ON THE NULL-FIELD EQUATIONS FOR THE EXTERIOR PROBLEMS OF ACOUSTICS

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SUMMARY

A familiar method for solving the exterior Neumann problem of acoustics in two dimensions is to derive an integral equation of the second kind over the boundary curve for the unknown potential, u, say. One way of doing this is to represent u as a continuous distribution of simple wave sources over the boundary, leading to an integral equation for the unknown source strength. Another way is to apply Green's theorem to u and a simple wave source (Helmholtz representation); when the field point lies on the boundary, this gives an integral equation for the unknown boundary values of u. It is well-known that both of these methods yield integral equations which have unique solutions, except at the same discrete set of wave numbers (the irregular values), corresponding to the eigenfrequencies of the interior Dirichlet problem. The same methods can be modified to solve the exterior Dirichlet problem, and both yield integral equations of the second kind which have unique solutions except at the eigenfrequencies of the interior Neumann problem.

When the field point lies inside the boundary curve, the Helmholtz representation gives an integral relation. Using the known bilinear expansion for the simple wave source (in cylindrical polar coordinates), this integral relation may be reduced to an infinite set of equations, called the 'null-field' equations; equations of this type were first derived by Waterman, in 1965. In two dimensions, we show that the null-field equations always have a unique solution—irregular values do not occur. This result is proved here for both the exterior Neumann problem and the exterior Dirichlet problem. Similar results may be obtained in three dimensions.

1. Introduction

In the exterior boundary-value problems of acoustics, one is concerned with finding solutions of the Helmholtz equation,

$$(\nabla^2+k^2)\phi=0,$$

in the infinite region D, exterior to a simple closed Lyapunov curve ∂D, such that ϕ also satisfies certain boundary conditions. A general problem in two dimensions is the following:

To find a potential $u(P)$ satisfying the Helmholtz equation in D, the radiation condition

$$r_P^{\frac{1}{2}}\left(\frac{\partial u}{\partial r_P}-iku\right)\to 0 \quad \text{as} \quad r_P\to\infty,$$

and a boundary condition on ∂D; for the Neumann problem, this boundary condition is

$$\frac{\partial u}{\partial n_p}(p) = f(p) \quad \text{on} \quad \partial D, \tag{1.1}$$

whilst for the Dirichlet problem, it is

$$u(p) = g(p) \quad \text{on} \quad \partial D. \tag{1.2}$$

The functions $f(p)$ and $g(p)$ are prescribed on ∂D. The notation is the same as that used by Ursell (**1**, **2**): capital letters P, Q denote points of D; small letters p, q denote points of ∂D; the origin O is taken at an arbitrary point inside D_-, the complement of $D \cup \partial D$; P_-, Q_- denote points of D_-; r_P is the length OP; $\partial/\partial n_P$ denotes normal differentiation at the point p, in the direction from D towards ∂D.

The usual approach for solving these two exterior problems of acoustics is to derive an integral equation of the second kind, over the boundary ∂D. One way of doing this is to assume that $u(P)$ can be represented as a distribution of sources (for the Neumann problem) or dipoles (for the Dirichlet problem) over ∂D; the source and dipole strengths are then found to be solutions of Fredholm integral equations of the second kind. Alternatively, integral equations of the second kind can be derived from Green's theorem (i.e. the Helmholtz formulae). It is well-known that both of these methods (which will be described, briefly, in section 2) lead to boundary integral equations of the second kind which are singular at a certain discrete set of frequencies, corresponding to eigenvalues of the related interior problems. This phenomenon is a consequence of the method of solution, for it is known that the boundary-value problems have unique solutions at all frequencies.

A different approach to this problem has been employed by Waterman (**3**). His method is based on solving the Helmholtz formula in the interior, D_-, and leads to an infinite system of equations, rather than a single (integral) equation; these equations, called the null-field equations, are derived in section 3. In section 4, we show that the null-field equations always have a unique solution, i.e. difficulties at interior eigenvalues do not occur with this method. Finally, in section 5, we consider methods for solving the null-field equations.

