

1. a) $\text{Tr}(\gamma^\mu \gamma^\nu) = 4n^{\mu\nu}$

Use: $\{\gamma^\mu, \gamma^\nu\} = 2n^{\mu\nu} \Rightarrow \text{Tr}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = 2n^{\mu\nu} \text{Tr} I = 8n^{\mu\nu}$
 $\text{Tr}(\gamma^\mu \gamma^\nu) + \text{Tr}(\gamma^\nu \gamma^\mu) = 8n^{\mu\nu}$
 $2 \text{Tr}(\gamma^\mu \gamma^\nu) = 8n^{\mu\nu}$
 $\text{Tr}(\gamma^\mu \gamma^\nu) = 4n^{\mu\nu} \quad \checkmark$

b) $\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho) = 4(n^{\mu\nu} n^{\lambda\rho} - n^{\mu\lambda} n^{\nu\rho} + n^{\mu\rho} n^{\nu\lambda})$

Use: $\{\gamma^\mu, \gamma^\nu\} = 2n^{\mu\nu} \Rightarrow \gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu + 2n^{\mu\nu}$
 $\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho) = \text{Tr}(-\gamma^\nu \gamma^\mu \gamma^\lambda \gamma^\rho) + \text{Tr}(2n^{\mu\nu} \gamma^\lambda \gamma^\rho)$
 $= -\text{Tr}(\gamma^\nu \gamma^\mu \gamma^\lambda \gamma^\rho) + 2n^{\mu\nu} \text{Tr}(\gamma^\lambda \gamma^\rho)$
 $= -\text{Tr}(\gamma^\nu \gamma^\mu \gamma^\lambda \gamma^\rho) + 8n^{\mu\nu} n^{\lambda\rho}$
 $= -\text{Tr}(\gamma^\rho \gamma^\nu \gamma^\mu \gamma^\lambda) + 8n^{\mu\nu} n^{\lambda\rho}$
 $= \text{Tr}(\gamma^\rho \gamma^\nu \gamma^\mu \gamma^\lambda) - 2n^{\rho\nu} \text{Tr}(\gamma^\mu \gamma^\lambda) + 8n^{\mu\nu} n^{\lambda\rho}$
 $= \text{Tr}(\gamma^\nu \gamma^\rho \gamma^\mu \gamma^\lambda) - 8n^{\rho\nu} n^{\mu\lambda} + 8n^{\mu\nu} n^{\lambda\rho}$
 $= -\text{Tr}(\gamma^\nu \gamma^\rho \gamma^\mu \gamma^\lambda) + 2n^{\rho\nu} \text{Tr}(\gamma^\mu \gamma^\lambda) - 8n^{\rho\nu} n^{\mu\lambda} + 8n^{\mu\nu} n^{\lambda\rho}$
 $2 \text{Tr}(\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho) = 8n^{\rho\nu} n^{\mu\lambda} - 8n^{\rho\nu} n^{\mu\lambda} + 8n^{\mu\nu} n^{\lambda\rho}$
 $\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho) = 4(n^{\mu\nu} n^{\lambda\rho} - n^{\mu\lambda} n^{\nu\rho} + n^{\mu\rho} n^{\nu\lambda}) \quad \checkmark$

c) $\gamma^5 \gamma^5 = 1$

($-\gamma^0 \gamma^1 \gamma^2 \gamma^3$) $\equiv \gamma^5$ and use $(\gamma^0)^2 = -1, (\gamma^i)^2 = +1$ and $\gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu \quad \mu \neq \nu$

$\gamma^5 \gamma^5 = (-\gamma^0 \gamma^1 \gamma^2 \gamma^3)(-\gamma^0 \gamma^1 \gamma^2 \gamma^3) = -\underbrace{\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^0 \gamma^1 \gamma^2 \gamma^3}_{\substack{\gamma^0 \gamma^0 \gamma^1 \gamma^1 \gamma^2 \gamma^2 \gamma^3 \gamma^3}} = \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^1 \gamma^2 \gamma^3 \gamma^0 = \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^2 \gamma^3 \gamma^1 \gamma^0$
 $= -\gamma^0 \gamma^1 \gamma^2 \gamma^1 \gamma^2 \gamma^3 \gamma^3 \gamma^0 = 1 \quad \checkmark$

d) $\{\gamma^\mu, \gamma^5\} = 0$

$\{\gamma^\mu, \gamma^5\} = -\gamma^\mu \gamma^0 \gamma^1 \gamma^2 \gamma^3 - \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^\mu$

For $\mu=0 \quad \{\gamma^0, \gamma^5\} = -\gamma^0 \gamma^0 \gamma^1 \gamma^2 \gamma^3 + \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^0 = 0$

$\mu=1 \quad \{\gamma^1, \gamma^5\} = -\gamma^1 \gamma^0 \gamma^1 \gamma^2 \gamma^3 - \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^1 = -\gamma^0 \gamma^1 \gamma^1 \gamma^2 \gamma^3 + \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^1 = 0$

Similarly for $\mu=2, 3$.

