ and also determining how $[\gamma^i, \gamma^j]\gamma^0$ relates to $\gamma^0[\gamma^i, \gamma^j]$ and how $[\gamma^0, \gamma^j]\gamma^0$ relates to $\gamma^0[\gamma^0, \gamma^j]$.

   The last three questions will be more approachable after Tuesdays lecture.

3. Consider the functional $G[f(x)] = \int_0^1 ((f-1)^2 + f'^2)\, dx$ where $f' = \frac{df}{dx}$ and the unknown function is subject to the boundary conditions $f(0) = f(1) = 1$. I want you to find the function $f(x)$ which minimizes this functional subject to the prescribed boundary conditions in two ways:
   a) First by inspection of the functional itself. Consider what its minimum value could be and see if you can guess what function would yield that value. **Hint:** Consider the signs of various terms.

b) Now compute the variation of the functional under a change in $f(x)$ to obtain a differential equation that the extremizing function $f(x)$ should satisfy, then showing that your answer from part (a) satisfies this differential equation.

Congratulations! You are now an action superstar!

4. As promised in class, you get to work though the Euler Lagrange equation of motion for a (massless) vector field starting with the Proca Lagrangian given in class (we didn't give it a name, but Proca is what you should call it). But....this time I want you to consider the field A_μ to be "coupled" to a 4-current J^μ so the total Lagrangian density is

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu} - \frac{1}{c} A_\mu J^\mu$$

What you will be deriving is essentially the relativistic form of two of Maxwell's equations, this time including sources (since J^μ contains both 3-current and charge density).

Hint: The additional term should actually be a straightforward modification. The first term is where the real work needs to be done. It helps to write out $F_{\mu\nu}F^{\mu\nu}$ explicitly and use factors of the inverse metric to get all of the indices on ∂ and A lowered before taking derivatives. The tricky part is finding a factor of "4" that is not obvious. This will arise from considering all possible relabeling of "dummy" or repeated indices that get each term into the form of what you are taking the derivative with respect to. Note: You should be considering how $\mathcal{L}$ changes under a variation of A_μ not A^μ.

5. Show that any solution of the Dirac equation is also a solution of the Klein-Gordon equation.
Hint: Consider the Dirac equation as one operator acting on ψ. Now act on the left with the same operator, but with the sign reversed and then use gamma matrix and index gymnastics to reduce the result to the KG equation.