2. Boundary integral equations

Let $G(P, Q)$ be any fundamental solution, i.e. $G(P, Q)$ satisfies the Helmholtz equation in D and the radiation condition at infinity, and has a suitably normalised logarithmic singularity at Q (see, e.g., (**1**)). The simplest choice for $G(P, Q)$ is the free-space wave source,

$$G_0(P, Q) = \tfrac{1}{2} i\pi H_0^{(1)}(k\,|\mathbf{r}_P - \mathbf{r}_Q|), \tag{2.1}$$

where $H_0^{(1)}(z)$ denotes the Hankel function of the first kind. If we apply

Green's theorem in D, to $u(P)$ and $G_0(P, Q)$, we obtain the following equations:

$$2\pi u(P) = \int_{\partial D} \left\{ G_0(P, q) \frac{\partial}{\partial n_q} u(q) - u(q) \frac{\partial}{\partial n_q} G_0(P, q) \right\} ds_q, \tag{H.1}$$

$$\pi u(p) = \int_{\partial D} \left\{ G_0(p, q) \frac{\partial}{\partial n_q} u(q) - u(q) \frac{\partial}{\partial n_q} G_0(p, q) \right\} ds_q, \tag{H.2}$$

and

$$0 = \int_{\partial D} \left\{ G_0(P_-, q) \frac{\partial}{\partial n_q} u(q) - u(q) \frac{\partial}{\partial n_q} G_0(P_-, q) \right\} ds_q. \tag{H.3}$$

These equations are known as (Weber's analogue of) Helmholtz's formulae (see, e.g., (**4**), §§4.2, 6.2). Similar equations may be derived when $G_0(P, Q)$ is replaced by any fundamental solution, $G(P, Q)$.

(H.1) is a representation for the radiated field in D as a distribution of sources and dipoles over ∂D; this integral representation may be used to evaluate u everywhere in D when both u and $\partial u/\partial n$ are known on ∂D.

For the Neumann problem, use of (1.1) in (H.2) yields

$$\pi u(p) + \int_{\partial D} u(q) \frac{\partial}{\partial n_q} G_0(p, q)\, ds_q = \int_{\partial D} G_0(p, q) f(q)\, ds_q, \tag{2.2}$$

which is an integral equation of the second kind for the unknown boundary values of u. This integral equation possesses a unique solution unless the corresponding homogeneous integral equation,

$$\pi u(p) + \int_{\partial D} u(q) \frac{\partial}{\partial n_q} G_0(p, q)\, ds_q = 0, \tag{2.3}$$

has a non-trivial solution. It is known that (2.3) does have non-trivial solutions whenever k^2 is an eigenvalue of the interior Dirichlet problem. At these values of k^2 (called the irregular values), the integral equation (2.2) does not have a unique solution for general $f(p)$. However, this difficulty may be overcome by using a different fundamental solution in place of $G_0(P, Q)$ (**1, 2, 5**).

A different approach for solving the Neumann problem is to represent $u(P)$ by a distribution of sources over ∂D (single layer),

$$u(P) = \int_{\partial D} \mu(q) G_0(P, q)\, ds_q. \tag{2.4}$$

On applying the boundary condition (1.1), we find that the unknown source strength $\mu(q)$ satisfies the integral equation

$$\pi \mu(p) + \int_{\partial D} \mu(q) \frac{\partial}{\partial n_p} G_0(p, q)\, ds_q = f(p). \tag{2.5}$$

This integral equation is of the same form as (2.2), except that the kernel of (2.5) is the transpose of the kernel appearing in (2.2) (here, we have used the symmetry of the fundamental solution (2.1)). Hence, (2.5) has the same irregular values as (2.2).

When k^2 is not an irregular value, the unique solution of (2.5) is seen to solve the exterior Neumann problem. For the boundary condition (1.1) is automatically satisfied by (2.4) if $\mu(p)$ satisfies (2.5). This is not the case with the integral equation (2.2). If we substitute the unique solution of (2.2) into (H.1), we can define a function $U(P)$, say, by

$$2\pi U(P) = \int_{\partial D} \left\{ G_0(P, q) f(q) - u(q) \frac{\partial}{\partial n_q} G_0(P, q) \right\} ds_q.$$

$U(P)$ certainly satisfies the Helmholtz equation in D, and the radiation condition, but there is no *a priori* guarantee that $U(P)$ satisfies the boundary condition (1.1). However, it can be shown that, provided k^2 is not an irregular value, $U(P)$ does indeed satisfy (1.1) and hence $U(P)$ is the solution of the exterior Neumann problem (see Kleinman and Roach (**6**), who review the corresponding problems in three dimensions).

