Polarization and group velocity of P-waves in orthorhombic media

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ABSTRACT
Analytic description of seismic signatures in azimuthally anisotropic media is of primary importance in characterization of fractured reservoirs. The formalism developed here provides a convenient way of describing the behavior of group velocity and polarization vector in models with orthorhombic symmetry. The expressions for group and polarization vectors become particularly simple in the coordinate system associated with the vertical plane that contains the phase-velocity vector. For instance, two "in-plane" components of the group-velocity vector can be obtained directly from the well-known equations for vertical transverse isotropy (VTI media). Due to the presence of azimuthal anisotropy, the group-velocity vector acquires an out-of-plane component that also has a concise analytic representation.

To understand the influence of the anisotropic coefficients on the orientation of the group and polarization vectors, we derived linearized weak-anisotropy approximations based on the notation for orthorhombic media introduced by Tsvankin (1996a). The relationship between the group and polarization vectors in weakly orthorhombic models turned out to have the same form as in VTI media (Rommel, 1994), although both vectors deviate from the vertical phase plane. Our analytic results show that polarization and group directions usually are close to each other, and in this sense P-wave polarization in orthorhombic media is almost "isotropic." This conclusion is in agreement with existing numerical results (Tsvankin and Chesnokov, 1990) and was further verified here by modeling for orthorhombic media with substantial anisotropy.

Key words: orthorhombic anisotropy, group velocity, polarization

Introduction

Azimuthally anisotropic models are commonly used to describe fractured reservoirs that contain one or more systems of vertical or dipping cracks. It was realized more than a decade ago that seismic methods can provide valuable information about the orientation and physical properties of crack systems (Crampin, 1985). Although most existing investigations of fractured media are limited to the analysis of split shear waves at near-vertical incidence (e.g., Thomsen, 1988), some recent experimental studies have demonstrated the sensitivity of P-wave ⋆ data to the presence of azimuthal anisotropy (Lynn et al., 1996). Clearly, interpretation of seismic signatures in azimuthally anisotropic media is impossible without an analytic insight into the relation between the anisotropic parameters and seismic wavefields.

The behavior of seismic signatures is relatively well understood for the transversely (azimuthally) isotropic model with a vertical symmetry axis (VTI media). A summary of recent advances in the analytic description of seismic velocities and amplitudes in VTI models can be found in an overview paper by Tsvankin (1996a). Some of the results obtained for vertical transverse isotropy can be directly applied to TI models with a *horizontal* symmetry axis (HTI media) used to describe the simplest fractured formations that contain parallel

⋆ The qualifiers in "quasi-P–wave" and "quasi-S–wave" will be omitted.

vertical penny-shaped cracks in a purely isotropic background. Evidently, such signatures as phase and group velocity, polarization vector, and point-source radiation pattern can be expressed in the same form for any homogeneous TI model, whether the symmetry axis is vertical or horizontal.

Horizontal transverse isotropy, however, is a relatively restrictive model that cannot be used to characterize fractured reservoirs with two vertical crack systems, non-aligned cracks, or an anisotropic background medium. Realistic fractured media may well have orthorhombic (Wild and Crampin, 1991), monoclinic or even the lowest, triclinic symmetry. Orthorhombic models have three mutually orthogonal symmetry planes, in which the Christoffel equation has the same form as in transversely isotropic media (Musgrave, 1970; Schoenberg and Helbig, 1997). Therefore, body-wave velocities and polarizations in the *symmetry planes* of orthorhombic media are described by the same equations as for VTI media. Tsvankin (1996b) used the limited analogy with VTI media to introduce dimensionless anisotropic parameters for orthorhombic media defined similarly to Thomsen's (1986) VTI coefficients ϵ, δ, and γ. This notation provides a simple way of obtaining kinematic signatures and polarizations in the symmetry planes of orthorhombic media by adapting the VTI equations represented through Thomsen parameters. The dimensionless parameters also proved to be well-suited for developing a concise weak-anisotropy approximation for phase velocity *outside* the symmetry planes of orthorhombic models. Another advantage of this notation is the reduction in the number of independent parameters responsible for P-wave *kinematic* signatures from nine (in the conventional notation) to six (Tsvankin, 1996b).

Here, we present an exact expression for the group-velocity vector, as well as concise weak-anisotropy approximations for the P-wave group and polarization angles valid *outside* the symmetry planes of orthorhombic media. By using an auxiliary coordinate system associated with the phase-velocity vector, we obtain the "in-plane" components of the group and polarization vector in the same form as for vertical transverse isotropy. Similar approximations, derived for the "out-of-plane" components, show that the group and polarization vector deviate from the phase vector in the same direction and are typically close to each other. Our equation for the polarization vector has a much simpler form than the more general expression developed by Pšenčík and Gajewski (1997) for arbitrary anisotropic media and specified for orthorhombic models.

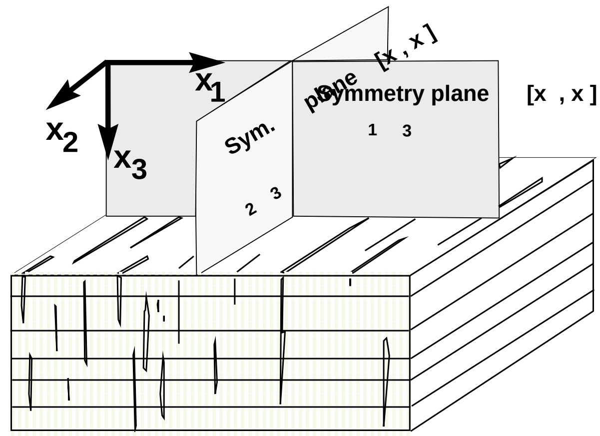

Figure 1. Orthorhombic media have three mutually orthogonal planes of mirror symmetry. One of the reasons for orthorhombic anisotropy is a combination of parallel vertical cracks and vertical transverse isotropy (e.g., due to thin horizontal layering) in the background.

Orthorhombic Symmetry System

Orthorhombic anisotropy describes several typical fractured models including those containing a system of parallel vertical cracks in a VTI background medium or two orthogonal crack systems (Figure 1). Media with orthorhombic symmetry have three mutually orthogonal planes of mirror symmetry; for the model with a single crack system shown in Figure 1, the vertical symmetry planes are defined by the directions parallel and normal to the cracks. The velocities and polarizations in the symmetry planes of orthorhombic media are given by the same equations as for vertical transverse isotropy. (Body-wave amplitudes in the symmetry planes, however, are influenced by the azimuthal velocity variations and require a special treatment.) Tsvankin (1996b) has taken advantage of the limited equivalence between orthorhombic and VTI media to introduce the following dimensionless anisotropic parameters defined similarly to the well-known Thomsen's (1986) coefficients ϵ, δ and γ for vertical transverse isotropy:

- V_{P0} – the vertical velocity of the P-wave:

$$V_{P0} \quad \equiv \quad \sqrt{\frac{c_{33}}{\rho}}\,. \tag{1}$$

- V_{S0} – the vertical velocity of the S-wave polarized in the x_1 direction:

$$V_{S0} \quad \equiv \quad \sqrt{\frac{c_{55}}{\rho}}\,. \tag{2}$$

- $\epsilon^{(2)}$ – the VTI parameter ϵ in the symmetry plane $[x_1, x_3]$; the superscript "2" refers to the x_2-axis that is normal to the $[x_1, x_3]$-plane:

$$\epsilon^{(2)} \equiv \frac{c_{11} - c_{33}}{2c_{33}}. \quad (3)$$

$\epsilon^{(2)}$ is close to the fractional difference between the P-wave velocities in the x_1- and x_3-directions.