e) $\text{Tr}(\gamma^\mu \gamma^\nu \dots) = 0$ for odd # of γ 's

Recall that $\gamma^5 \gamma^5 = 1$ then $\text{Tr}(\gamma^\mu) = \text{Tr}(\gamma^\mu \gamma^5 \gamma^5) = \text{Tr}(\gamma^5 \gamma^\mu \gamma^5) = -\text{Tr}(\gamma^\mu \gamma^5 \gamma^5)$
 and $\gamma^\mu \gamma^5 = -\gamma^5 \gamma^\mu$

But if $\text{Tr}(\gamma^\mu \gamma^5 \gamma^5) = -\text{Tr}(\gamma^\mu \gamma^5 \gamma^5)$ then $\text{Tr}(\gamma^\mu \gamma^5 \gamma^5) = \text{Tr}(\gamma^\mu) = 0$

This argument can be iterated for higher odd numbers of γ 's.

f) $\text{Tr}(\gamma^5 \gamma^\mu \gamma^\nu) = -\text{Tr}(\gamma^5 \gamma^\nu \gamma^\mu)$

$\text{Tr}(\gamma^5 \gamma^\mu \gamma^\nu) = \text{Tr}(\gamma^\nu \gamma^5 \gamma^\mu) = \text{Tr}(\gamma^5 \gamma^\nu \gamma^\mu \gamma^5 \gamma^5) = -\text{Tr}(\gamma^5 \gamma^\nu \gamma^5 \gamma^\mu \gamma^5 \gamma^5) = -\text{Tr}(\gamma^5 \gamma^\nu \gamma^\mu) \quad \checkmark$



2. We want to show that $(e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}})^{\dagger}\gamma^0 = \gamma^0 e^{-\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}$ where $\sigma^{\mu\nu} = \frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}]$

Expanding: $(e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}})^{\dagger}\gamma^0 = (1 + \frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu} + \frac{1}{2!}(\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu})^2 + \dots)^{\dagger}\gamma^0$

To move γ^0 to the left, we can freely move it across everything except $\sigma^{\mu\nu}$.

Since $\sigma^{\mu\nu} = \frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}] = \frac{i}{4}\gamma^{\mu}\gamma^{\nu} - \frac{i}{4}\gamma^{\nu}\gamma^{\mu}$ and $\gamma^i\gamma^0 = -\gamma^0\gamma^i$

We find: $\sigma^{ij}\gamma^0 = \gamma^0\sigma^{ij}$ since γ^0 moves across 2 γ 's

$\sigma^{0i}\gamma^0 = -\gamma^0\sigma^{0i}$ since γ^0 moves across 1 γ

So moving γ^0 to the left depends on whether we have σ^{ij} or σ^{0i} (note $\sigma^{00} = 0$).

But when we hermitian conjugate, the result also depends on the $\sigma^{\mu\nu}$, i.e. $\sigma^{ij\dagger} = \sigma^{ij}$, $\sigma^{0i\dagger} = -\sigma^{0i}$

So when we do both, the negative sign from moving γ^0 left across σ^{0i} cancels the negative from conjugating.

Then: $(1 + \frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu} + \frac{1}{2!}(\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu})^2 + \dots)^{\dagger}\gamma^0 = \gamma^0(1 - \frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu} + \frac{1}{2!}(\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu})^2 + \dots)$

↑ remember \dagger includes $*$

Or: $\gamma^0(e^{-\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}})$ ✓

$$3. \text{ For } G[f(x)] = \int_0^1 \overbrace{(f-1)^2 + \underbrace{f'^2}_{\geq 0}}^{L(x)} dx$$

so we expect a minimum when $\left. \begin{array}{l} f-1=0 \\ f'=0 \end{array} \right\} \begin{array}{l} f(x)=1 \\ f'(x)=0 \end{array}$ $f(0)=f(1)=1$

$$\text{The e.o.v. is: } \frac{\partial L}{\partial f} - \frac{d}{dx} \left(\frac{\partial L}{\partial f'} \right) = 2(f-1) - \frac{d}{dx} 2f' = 2f-2-2f'' = 0 = f-1-f''$$

For $f(x)=1$ this is clearly satisfied.

$$4. \quad \mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} A_\mu \bar{J}^\mu = -\frac{1}{16\pi} (\partial_\mu A_\nu - \partial_\nu A_\mu) (\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2} A_\mu \bar{J}^\mu$$

$$= \underbrace{-\frac{1}{16\pi} \eta^{\alpha\lambda} \eta^{\beta\rho} (\partial_\mu A_\nu - \partial_\nu A_\mu) (\partial_\lambda A_\rho - \partial_\rho A_\lambda)}_{\mathcal{I}_1} - \frac{1}{2} A_\mu \bar{J}^\mu_{\mathcal{I}_2}$$

Varying w.r.t. A_μ : $\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\nu A_\mu)} \right)$

$$\hookrightarrow -\frac{1}{2} \bar{J}^\mu$$

To compute $\frac{\partial \mathcal{L}}{\partial (\partial_\nu A_\mu)}$ start w/ \mathcal{I}_1 above and use new letters for all of the dummy indices.