For the Dirichlet problem, use of (1.2) in (H.2) yields an integral equation of the first kind for the unknown boundary values of $\partial u/\partial n$. To obtain an integral equation of the second kind, we take the normal derivative of (H.1), let P approach ∂D and then use (1.2) to give (**6**)

$$\pi \frac{\partial u}{\partial n_p}(p) - \int_{\partial D} \frac{\partial u(q)}{\partial n_q} \frac{\partial}{\partial n_p} G_0(p, q)\, ds_q = -\frac{\partial}{\partial n_p} \int_{\partial D} g(q) \frac{\partial}{\partial n_q} G_0(p, q)\, ds_q. \tag{2.6}$$

(Note that a sufficient condition for the existence of the right-hand side of (2.6) is that $g(q)$ should be differentiable on the Lyapunov curve, ∂D; see (**6**), §2.) This integral equation for $\partial u/\partial n$ possesses a unique solution unless the corresponding homogeneous integral equation,

$$\pi v(p) - \int_{\partial D} v(q) \frac{\partial}{\partial n_p} G_0(p, q)\, ds_q = 0, \tag{2.7}$$

has a non-trivial solution. It is known that (2.7) does have non-trivial solutions whenever k^2 is an eigenvalue of the interior Neumann problem. As before, the difficulty of non-uniqueness at irregular values of k^2 may be overcome by using a different fundamental solution in place of $G_0(P, Q)$ (**1**).

An alternative approach for the Dirichlet problem is provided by representing $u(P)$ as a distribution of dipoles over ∂D (double layer). This leads to an integral equation of the second kind for the unknown dipole strength. This equation is of the same form as (2.6), except that the kernel is transposed, and hence has the same irregular values. Again, it can be shown that the solution of (2.6) does lead to a solution of the exterior Dirichlet problem, provided that k^2 is not an irregular value. For further details, see the papers by Ursell (**1**) and Kleinman and Roach (**6**).

The integral relation (H.3) asserts that the field induced in D_- by the sources on ∂D is exactly cancelled by the field induced by the dipoles on ∂D. In other words, although the continuation of the actual exterior field (i.e. the solution of the boundary-value problem), across ∂D, does not vanish in D_- (otherwise, it would vanish everywhere), the field generated by the source and dipole distributions over ∂D (which are used to represent the actual field in D) does vanish throughout D_-. Waterman (**3, 7**) calls this the 'extended boundary condition', and (H.3) the 'extended integral equation'. In the next section, we shall derive, from the interior integral relation (H.3), the infinite system of null-field equations.

3. The null-field equations

The free-space wave source, (2.1), may be written as (**1, 3**)

$$G_0(P, Q) = \tfrac{1}{2}i\pi \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \epsilon_m \psi_m^\sigma(Q) \hat{\psi}_m^\sigma(P), \tag{3.1}$$

for $r_Q > r_P$, where

$$\psi_m^\sigma(Q) = H_m^{(1)}(kr_Q) E_m^\sigma(\theta_Q),$$
$$\hat{\psi}_m^\sigma(Q) = J_m(kr_Q) E_m^\sigma(\theta_Q),$$
$$E_m^1(\theta) = \cos m\theta, \qquad E_m^2(\theta) = \sin m\theta,$$

and ϵ_m is the Neumann factor defined by $\epsilon_0 = 1$, $\epsilon_m = 2$ for $m > 0$.