- $\delta^{(2)}$ – the VTI parameter δ in the $[x_1, x_3]$-plane (responsible for near-vertical P-wave velocity variations, also influences SV-wave velocity anisotropy):

$$\delta^{(2)} \equiv \frac{(c_{13} + c_{55})^2 - (c_{33} - c_{55})^2}{2c_{33}\,(c_{33} - c_{55})}. \quad (4)$$

- $\gamma^{(2)}$ – the VTI parameter γ in the $[x_1, x_3]$-plane (close to the fractional difference between the SH-wave velocities in the x_1- and x_3-directions):

$$\gamma^{(2)} \equiv \frac{c_{66} - c_{44}}{2c_{44}}. \quad (5)$$

- $\epsilon^{(1)}$ – the VTI parameter ϵ in the $[x_2, x_3]$-plane:

$$\epsilon^{(1)} \equiv \frac{c_{22} - c_{33}}{2c_{33}}. \quad (6)$$

- $\delta^{(1)}$ – the VTI parameter δ in the $[x_2, x_3]$-plane:

$$\delta^{(1)} \equiv \frac{(c_{23} + c_{44})^2 - (c_{33} - c_{44})^2}{2c_{33}\,(c_{33} - c_{44})}. \quad (7)$$

- $\gamma^{(1)}$ – the VTI parameter γ in the $[x_2, x_3]$-plane:

$$\gamma^{(1)} \equiv \frac{c_{66} - c_{55}}{2c_{55}}. \quad (8)$$

- $\delta^{(3)}$ – the VTI parameter δ in the $[x_1, x_2]$-plane (x_1 plays the role of the symmetry axis):

$$\delta^{(3)} \equiv \frac{(c_{12} + c_{66})^2 - (c_{11} - c_{66})^2}{2c_{11}\,(c_{11} - c_{66})}. \quad (9)$$

Alternatively, the δ coefficient in the horizontal plane can be expressed through the average of c_{11} and c_{22}:

$$\overline{\delta}^{(3)} \equiv \frac{(c_{12} + c_{66})^2 - (\overline{c} - c_{66})^2}{2\overline{c}\,(\overline{c} - c_{66})}, \quad (10)$$

$$\text{with } \overline{c} = (c_{11} + c_{22})/2.$$

The parameter $\overline{\delta}^{(3)}$ makes weak-anisotropy approximations symmetric with respect to the vertical symmetry planes but it does not allow to preserve the uniformity in the definition of the δ coefficients.

Therefore, the two vertical velocities and seven dimensionless anisotropic coefficients can replace the nine independent stiffness components of the orthorhombic model. Due to the equivalence with VTI media, the new parameters can be conveniently used to describe seismic velocities and polarizations in the symmetry planes of orthorhombic media using the known VTI equations expressed through Thomsen parameters (Tsvankin, 1996b). Advantages of the new notation, however, are not limited to the symmetry planes. All *kinematic* signatures of P-waves in orthorhombic models are determined by the vertical velocity (a scaling coefficient in homogeneous media) and only *five* anisotropy parameters: $\epsilon^{(1)}$, $\delta^{(1)}$, $\epsilon^{(2)}$, $\delta^{(2)}$, and $\delta^{(3)}$ (Tsvankin, 1996b), as compared to nine stiffnesses in the conventional notation. Also, Tsvankin (1996b) presented a concise weak-anisotropy approximation for P-wave phase velocity in terms of these five relevant coefficients. Below, we use these results to give an analytic description of the P-wave group velocity and polarization vector *outside* the symmetry planes of orthorhombic media.

Group Velocity

General description

The group-velocity vector determines the direction of energy propagation (i.e., seismic rays) and, therefore, is of primary importance in seismic traveltime methods. While the phase-velocity vector (normal to the wavefront) can be obtained directly from the Christoffel equation, evaluation of group velocity involves differentiating the phase-velocity function with respect to the components of the wave vector. For transversely isotropic media, group velocity represents a relatively simple function of phase velocity and phase angle with the symmetry axis (Berryman, 1979). However, expressions for group velocity conventionally used for orthorhombic and lower symmetries are much more complicated and include the components of the polarization vector in addition to phase velocity (Musgrave, 1970). Here, the group-velocity vector in arbitrary anisotropic media is expressed directly through phase velocity and its derivatives with respect to the vertical (θ) and azimuthal (ϕ) phase angles.

Let us introduce an auxiliary Cartesian coordinate system $[x, y, z]$ with the horizontal axes rotated by the angle ϕ around the x_3-axis of the original coordinate system $[x_1, x_2, x_3]$, so that the phase-velocity vector lies in the $[x, z]$-coordinate plane (Figure 2). The vertical (z)-axis coincides with the axis x_3 of the old system, while the y-axis of the new coordinate system is normal to the plane $\phi = const$ and points counterclockwise from the original x_1-axis.

The group velocity for any anisotropic medium can be represented as follows (e.g., Berryman, 1979):

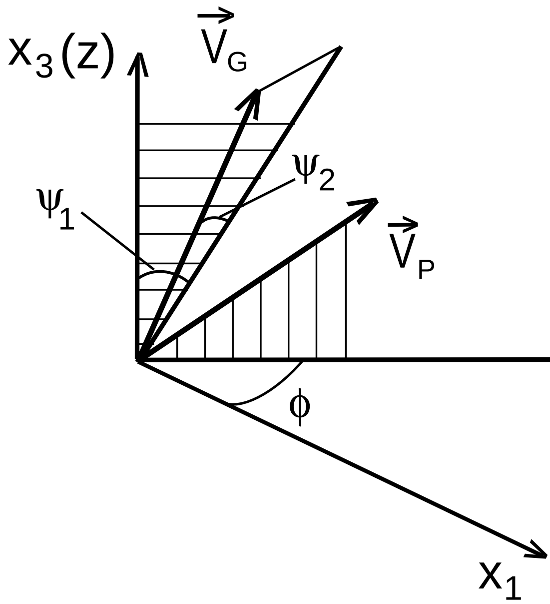

Figure 2. The vectors of group ($\vec{V}_G$) and phase ($\vec{V}_P$) velocity in azimuthally anisotropic media (hereafter, we omit the subscript in $\vec{V}_P$). The phase-velocity vector lies in the $[x, z]$-plane of an auxiliary coordinate system $[x, y, z]$ and makes the angle θ with vertical. To describe the orientation of the group-velocity vector, it is convenient to introduce the "in-plane" (ψ_1) and "out-of-plane" (ψ_2) group angles.