$$\frac{\partial}{\partial (\partial_\nu A_\mu)} \left[-\frac{1}{16\pi} \eta^{\alpha\lambda} \eta^{\beta\rho} (\partial_\alpha A_\beta - \partial_\beta A_\alpha) (\partial_\lambda A_\rho - \partial_\rho A_\lambda) \right]$$

$$= -\frac{1}{16\pi} \eta^{\alpha\lambda} \eta^{\beta\rho} \left[(\delta_\alpha^\nu \delta_\beta^\mu - \delta_\beta^\nu \delta_\alpha^\mu) (\partial_\lambda A_\rho - \partial_\rho A_\lambda) + (\partial_\alpha A_\beta - \partial_\beta A_\alpha) (\delta_\lambda^\nu \delta_\rho^\mu - \delta_\rho^\nu \delta_\lambda^\mu) \right]$$

$$= -\frac{1}{16\pi} \left[(\delta_\alpha^\nu \delta_\beta^\mu - \delta_\beta^\nu \delta_\alpha^\mu) F^{\alpha\beta} + F^{\lambda\rho} (\delta_\lambda^\nu \delta_\rho^\mu - \delta_\rho^\nu \delta_\lambda^\mu) \right]$$

$$= -\frac{1}{16\pi} \left[F^{\nu\mu} - F^{\mu\nu} + F^{\nu\mu} - F^{\mu\nu} \right] \quad \text{but } F^{\nu\mu} = -F^{\mu\nu}$$

$$= -\frac{1}{4\pi} F^{\nu\mu}$$

$$= \frac{1}{4\pi} F^{\mu\nu}$$

So in total $\frac{\partial \mathcal{I}_1}{\partial (\partial_\nu A_\mu)} = \frac{1}{4\pi} F^{\mu\nu}$

Altogether: $\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\nu A_\mu)} \right) = 0 \Rightarrow -\frac{1}{2} \bar{J}^\mu - \frac{1}{4\pi} \partial_\nu F^{\mu\nu} = 0 = -\frac{1}{2} \bar{J}^\mu + \frac{1}{4\pi} \partial_\nu F^{\nu\mu}$

$$\boxed{\partial_\nu F^{\nu\mu} = \frac{4\pi}{2} \bar{J}^\mu}$$

The best way to understand the lengthy procedure above is as follows.

Suppose we start w/ $\mathcal{L}_1 = -\frac{1}{16\pi} \eta^{\alpha\lambda} \eta^{\beta\rho} (\partial_\alpha A_\beta - \partial_\beta A_\alpha) (\partial_\lambda A_\rho - \partial_\rho A_\lambda)$

If we were taking the derivative w.r.t. $\partial (\partial_\nu A_\mu)$ we would have to find every place that this term appears. But since every index in sight is summed over, we should expect $\partial_\nu A_\mu$ to appear once in each term when we do the sum over indices.

A simpler example is: $\mathcal{L} = A_\nu A^\nu \Rightarrow \frac{\partial \mathcal{L}}{\partial A_\nu} = \frac{\partial (\eta^{\mu\nu} A_\mu A_\nu)}{\partial A_\nu} = \eta^{\mu\nu} A_\mu + \eta^{\nu\mu} A_\mu = 2\eta^{\mu\nu} A_\mu$

Then for example: $\frac{\partial \mathcal{L}}{\partial A_0} = \frac{\partial}{\partial A_0} (-A_0 A_0 + A_1 A_1 + A_2 A_2 + A_3 A_3)$

$$= -2A_0$$

5. The Dirac equation: $\gamma^\mu \partial_\mu \psi + \frac{mc}{\hbar} \psi = 0$

The Klein-Gordon equation: $\partial_\mu \partial^\mu \psi - \left(\frac{mc}{\hbar}\right)^2 \psi = 0$

Let's go... If $(\gamma^\mu \partial_\mu + \frac{mc}{\hbar}) \psi = 0$

then $(\gamma^\nu \partial_\nu - \frac{mc}{\hbar})(\gamma^\mu \partial_\mu + \frac{mc}{\hbar}) \psi = 0$

so $\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu \psi - \left(\frac{mc}{\hbar}\right)^2 \psi = 0$

but $-\gamma^\mu \gamma^\nu \partial_\nu \partial_\mu \psi + 2\eta^{\mu\nu} \partial_\nu \partial_\mu \psi - \left(\frac{mc}{\hbar}\right)^2 \psi = 0$

rebellious
w/ $\mu \leftrightarrow \nu$



$\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu \psi - \left(\frac{mc}{\hbar}\right)^2 \psi = 0$ adding this line to

these can be freely switched

$2\eta^{\mu\nu} \partial_\mu \partial_\nu \psi - 2\left(\frac{mc}{\hbar}\right)^2 \psi = 0$

or

$\partial_\mu \partial^\mu \psi - \left(\frac{mc}{\hbar}\right)^2 \psi = 0$ Boon!!