Let C_- be the inscribed circle to ∂D, which is centred on O. Similarly, let C_+ be the escribed circle to ∂D. Denote the interior of C_- by D_N. When P_- lies in D_N (where $r_p < r_q$), we may substitute (3.1) into (H.3) to give

$$\sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \epsilon_m \hat{\psi}_m^\sigma(P_-) \int_{\partial D} \left\{ u(q) \frac{\partial}{\partial n_q} \psi_m^\sigma(q) - \frac{\partial u(q)}{\partial n_q} \psi_m^\sigma(q) \right\} ds_q = 0. \tag{3.2}$$

Since the regular functions $\hat{\psi}_m^\sigma$ are orthogonal, it follows that each term in (3.2) must vanish and so we obtain the following set of equations:

$$\int_{\partial D} \left\{ u(q) \frac{\partial}{\partial n_q} \psi_m^\sigma(q) - \frac{\partial u(q)}{\partial n_q} \psi_m^\sigma(q) \right\} ds_q = 0.$$
$$\sigma = 1, 2, \quad m = 0, 1, 2, \ldots \tag{N.F}$$

These are the so-called "null-field equations" of acoustics (**8, 9**). Equations of this type were first derived by Waterman (**7**) for electromagnetic scattering problems and later, for acoustic scattering problems (**3**). More recently, several authors have derived corresponding equations for elastodynamic problems (**10, 11**).

For the Neumann problem, use of (1.1) in (N.F) results in an infinite set of equations for the boundary values of u, whilst for the Dirichlet problem, use of (1.2) in (N.F) leads to an infinite set of equations for the boundary values of $\partial u/\partial n$. Once u and $\partial u/\partial n$ are both known on ∂D, we may use the integral representation (H.1) to evaluate u everywhere in D. In particular, if P lies outside C_+, we can use (3.1) in (H.1) to obtain

$$u(P)=\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2} c_m^\sigma \psi_m^\sigma(P), \tag{3.3}$$

where the coefficients c_m^σ are given by

$$4ic_m^\sigma=\int_{\partial D}\left\{u(q)\frac{\partial}{\partial n_q}\hat{\psi}_m^\sigma(q)-\frac{\partial u(q)}{\partial n_q}\hat{\psi}_m^\sigma(q)\right\}ds_q, \qquad \sigma=1,2, \quad m=0,1,2,\ldots. \tag{3.4}$$

The representation (3.3) implies that $u(P)$ corresponds to outgoing waves, everywhere exterior to C_+. If we assume that $u(P)$ may be represented in this way, then we can derive the null-field equations more simply, as follows (**11**). Apply Green's theorem to $u(P)$ and $\psi_m^\sigma(P)$ in the region bounded by ∂D and S, where S is a large circle enclosing ∂D and centred on O. The contribution from integrating over S may be evaluated by using (3.3) and noting that, on S, $\partial/\partial n_q=\partial/\partial r_q$. Hence, using the orthogonality of $E_m^\sigma(\theta)$, it follows that this contribution must be zero. Choosing appropriate values for m and σ yields the complete set of null-field equations, (N.F). If we also apply Green's theorem in the same region to $u(P)$ and $\hat{\psi}_m^\sigma(P)$ (i.e. the real part of $\psi_m^\sigma(P)$, for k real), and then use asymptotic properties of $\psi_m^\sigma(r,\theta)$ and $\hat{\psi}_m^\sigma(r,\theta)$, for large r, we obtain (3.4).

This derivation is illuminating, for it demonstrates that the null-field equations do not depend, essentially, on the bilinear expansion of the free-space wave source (3.1), or on the interior integral relation (H.3), but on the expansion of fields, which satisfy a radiation condition, as (3.3).

4. Solvability of the null-field equations

In this section, we shall prove that the set of null-field equations, (N.F), possesses a unique solution, $v(q)$, for all real values of k^2, where $v(q)=u(q)$ for the exterior Neumann problem and $v(q)=\partial u(q)/\partial n_q$ for the exterior Dirichlet problem. To do this, we show that $v(q)$ satisfies (N.F) if and only if $v(q)$ satisfies an integral equation of the second kind, which is known to possess a unique solution.