$$\vec{V}_G = \frac{\partial(kV)}{\partial k_x}\vec{x} + \frac{\partial(kV)}{\partial k_y}\vec{y} + \frac{\partial(kV)}{\partial k_z}\vec{z}, \quad (11)$$

where V is the phase velocity, $\vec{k}$ is the wave vector, which is parallel to the phase-velocity vector and has the magnitude $k = \omega/V$ (ω is the angular frequency), and $\vec{x}$, $\vec{y}$, and $\vec{z}$ are the unit coordinate vectors. Differentiation with respect to each component of the wave vector has to be performed with the other two components held constant. Since both group-velocity components in the $[x, z]$-plane (V_{Gx} and V_{Gz}) are calculated for $k_y = 0$, they are independent from out-of-plane phase-velocity variations. As further confirmed by the derivation in Appendix A, V_{Gx} and V_{Gz} represent the same functions of phase velocity as in any symmetry plane in arbitrary anisotropic media (Tsvankin, 1995):

$$V_{Gx} = \frac{\partial(kV)}{\partial k_x} \quad (12)$$

$$= V\sin\theta + \left.\frac{\partial V}{\partial\theta}\right|_{\phi=const}\cos\theta\,. \quad (13)$$

$$V_{Gz} = \frac{\partial(kV)}{\partial k_z} \quad (14)$$

$$= V\cos\theta - \left.\frac{\partial V}{\partial\theta}\right|_{\phi=const}\sin\theta\,. \quad (15)$$

Note that equations (13) and (15) are identical to the well-known expressions for the group-velocity vector in VTI media (Berryman, 1979; Thomsen, 1986).

The transverse component of the group-velocity vector V_{Gy} depends solely on *azimuthal* phase-velocity variations and exists only outside the symmetry planes. As shown in Appendix A, V_{Gy} is determined by the first derivative of phase velocity with respect to the azimuthal phase angle ϕ:

$$V_{Gy} = \frac{\partial(kV)}{\partial k_y} \quad (16)$$

$$= \frac{1}{\sin\theta}\left.\frac{\partial V}{\partial\phi}\right|_{\theta=const}. \quad (17)$$

Equations (13)-(17) conveniently express the group-velocity vector in arbitrary anisotropic media through 3-D variations in the phase-velocity function. Next, we use this representation to gain insight into the behavior of group velocity in orthorhombic media.

Weak-anisotropy approximation for group velocity

Due to the complexity of the phase-velocity function, equations (13)-(17) do not provide an easy insight into the behavior of group velocity in orthorhombic media. Here, we transform the expression for P-wave group-velocity in orthorhombic media under the assumption of weak anisotropy. The weak-anisotropy approximation for P-wave phase velocity, linearized in the dimensionless anisotropic coefficients, was given by Tsvankin (1996b):

$$V_P(\theta,\phi) = V_{P0} \quad (18)$$

$$\left[1 + \delta(\phi)\sin^2\theta\cos^2\theta + \epsilon(\phi)\sin^4\theta\right]\,; \quad (19)$$

$$\delta(\phi) = \delta^{(2)}\cos^2\phi + \delta^{(1)}\sin^2\phi\,, \quad (20)$$

$$\epsilon(\phi) = \epsilon^{(2)}\cos^4\phi + \epsilon^{(1)}\sin^4\phi + (2\epsilon^{(2)} + \delta^{(3)})\sin^2\phi\cos^2\phi\,, \quad (21)$$

$$= \epsilon^{(2)}\cos^2\phi + \epsilon^{(1)}\sin^2\phi + \bar{\delta}^{(3)}\sin^2\phi\cos^2\phi\,. \quad (22)$$

Equation (19) has exactly the *same* form as the Thomsen's (1986) weak-anisotropy approximation for vertical transverse isotropy, but the parameters ϵ and δ in orthorhombic media become azimuthally dependent. In both vertical symmetry planes, equation (19) reduces to the VTI expression that includes the appropriate pair of the parameters ϵ and δ ($\epsilon^{(2)}$ and $\delta^{(2)}$ in the $[x_1, x_3]$-plane and $\epsilon^{(1)}$ and $\delta^{(1)}$ in the $[x_2, x_3]$-plane). Substituting equation (19) into the group-velocity expressions (13)-(17), we can obtain an explicit approximation for group velocity in terms of the anisotropy parameters.

The magnitude of the group-velocity vector can be found from equations (13)-(17):

$$V_G = V\sqrt{1+\left(\frac{1}{V}\frac{\partial V}{\partial \theta}\right)^2+\left(\frac{1}{V\sin\theta}\frac{\partial V}{\partial \phi}\right)^2}. \quad (23)$$

Clearly, all anisotropic terms in equation (23) are quadratic in the anisotropy parameters. This means that in the linearized weak-anisotropy approximation the absolute values of phase and group velocity are identical, as in VTI media. However, the difference between the *directions* of the group- and phase-velocity vectors cannot be ignored even in the weak-anisotropy approximation. It is convenient to describe the orientation of the group-velocity vector in terms of the "in-plane" and "out-of-plane" group angles. The in-plane group angle ψ_1 (Figure 2) is defined as

$$\tan\psi_1 = \frac{V_{Gx}}{V_{Gz}}. \quad (24)$$

In the weak-anisotropy limit, P-wave phase velocity in each vertical $[x,z]$-plane is given by equation (19) that has the same form as for vertical transverse isotropy. Since both V_{Gx} and V_{Gz} represent the same functions of the in-plane phase velocity as in VTI media, we can obtain the weak-anisotropy approximation for the angle ψ_1 by substituting $\epsilon(\phi)$ and $\delta(\phi)$ into the known VTI equation (Thomsen, 1986):

$$\tan\psi_1 = (1+2p)\tan\theta, \quad (25)$$

where

$$p = \delta(\phi)\cos 2\theta + 2\epsilon(\phi)\sin^2\theta \quad (26)$$

$$= \delta(\phi) + 2\left[\epsilon(\phi)-\delta(\phi)\right]\sin^2\theta \quad (27)$$

In the $[x_1,x_3]$-symmetry plane, $\delta(\phi)=\delta^{(2)}$, $\epsilon(\phi)=\epsilon^{(2)}$, while in the $[x_2,x_3]$-plane $\delta(\phi)=\delta^{(1)}$, $\epsilon(\phi)=\epsilon^{(1)}$.

Expanding $\tan^{-1}\psi_1$ in a Taylor series in the anisotropic factor p and keeping just the linear term allows us to rewrite equation (25) as

$$\psi_1 = \theta + p\sin 2\theta. \quad (28)$$

The second group angle, ψ_2, describes the deviation of the group-velocity vector from the vertical $[x,z]$-plane $\phi = const$ (Figure 2):

$$\tan\psi_2 = \frac{V_{Gy}}{\sqrt{V_{Gx}^2+V_{Gz}^2}}. \quad (29)$$

In the linearized weak-anisotropy approximation,

$$\tan\psi_2 = \frac{V_{Gy}}{V_{P0}}. \quad (30)$$

Substituting the phase-velocity expression (19) into equation (17) for V_{Gy} and dividing by V_{P0} [equation (30)], we find

$$\tan\psi_2 = q\sin 2\phi\sin\theta, \quad (31)$$

where

$$q = \left[\delta^{(1)}-\delta^{(2)}\right]\cos^2\theta + \left[2\left(\epsilon^{(1)}-\epsilon^{(2)}\right)\sin^2\phi + \delta^{(3)}\cos 2\phi\right]\sin^2\theta, \quad (32)$$

$$= \left[\delta^{(1)}-\delta^{(2)}\right]\cos^2\theta + \left[\epsilon^{(1)}-\epsilon^{(2)}+\overline{\delta}^{(3)}\cos 2\phi\right]\sin^2\theta. \quad (33)$$

Since $\tan\psi_2$ in equation (31) is linear in the anisotropic coefficients, it can be replaced with the angle ψ_2 itself.

If the medium is azimuthally isotropic (VTI), then $\delta^{(1)}=\delta^{(2)}$, $\epsilon^{(1)}=\epsilon^{(2)}$, $\delta^{(3)}=0$, and the angle ψ_2 is identically zero for all phase directions. Also, $\psi_2=0$ in both vertical symmetry planes of orthorhombic media corresponding to $\phi=0$ and $\phi=90°$. Evidently, the group-velocity vector deviates from the vertical plane that contains the phase vector only outside the symmetry planes of azimuthally anisotropic media.

Equations (25) and (31) can be efficiently used in seismic tomography to relate the traveltimes in orthorhombic media to the anisotropy parameters. For instance, the equations for the group angles can be combined with the weak-anisotropy approximation for the magnitude of the group-velocity vector [given just by the phase-velocity equation (19)] to calculate explicit analytic expressions for the Frechét derivatives of the traveltime needed in the singular value decomposition of the tomographic inverse problem.

Weak-anisotropy approximation for polarization

Plane-wave polarization vector in isotropic media is either parallel (for P-waves) or orthorgonal (for S-waves) to the phase (slowness) direction. In the presence of anisotropy, polarization is governed not only by the slowness vector, but also by the elastic constants of the medium. For a given slowness direction, the polarization vectors of the three plane waves define the eigenvectors of the symmetric Christoffel matrix and, therefore, are always mutually orthogonal. (Note that this property is no longer valid for *non-planar* wavefronts because the three body waves recorded at any receiver location correspond to *different* slowness directions.) The polarization vector of the plane P-wave, however, is not necessarily aligned with either phase (slowness) or group (ray) vector; this explains the term "quasi-P"-wave.

The weak-anisotropy approximation for the P-wave polarization vector in orthorhombic media is derived in

Appendix B. Using the auxiliary coordinate system introduced above for the group-velocity vector, we find the following expressions for the components of the polarization vector:

$$P_x = \left(1 + 2B\,p\cos^2\theta\right)\sin\theta, \quad (34)$$

$$P_y = B\,q\sin 2\phi\sin\theta, \quad (35)$$

$$P_z = \left(1 - 2B\,p\sin^2\theta\right)\cos\theta, \quad (36)$$

where p and q were defined in equations (26) and (33), and B is the coefficient suggested by Tsvankin (1996a):

$$B \equiv \frac{1}{2(1 - V_{S0}^2/V_{P0}^2)}\,. \quad (37)$$

Alternatively, the polarization vector can also be expressed in polar coordinates, with the radial and vertical components (the azimuthal angle is not shown here) given by

$$P_r = \left(1 + 2B\,p\cos^2\theta\right)\sin\theta\,, \quad (38)$$

$$P_z = \left(1 - 2B\,p\sin^2\theta\right)\cos\theta\,. \quad (39)$$

The formal structure of P_r and P_z is the same in VTI media (Rommel, 1994), but the factor p for orthorhombic anisotropy is azimuthally dependent and should be recalculated for every azimuthal phase angle ϕ.

The influence of anisotropy on P-wave polarization is absorbed by the factors p and q, which depend on the anisotropic coefficients and phase angles. For instance, p is mostly controlled by the following parameter combinations: $\delta(\phi)$ near vertical, $2\epsilon(\phi)$ for $\theta \approx 45^\circ$ and $2\epsilon(\phi) - \delta(\phi)$ near horizontal (for a detailed discussion, see the section on numerical examples). The anisotropic terms in the expressions for P_x and P_z are also scaled by a constant factor B that depends on the ratio of the vertical velocities.

Note that the transverse component P_y is proportional to the linear anisotropic term q and, therefore, vanishes in isotropic media (and in VTI media as well). In orthorhombic models the P-wave polarization vector cannot have the out-of-plane component P_y in both vertical symmetry planes ($\phi = 0^\circ$ and $\phi = 90^\circ$). Likewise, in the horizontal symmetry plane ($\theta = 90^\circ$) the vertical polarization P_z goes to zero.

Clearly, the orientation of the polarization and phase vectors is generally different, with the deviation dependent on the phase vector. In some directions, however, the polarization and phase vectors may be parallel to each other. These pure-mode directions are sometimes called "longitudinal" (Schoenberg and Helbig, 1997) and include, for example, the intersections of the symmetry planes of orthorhombic media. Within the vertical symmetry planes, P-wave polarization in weakly orthorhombic media becomes purely longitudinal if the factor p (26) vanishes, and the equation

$$\tan\theta = \sqrt{\frac{-\delta(\phi)}{2\epsilon(\phi) - \delta(\phi)}} \quad (40)$$

has a real solution (Brugger, 1964). Outside the symmetry planes a longitudinal direction occurs if not only p, but also the q goes to zero. Using equation (33) and setting $q = 0$, we find the following relationship between the phase angles:

$$\tan\phi = \sqrt{\frac{[\delta^{(1)} - \delta^{(2)}] + \left[\overline{\delta}^{(3)} + (\epsilon^{(1)} - \epsilon^{(2)})\right]\tan^2\theta}{-[\delta^{(1)} - \delta^{(2)}] + \left[\overline{\delta}^{(3)} - (\epsilon^{(1)} - \epsilon^{(2)})\right]\tan^2\theta}}\,. \quad (41)$$

If conditions (40) and (41) are simultaneously satisfied ($p = q = 0$), the P-wave polarization vector is parallel to the phase-velocity vector (to the first order in the anisotropy coefficients). From equations (25) and (31) it is clear that for $p = q = 0$ the group- and phase-velocity vectors coincide with each other.

Equations (34)–(36) allow us to determine the "in-plane" and "out-of-plane" polarization angles defined in the same way as the corresponding group angles. In the weak-anisotropy approximation, the tangent of the in-plane polarization angle ν_1 with vertical is given by

$$\tan\nu_1 \equiv \frac{P_x}{P_z} \quad (42)$$

$$= (1 + 2B\,p)\tan\theta\,, \quad (43)$$

or, expanding $\tan^{-1}\nu_1$,

$$\nu_1 = \theta + B\,p\sin 2\theta\,. \quad (44)$$

The "in-plane" polarization angle for orthorhombic anisotropy has the same form as in VTI media (Rommel, 1994; Tsvankin, 1996a), but the coefficients ϵ and δ in the expression for p [equation (26)] are azimuthally dependent.