As an initial approach, we might try to reverse the (first) derivation of the null-field equations: multiply each of (N.F) by $\hat{\psi}_m^\sigma(P_-)$, where $P_-\in D_N$ and then use (3.1) to yield (H.3), i.e.

$$w_0(P_-)\equiv\int_{\partial D}\left\{u(q)\frac{\partial}{\partial n_q}G_0(P_-,q)-\frac{\partial u(q)}{\partial n_q}G_0(P_-,q)\right\}ds_q=0. \tag{4.1}$$

This equation is valid for all $P_- \in D_N$. However, w_0 is a solution of the Helmholtz equation in D_- which vanishes in D_N. Thus, we can assert that (4.1) holds for all $P_- \in D_-$ (continuation arguments of this type were also used by Waterman (**3**, **7**)).

Let us now consider the Neumann problem. Using (1.1) in (4.1) and letting P_- approach ∂D, we obtain (2.2), which is an integral equation of the second kind for $u(q)$. As we have already remarked, (2.2) has a unique solution, except at the irregular values of k^2 (i.e. at the eigenfrequencies of the interior Dirichlet problem). Conversely, if we are not at an irregular value, it follows that the unique solution of (2.2) also solves the null-field equations. For $w_0(P_-)$ satisfies the Helmholtz equation in D_- and, by (2.2), w_0 vanishes on ∂D. Hence, w_0 vanishes everywhere in D_- and so $u(q)$ satisfies (4.1), i.e. $u(q)$ satisfies (N.F), together with the boundary conditions (1.1).

At the irregular values, this argument must be modified. There are, in fact, infinitely many integral equations satisfied by $u(q)$, each one corresponding to a different choice of fundamental solution, $G(P, Q)$. In order to derive a different integral equation of the second kind for $u(q)$, we need only take a different linear combination of the null-field equations. Suppose we multiply the first $N+1$ of (N.F) by $a_m^\sigma \psi_m^\sigma(P_-)$, where the a_m^σ are constants. Adding the resulting equations to (4.1) gives

$$w_1(P_-) \equiv \int_{\partial D} \left\{ u(q) \frac{\partial}{\partial n_q} G_1(P_-, q) - \frac{\partial u(q)}{\partial n_q} G_1(P_-, q) \right\} ds_q = 0, \tag{4.2}$$

where $G_1(P, Q)$ is a new (symmetric) fundamental solution, defined by

$$G_1(P, Q) = G_0(P, Q) + \sum_{m=0}^{N} \sum_{\sigma=1}^{2} a_m^\sigma \psi_m^\sigma(P) \psi_m^\sigma(Q). \tag{4.3}$$

Proceeding as before, we let P_- approach ∂D and use (1.1) to obtain

$$\pi u(p) + \int_{\partial D} u(q) \frac{\partial}{\partial n_q} G_1(p, q)\, ds_q = \int_{\partial D} G_1(p, q) f(q)\, ds_q, \tag{4.4}$$

which is another integral equation of the second kind for $u(q)$.

Fundamental solutions of the form (4.3) have been considered by several authors. Ursell (**1**) found certain values for a_m^σ, $\sigma = 1, 2$, $m = 0, 1, 2, \ldots$, which ensured that $G_1(P, Q)$ satisfied a dissipative boundary condition on a circle lying inside D_-; the corresponding integral equation (4.4) was then shown to possess a unique solution for all real values of k^2. More generally, Jones (**5**) has considered (4.3) with a_m^σ, $\sigma = 1, 2$, $m = 0, 1, \ldots, N$, all real and non-zero. By choosing N large enough, he showed that any given value of k^2 was not an irregular value for the fundamental solution, $G_1(P, Q)$, i.e. at any given value of k^2, we can always ensure that (4.4) has a unique solution. The restrictions on a_m^σ have been further relaxed by Ursell (**2**), who showed that the conditions $|a_m^\sigma + \frac{1}{2} i\pi| > \frac{1}{2}\pi$ are sufficient for Jones's results to hold.

In summary, if $u(q)$ satisfies the null-field equations, then, by taking a suitable linear combination of these equations, we can see that $u(q)$ also satisfies an integral equation of the second kind, (4.4), which always has a unique solution. Moreover, the solution of this integral equation also solves the null-field equations and so we have proved the following theorem.

THEOREM 1. *The null-field equations for the exterior Neumann problem of acoustics (i.e.* (N.F), *together with the boundary condition* (1.1)) *possess a unique solution for all real values of* k^2.

Let us now consider the Dirichlet problem. If we take the normal derivative of (4.2), use the boundary condition (1.2) and then let P_- approach ∂D, we obtain

$$\pi\frac{\partial u(p)}{\partial n_p}-\int_{\partial D}\frac{\partial u(q)}{\partial n_q}\frac{\partial}{\partial n_p}G_1(p,q)\,ds_q=-\frac{\partial}{\partial n_p}\int_{\partial D}g(q)\frac{\partial}{\partial n_q}G_1(p,q)\,ds_q, \tag{4.5}$$

which is an integral equation of the second kind for $\partial u(q)/\partial n_q$. The same argument as before now shows that (4.5) always has a unique solution, when the coefficients appearing in (4.3) take on suitable values. Although Jones (**5**) did not consider the exterior Dirichlet problem, the particular fundamental solution constructed by Ursell (**1**), for the Neumann problem, also solves the Dirichlet problem. Using this particular fundamental solution, we see that any solution of the null-field equations, for the Dirichlet problem, also satisfies (4.5), an integral equation with a unique solution. Moreover, this unique solution also satisfies the null-field equations. For $w_1(P_-)$ satisfies the Helmholtz equation in D_- and, by (4.5), $\partial w_1/\partial n$ vanishes on ∂D. Applying Green's theorem in D_- to w_1 and G_1, we obtain

$$2\pi w_1(P_-)=\int_{\partial D}w_1(q)\frac{\partial}{\partial n_q}G_1(P_-,q)\,ds_q. \tag{4.6}$$

Letting P_- approach ∂D, (4.6) gives

$$\pi w_1(p)-\int_{\partial D}w_1(q)\frac{\partial}{\partial n_q}G_1(p,q)\,ds_q=0. \tag{4.7}$$

Ursell (**1**) has shown that the only solution of (4.7) is the trivial one, $w_1(q)=0$, whence (4.6) shows that $w_1(P_-)\equiv 0$ and so $\partial u(q)/\partial n_q$ satisfies (N.F), together with the boundary condition (1.2). Hence, we have proved the following theorem.

THEOREM 2. *The null-field equations for the exterior Dirichlet problem of acoustics (i.e.* (N.F), *together with the boundary condition* (1.2)) *possess a unique solution for all real values of* k^2.

As a corollary to Theorems 1 and 2, we can see that the interior integral relation (H.3) always has a unique solution. For example, consider the exterior Neumann problem. Use of (1.1) in (H.3) gives

$$\int_{\partial D} u(q)\frac{\partial}{\partial n_q} G_0(P_-, q)\, ds_q = \int_{\partial D} f(q)G_0(P_-, q)\, ds_q. \tag{4.8}$$

If (4.8) is satisfied at all points $P_- \in D_-$, then it is equivalent to the set of null-field equations, and hence has a unique solution. In fact, if (4.8) holds in a finite region of D_-, then continuation arguments ensure that it holds throughout D_-. Similar results can be obtained for the exterior Dirichlet problem.

5. Solution of the null-field equations

In the previous section we showed that the set of null-field equations and the interior integral relation are equivalent, in the sense that any solution of one is also a solution of the other. In fact, solving the null-field equations could be interpreted as an indirect method of solving the interior integral relation. So, before considering the solution of the null-field equations, it may be worthwhile to examine what might happen if we attempted to solve the interior integral relation, directly.

A simple method for solving the interior integral relation (for the Neumann problem) is to satisfy (4.8) at a finite number of points in D_-. Choosing N points, $P_i \in D_-$, we find that

$$\int_{\partial D} u(q)\frac{\partial}{\partial n_q} G_0(P_i, q)\, ds_q = \int_{\partial D} f(q)G_0(P_i, q)\, ds_q, \quad i = 1, 2, \ldots, N. \tag{5.1}$$