Comparison with the results of Rommel (1994) and Tsvankin (1996a) shows that both group and polarization angles in the vertical plane that contains the phase vector are the same functions of $\epsilon(\phi)$, $\delta(\phi)$ and B as in VTI media with the coefficients ϵ and δ and the ratio of the vertical velocities V_{S0}/V_{P0}. Therefore, conclusions about the relative positions of the phase, group, and polarization vector obtained for vertical transverse isotropy remain fully valid for the in-plane components of these vectors in weakly orthorhombic media. Combining equations (28) and (44), we find the VTI relationship

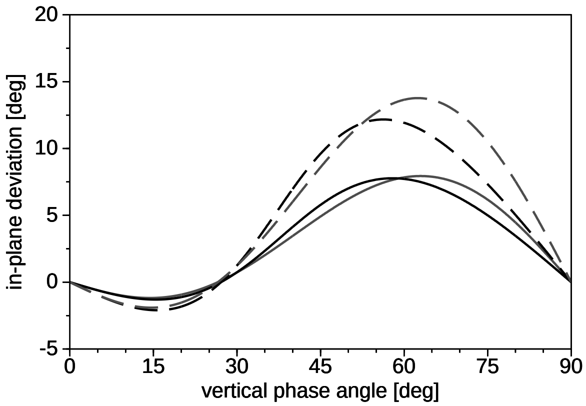

Figure 3. In-plane deviation of the polarization vector (solid black line) and the group-velocity vector (dashed black) from the phase direction (i.e., $\nu_1 - \theta$ and $\psi_1 - \theta$, respectively) in the $[x_1, x_3]$-plane ($\phi = 0$). The weak-anisotropy approximations are shown by the solid gray line (polarization) and dashed gray line (group velocity). The medium parameters are $V_{P0} =$ 3000 m/s, $V_{S0} = 1200$ m/s, $\epsilon^{(1)} = 0.25$, $\epsilon^{(2)} = 0.15$, $\delta^{(1)} =$ 0.05, $\delta^{(2)} = -0.1$, and $\delta^{(3)} = 0.15$. In the $[x_1, x_3]$-plane $\epsilon(\phi) =$ $\epsilon^{(2)}$ and $\delta(\phi) = \delta^{(2)}$. A pure mode exists at a vertical phase angle $\theta \approx 27°$ (longitudinal direction).

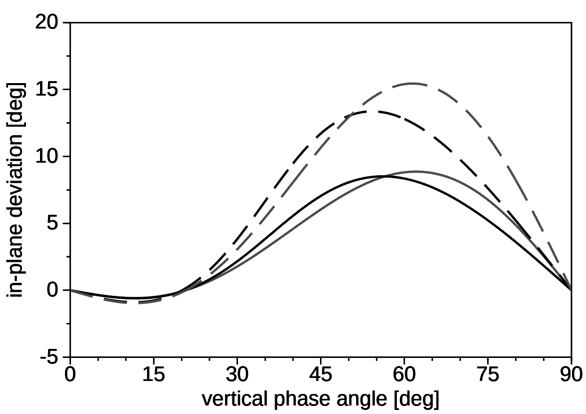

Figure 4. Same as Figure 3, but for the vertical plane $\phi =$ $30°$. The effective anisotropy coefficients are $\epsilon(30°) = 0.19$ and $\delta(30°) = -0.06$. A longitudinal direction occurs at $\theta \approx 20°$ which is smaller than in Figure 3.

$$\nu_1 - \theta \quad = \quad B\,(\psi_1 - \theta)\,. \tag{45}$$

From equation (45) it follows that the in-plane polarization vector lies between the group and phase directions, but is closer to the group vector. Indeed, for plausible values of the V_{S0}/V_{P0} ratio, B belongs to the interval $0.5 < B < 1$ (Tsvankin, 1996a).

Outside of the vertical symmetry planes, the polarization vector deviates from the vertical plane that contains the phase vector due to the non-zero polarization

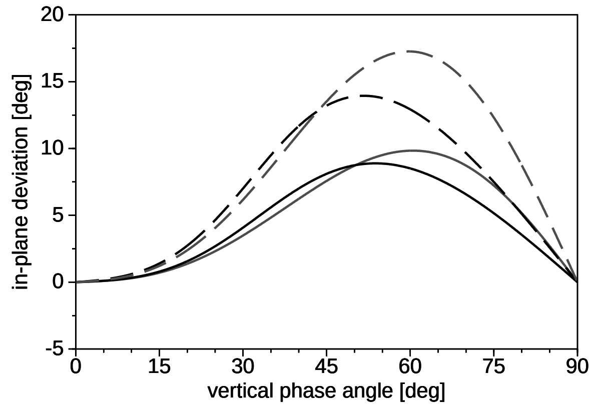

Figure 5. Same as Figure 3, but for the vertical plane $\phi =$ $60°$. The effective anisotropy coefficients are $\epsilon(60°) = 0.25$ and $\delta(60°) = 0.01$. Here, the longitudinal direction disappears.

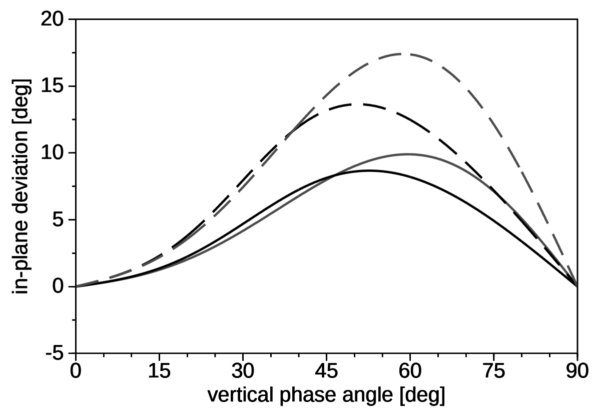

Figure 6. Same as Figure 3, but for the $[x_2, x_3]$ symmetry plane ($\phi = 90°$). The effective anisotropy coefficients are $\epsilon^{(1)} = 0.25$ and $\delta^{(1)} = 0.05$. There is no longitudinal direction at intermediate angles.

component P_y. The "out-of-plane" polarization angle is defined as

$$\tan\nu_2 \quad \equiv \quad \frac{P_y}{\sqrt{P_x^2 + P_z^2}} \tag{46}$$

$$\quad = \quad B\,q\,\sin 2\phi \sin\theta\,, \tag{47}$$

or

$$\nu_2 \quad = \quad B\,q\,\sin 2\phi \sin\theta \tag{48}$$

Using equations (31) and (48), we find the relation between the out-of-plane ray and polarization angles:

$$\nu_2 \quad = \quad B\,\psi_2\,. \tag{49}$$

Therefore, the polarization and group vector deviate from the vertical phase plane in the same direction, and the out-of-plane angles are scaled by the same factor B as the deviations from the phase vector in the vertical plane.

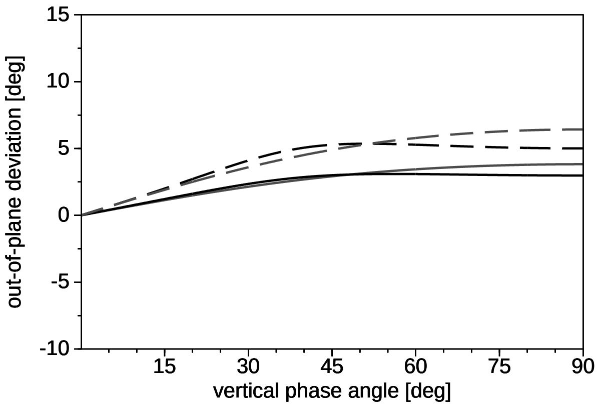

Figure 7. Out-of-plane deviation of the polarization vector (solid black line) and the group-velocity vector (dashed black) (i.e., ν_2 and ψ_2, respectively) in the vertical plane $\phi = 30°$). The weak-anisotropy approximations are shown by the solid gray line (polarization) and dashed gray line (group velocity). The medium is the same as in Figure 3; the effective anisotropy coefficients are $\epsilon(30°) = 0.19$ and $\delta(30°) = -0.06$.

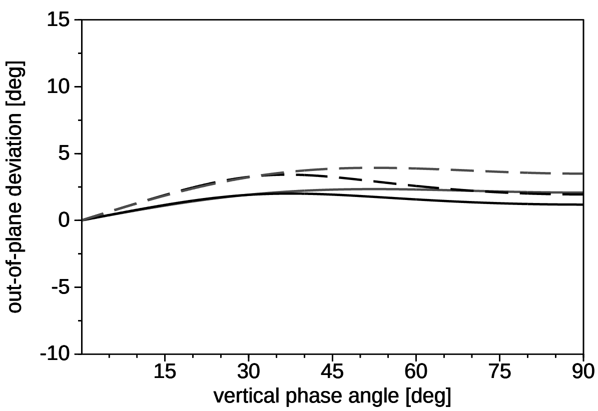

Figure 8. Same as Figure 7, but for the vertical plane $\phi = 60°$. The effective anisotropy coefficients are $\epsilon(60°) = 0.25$ and $\delta(60°) = 0.01$.

We conclude that the P-wave polarization vector in orthorhombic media always deviates from the phase direction towards the group vector, and the angle between the polarization and group vectors is relatively small. This analytic result is in good agreement with the numerical analysis of P-wave polarizations in orthorhombic media given in Tsvankin and Chesnokov (1990).

Numerical study of group velocity and polarization

Here, we present numerical examples illustrating the accuracy of the analytic approximations and the behavior of the group and polarization vectors in orthorhombic media. The whole suite of plots shows that the weak-anisotropy approximations for the group and polarization angles are sufficiently close to the exact solutions, even if the magnitude of angular velocity variations is sunbstantial. As could be expected from the design of the medium parameters, the highest accuracy is achieved near the vertical direction. Still, despite increasing deviations with the vertical phase angle, the maximum error of the weak-anisotropy approximations is limited by several degrees. It should be emphasized that the model used in this section cannot be considered weakly anisotropic since the largest ϵ coefficient reaches 0.25.

Figures 3–6 show the in-plane deviation of the polarization and group vectors from the phase vector ($\nu_1 - \theta$ and $\psi_1 - \theta$, respectively). The behavior of both differences is controlled by the factor p [equation (26)] dependent on the azimuthally varying parameters $\delta(\phi)$ and $\epsilon(\phi)$. For a fixed azimuthal phase angle ϕ, the polarization and group angles are governed by $\delta(\phi)$ (near vertical), $2\epsilon(\phi)$ (near $\theta = 45°$), and $2\epsilon(\phi) - \delta(\phi)$ (near horizontal). In the vertical direction ($\theta = 0°$) and the horizontal plane ($\theta = 90°$) the in-plane components of the group and polarization vector are aligned with the phase vector, so $\nu_1 = \psi_1 = \theta$.

For intermediate phase angles, the orientations of the vectors are different, with the deviations determined by the sign and magnitude of the above-listed terms. If both $\delta(\phi)$ and $2\epsilon(\phi) - \delta(\phi)$ are positive, the polarization and ray angle are larger than the phase angle, which implies that the polarization and group vector are tilted from the phase vector towards horizontal (Figures 5 and 6). If $\delta(\phi)$ and $2\epsilon(\phi) - \delta(\phi)$ are negative, then the polarization and group vectors deviate from the phase vector towards vertical. Finally, each deviation has two extrema (a maximum and a minimum) if these two terms have opposite signs (Figures 3 and 4). At some angle between the extrema, corresponding to $p = 0$ [equation (26)], the in-plane components of the polarization and group vectors are parallel to the phase vector. If p vanishes for a certain angle θ in one of the symmetry planes (as in Figure 3), this angle corresponds to a longitudinal direction, where the polarization vector as a whole is parallel to the phase vector, and the phase and group vectors coincide with each other.

Since in orthorhombic media the term p is azimuthally dependent, a point where $\nu_1 = \psi_1 = \theta$ might disappear and reappear with changing azimuth. The properties in each vertical plane of a weakly orthorhombic medium are identical to those of the effective VTI medium with the parameters $\epsilon(\phi)$ and $\delta(\phi)$, which determine the character of azimuthal variations in polarization and group velocity. It should be mentioned that $\epsilon(\phi)$ [equation (21)] does not change in a fully monotonous way between the vertical symmetry planes due to the influence of the coefficient $\overline{\delta}^{(3)}$ (especially near horizontal).

The deviations of both the polarization and group-velocity vectors from the vertical phase plane are relatively small and, in agreement with the weak-anisotropy approximation, have the same sign (Figures 7 and 8). The out-of-plane deviation is governed by the term q [equation (33)] and depends on the differences $\delta^{(1)} - \delta^{(2)}$, $\epsilon^{(1)} - \epsilon^{(2)}$), as well as on the coefficient $\delta^{(3)}$. While $\delta^{(1)}$ and $\delta^{(2)}$ are responsible for q near vertical, $\epsilon^{(1)}$, $\epsilon^{(2)}$, and $\overline{\delta}^{(3)}$ determine q for large angles θ.

Another parameter governing both the in-plane and out-of-plane components of the polarization vector is a combination of the vertical velocities that we denoted as B [equation (37)]. With increasing ratio V_{P0}/V_{S0}, the value of B and the anisotropic term in expressions (44) and (48) become smaller, and the polarization vector moves closer to the phase vector and further away from the group vector. Therefore, the deviation of the polarization vector from the phase vector is more pronounced in hard rocks than in unconsolidated sediments. The parameter B has no influence the form of the deviation (e. g., on the position of the zeros, maxima and minima); rather, it scales the deviation as a whole.

Discussion and Conclusions

We have presented analytic approximations for group velocity and polarization vector of P-waves in media with orthorhombic symmetry. Group velocity for arbitrary anisotropic media can be conveniently expressed through phase velocity and its derivatives with respect to the phase angles. We show that the group-velocity components in the vertical plane that contains the phase-velocity vector can be evaluated by analogy with transversely isotropic models with a vertical symmetry axis (VTI media). The transverse component of the group-velocity vector, normal to this vertical phase plane, depends on the derivative of phase velocity with respect to the azimuthal phase angle and appears only outside the symmetry planes of azimuthally anisotropic media.

The exact group-velocity expression was transformed into a much simpler weak-anisotropy approximation for orthorhombic media using the phase-velocity equations given by Tsvankin (1996b). In the limit of weak anisotropy, the group and phase velocity are equal to each other, but the *angle* between the group- and phase-velocity vector is linear in the anisotropic coefficients. The analytic approximation for group angles can be used in traveltime tomography to perform singular-value decomposition or to build fast ray-tracing algorithms in weakly anisotropic orthorhombic media.

Using the same auxiliary coordinate system associated with the phase-velocity vector, we also obtained a concise weak-anisotropy approximation for the polarization angles. The relationship between the polarization, group and phase angles in the vertical phase plane has exactly the same form as in VTI media, but the anisotropic coefficients ϵ and δ are azimuthally dependent. It is interesting that the out-of-plane (azimuthal) deviations of the polarization and group vector are scaled by the same factors as their in-plane deviations from the phase vector. As a result, the polarization and group vectors diverge from the phase vector in the same direction and are usually close to each other. This analytic conclusion is supported by the numerical results of Tsvankin and Chesnokov (1990) who found the P-wave polarization vector in orthorhombic media to be surprisingly well-aligned with the group-velocity vector. In this sense, P-wave polarization in orthorhombic media typically is close to "isotropic" and will not create, for example, distortions in the radiation patterns (Tsvankin and Chesnokov, 1990). The polarization vector may still deviate substantially from the phase vector, and this phenomenon can be used to study anisotropy in VSP experiments.

The high accuracy of the analytic approximations was confirmed by calculating the exact group and polarization angles within and outside the symmetry planes in orthorhombic media with pronounced velocity variations. The modeling reveals the magnitude of the azimuthal variation in the group and polarization angles, verifies the position of the "longitudinal" directions, and illustrates the influence of various parameter combinations.

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APPENDIX A: Group velocity in arbitrarily anisotropic media

Here, the group-velocity vector in arbitrary anisotropic media is expressed as a function of phase velocity $V(\theta, \phi)$ represented through the spherical polar (θ) and azimuthal (ϕ) phase angles. For purposes of this derivation, it is convenient to use an auxiliary Cartesian coordinate system $[x, y, z]$ rotated by the angle ϕ around the x_3-axis of the original system $[x_1, x_2, x_3]$ (Figure 2). We start with the general expression for group velocity introduced in the main text (e.g., Berryman, 1979):

$$\vec{V}_G = \frac{\partial(kV)}{\partial k_x}\vec{x} + \frac{\partial(kV)}{\partial k_y}\vec{y} + \frac{\partial(kV)}{\partial k_z}\vec{z}, \tag{A1}$$

where V is the phase velocity and $\vec{k}$ is the wave vector. First, we obtain the two components of the group-velocity vector confined to the vertical plane $y = 0$ (V_{Gx} and V_{Gz}); in the following, we will call them the "in-plane" components.

From equation (A1) we have

$$V_{Gx} = \frac{\partial(kV)}{\partial k_x} \tag{A2}$$

$$= V\,\frac{\partial k}{\partial k_x} + k\,\frac{\partial V}{\partial k_x}\,. \tag{A3}$$

To evaluate V_{Gx}, both V and k_x can be represented as functions of the phase angle θ with the vertical (z)-axis. Keeping the components k_y and k_z constant and substituting $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$, we get

$$\left.\frac{\partial k}{\partial k_x}\right|_{k_y,k_z=const} = \frac{k_x}{k} = \sin\theta\,. \tag{A4}$$

The second term in the right-hand side of equation (A3) takes the form

$$k\left.\frac{\partial V}{\partial k_x}\right|_{k_y,k_z=const} = k\frac{\left.\frac{\partial V}{\partial \theta}\right|_{\phi=const}}{\left.\frac{\partial k_x}{\partial \theta}\right|_{k_z=const}} \tag{A5}$$

Since $k_x = k_z \tan\theta$,

$$\left.\frac{\partial k_x}{\partial \theta}\right|_{k_z=const} = \frac{k_z}{\cos^2\theta}\,. \tag{A6}$$

Taking into account that $k_z/k = \cos\theta$, we obtain

$$k\,\frac{\partial V}{\partial k_x} = \left.\frac{\partial V}{\partial \theta}\right|_{\phi=const}\cos\theta\,. \tag{A7}$$

Substituting equations (A4) and (A7) into equation (A3) yields

$$V_{Gx} = V\sin\theta + \left.\frac{\partial V}{\partial \theta}\right|_{\phi=const}\cos\theta\,. \tag{A8}$$

Similarly, the second in-plane component of the group-velocity vector is given by

$$V_{Gz} = V\cos\theta - \left.\frac{\partial V}{\partial \theta}\right|_{\phi=const}\sin\theta\,. \tag{A9}$$

Equations (A8) and (A9) for V_{Gx} and V_{Gz} are identical to the expressions for the vertical and horizontal components of the group-velocity vector in transversely isotropic media in any symmetry plane in anisotropic media (Tsvankin, 1995).

For the transverse component of the group-velocity vector we have

$$V_{Gy} = \left.\frac{\partial(kV)}{\partial k_y}\right|_{k_x,k_z=const} \tag{A10}$$

$$= V\frac{k_y}{k} + k\frac{\partial V}{\partial k_y}\,. \tag{A11}$$

The first term in the right-hand side of equation (A11) vanishes because the transverse component of the wave vector (k_y) in the xz-plane is zero. Both the phase velocity V and k_y in the second term are convenient to express through the azimuthal phase angle $\bar{\phi}$ defined with respect to the plane $y = 0$:

$$\frac{\partial V}{\partial k_y} = \frac{\frac{\partial V}{\partial \bar{\phi}}}{\frac{\partial k_y}{\partial \bar{\phi}}}\,. \tag{A12}$$

Expressing k_y through the second horizontal component k_x, we have

$$k_y = k_x\,\tan\bar{\phi}\,, \tag{A13}$$

Therefore,

$$\left.\frac{\partial k_y}{\partial \bar{\phi}}\right|_{k_x,k_z=const} = \frac{k_x}{\cos^2\bar{\phi}}\,. \tag{A14}$$

Since the derivative should be taken at $\bar{\phi} = 0$,

$$\left.\frac{\partial k_y}{\partial \bar{\phi}}\right|_{k_x,k_z=const} = k_x \,. \quad \text{(A15)}$$

When computing the derivative of phase velocity with respect to $\bar{\phi}$, we have to take into account that both the polar angle θ and the azimuthal angle ϕ change with k_y and $\bar{\phi}$. Therefore, we need to evaluate

$$\left.\frac{\partial V}{\partial \bar{\phi}}\right|_{k_x,k_z=const} = \left.\frac{\partial V}{\partial \theta}\right|_{\phi=const} \left.\frac{\partial \theta}{\partial \bar{\phi}}\right|_{k_x,k_z=const} + \left.\frac{\partial V}{\partial \phi}\right|_{\theta=const} \left.\frac{\partial \phi}{\partial \bar{\phi}}\right|_{k_x,k_z=const} . \quad \text{(A16)}$$

From simple geometry,

$$\tan\theta = \frac{k_x}{k_z}\frac{1}{\cos\bar{\phi}}\,, \quad \text{(A17)}$$

and

$$\left.\frac{\partial \theta}{\partial \bar{\phi}}\right|_{k_x,k_z=const} = \frac{k_x}{k_z}\frac{\cos^2\theta}{\cos^2\bar{\phi}}\sin\bar{\phi}\,. \quad \text{(A18)}$$

Substituting $\bar{\phi} = 0$, we find that

$$\left.\frac{\partial \theta}{\partial \bar{\phi}}\right|_{k_x,k_z=const} = 0\,. \quad \text{(A19)}$$

If the y-axis points in the positive ϕ-direction (counterclockwise from the x_1-axis),

$$\frac{\partial \phi}{\partial \bar{\phi}} = 1\,, \quad \text{(A20)}$$

and equation (A16) reduces to

$$\left.\frac{\partial V}{\partial \bar{\phi}}\right|_{k_x,k_z=const} = \left.\frac{\partial V}{\partial \phi}\right|_{\theta=const} . \quad \text{(A21)}$$

Using equations (A15) and (A21), we can rewrite equation (A12) as

$$\frac{\partial V}{\partial k_y} = \frac{1}{k_x}\left.\frac{\partial V}{\partial \phi}\right|_{\theta=const} . \quad \text{(A22)}$$

Finally, for the transverse component of the group vector [equation (A11)] we have

$$V_{Gy} = \frac{k}{k_x}\left.\frac{\partial V}{\partial \phi}\right|_{\theta=const} \quad \text{(A23)}$$

$$= \frac{1}{\sin\theta}\left.\frac{\partial V}{\partial \phi}\right|_{\theta=const} . \quad \text{(A24)}$$

The y-axis in equation (A24) points in the direction of increasing ϕ, i.e., counterclockwise from the x_1-axis of the original coordinate system.

APPENDIX B: Polarization

The phase velocity and polarization of plane waves in an unbounded anisotropic medium is described by the Christoffel equation. The squared phase velocities v represent the eigenvalues of the symmetric Christoffel matrix $\mathbf{T}$, while the polarizations $\mathbf{P}$ are the corresponding eigenvectors:

$$\left(\mathbf{T} - v^2\mathbf{I}\right)\mathbf{P} = \mathbf{0}, \quad \text{(B.1)}$$

where $\mathbf{I}$ the unit matrix. The elements of the Christoffel matrix $T_{ij} = a_{ikjl}n_k n_l$ depend on the density-normalized stiffness tensor $\mathbf{a}$ and the unit vector in the phase (slowness) direction $\mathbf{n}$. For an orthorhombic medium, the Christoffel matrix has the following form in Voigt's notation:

$$T_{11} = a_{11}{n_1}^2 + a_{66}{n_2}^2 + a_{55}{n_3}^2, \quad \text{(B.2)}$$
$$T_{12} = (a_{12} + a_{66})\,n_1 n_2, \quad \text{(B.3)}$$
$$T_{13} = (a_{13} + a_{55})\,n_1 n_3, \quad \text{(B.4)}$$
$$T_{22} = a_{66}{n_1}^2 + a_{22}{n_2}^2 + a_{44}{n_3}^2, \quad \text{(B.5)}$$
$$T_{23} = (a_{23} + a_{44})\,n_2 n_3, \quad \text{(B.6)}$$
$$T_{33} = a_{55}{n_1}^2 + a_{44}{n_2}^2 + a_{33}{n_3}^2. \quad \text{(B.7)}$$

The phase velocities v are computed by solving the characteristic determinant $\left|\mathbf{T} - v^2\mathbf{I}\right|$ of the Christoffel equation (B.1). Provided the three phase velocities (eigenvalues) v are distinct, the polarization $\mathbf{P}$ vector of any mode is given by (Rokhlin et al., 1986):

$$P_i P_j = \frac{W_{ij}}{W_{mm}\,,} \quad \text{(B.8)}$$

where

$$W_{ij} = \tfrac{1}{2}\epsilon_{ikl}\epsilon_{jmn}G_{km}G_{ln} \quad \text{(B.9)}$$

is the adjunct matrix with

$$G_{ij} = T_{ij} - v^2\delta_{ij}\,. \quad \text{(B.10)}$$

Here, ϵ_{ijk} is the Levi-Civita or permutation tensor, and δ_{ij} is the Kronecker symbolic δ. Direct calculation gives

$$W_{11} = G_{22}G_{33} - G_{32}G_{23}, \quad \text{(B.11)}$$
$$W_{22} = G_{33}G_{11} - G_{13}G_{31}, \quad \text{(B.12)}$$
$$W_{33} = G_{11}G_{22} - G_{21}G_{12}, \quad \text{(B.13)}$$
$$W_{12} = G_{31}G_{23} - G_{21}G_{33}, \quad \text{(B.14)}$$
$$W_{13} = G_{32}G_{21} - G_{22}G_{31}, \quad \text{(B.15)}$$
$$W_{23} = G_{31}G_{12} - G_{32}G_{11}. \quad \text{(B.16)}$$

The first component of the polarization has an arbitrary sign; it is obtained from the general equation (B.8) as

$$P_1^2 = \frac{W_{11}}{W_{mm}} \quad \text{(B.17)}$$

The other components of the polarization vector are computed through P_1 from equation B.8, with the correct relative sign being obtained automatically:

$$P_1 = \sqrt{\frac{W_{11}}{W_{mm}}}, \tag{B.18}$$

$$P_2 = \frac{W_{12}}{W_{mm} P_1}, \tag{B.19}$$

$$P_3 = \frac{W_{13}}{W_{mm} P_1}. \tag{B.20}$$

It may happen that for a certain slowness direction W_{11} and P_1 vanish. In this case, it is necessary to start with another (nonzero) polarization component.

The anisotropy coefficients can be inserted anywhere in the above expressions. We derived the weak-anisotropy approximation for the P-wave polarization vector by representing the Christoffel matrix $\mathbf{T}$ through Tsvankin's (1996b) anisotropy parameters, substituting $\mathbf{T}$ into equations (B.10)–(B.16) and carrying out linearization of the polarization components.

Unfortunately, in general Cartesian coordinates the polarization vector is rather complicated. Therefore, we used the auxiliary Cartesian coordinate system $[x, y, z]$ associated with the phase-velocity vector (Figure 2) and obtained the following concise expressions:

$$P_x = \left(1 + \frac{c_{33}}{c_{33} - c_{55}} p \cos^2\theta\right) \sin\theta, \tag{B.21}$$

$$P_y = \frac{c_{33}}{c_{33} - c_{55}} q \sin\phi \cos\phi \sin\theta, \tag{B.22}$$

$$P_z = \left(1 - \frac{c_{33}}{c_{33} - c_{55}} p \sin^2\theta\right) \cos\theta, \tag{B.23}$$

where the factors p and q are defined in equations (26) and (33) of the main text.