This is a system of N equations from which an approximation to $u(q)$ may be found. Since we have only satisfied (4.8) at a discrete set of points, continuation arguments are not applicable, and so we cannot use Theorem 1 to say anything about the solution of (5.1). Indeed, if k^2 is an eigenvalue of the interior Dirichlet problem, and all the points P_i lie on the nodal lines of the corresponding eigenfunction ($v(P_-)$, say), then (5.1) will not have a unique solution (**12**). This result follows by noting that $v(P_-)$ can be represented as a dipole distribution over ∂D, namely

$$v(P_-) = \int_{\partial D} \nu(q)\frac{\partial}{\partial n_q} G_0(P_-, q)\, ds_q,$$

where $\nu(q)$ is a non-trivial solution of

$$\pi\nu(p) + \int_{\partial D} \nu(q)\frac{\partial}{\partial n_q} G_0(p, q)\, ds_q = 0.$$

($v(P_-)$ can also be represented by a distribution of sources over ∂D; see (**6**), §4.) Thus, when all the points P_i lie on the nodal lines of v, there exists a non-trivial function $\nu(q)$, which satisfies

$$0 = \int_{\partial D} \nu(q)\frac{\partial}{\partial n_q} G_0(P_i, q)\, ds_q;$$

such a function can then always be added to any solution of (5.1). Since the eigenfunctions of the interior Neumann problem can also be represented as a dipole distribution over ∂D, it follows that the solution of (5.1) is also non-unique whenever k^2 is an eigenvalue of the interior Neumann problem and all the points P_i lie on the nodal lines of the corresponding eigenfunction.

In general, the position of the nodal lines in D_- is not known, *a priori*, and so the choice of the points P_i must be based largely on experience and intuition. It is also not known what would happen if only some of the points P_i were to lie on nodal lines. Finally, it can also be shown that this method of solution exhibits the same difficulties when applied to the exterior Dirichlet problem. Consequently, we see that this simple method for solving the interior integral relation may not yield the correct solution, and so we shall now briefly describe how the null-field equations can be solved.

The null-field equations for the exterior Neumann problem may be written as

$$\int_{\partial D} u(q) \frac{\partial}{\partial n_q} \psi_m^\sigma(q)\, ds_q = f_m^\sigma, \qquad \sigma = 1, 2, \qquad m = 0, 1, 2, \ldots, \tag{5.2}$$

where $u(q)$ is to be determined and the known constants f_m^σ are given by

$$f_m^\sigma = \int_{\partial D} f(q) \psi_m^\sigma(q)\, ds_q.$$

For the exterior Dirichlet problem, the null-field equations may be written as

$$\int_{\partial D} \frac{\partial u(q)}{\partial n_q} \psi_m^\sigma(q)\, ds_q = g_m^\sigma, \qquad \sigma = 1, 2, \qquad m = 0, 1, 2, \ldots, \tag{5.3}$$

where $\partial u(q)/\partial n_q$ is to be determined and

$$g_m^\sigma = \int_{\partial D} g(q) \frac{\partial}{\partial n_q} \psi_m^\sigma(q)\, ds_q.$$

If ∂D is a circle, centred on O, then (5.2) and (5.3) simply give the Fourier components of $u(q)$ and $\partial u(q)/\partial n_q$, respectively. For any other geometry, the null-field equations must be solved numerically.

One approach is to expand the unknown function as a series of basis functions. For example, consider the Neumann problem and expand $u(q)$ as

$$u(q) = \sum_{n=0}^{\infty} u_n \phi_n(q), \tag{5.4}$$

where the set of functions $\{\phi_n(q)\}$ is required to be complete over ∂D. Substituting (5.4) into (5.2), we obtain

$$\sum_{n=0}^{\infty} u_n \int_{\partial D} \phi_n(q) \frac{\partial}{\partial n_q} \psi_m^\sigma(q)\, ds_q = f_m^\sigma, \qquad \sigma = 1,2, \qquad m = 0, 1, 2, \ldots, \tag{5.5}$$

which is an infinite system for the unknown coefficients u_n; truncating this system leads to a numerical method for solving the null-field equations.

Different authors have advocated different choices for $\phi_n(q)$. For example, Waterman (**3**) chose $\phi_n = \hat{\psi}_n^\sigma$, and thus reintroduced the difficulties at irregular values, for the set $\{\hat{\psi}_n^\sigma\}$ is not complete at these values (**3**). Bates and Wall (**9**) have considered some other choices for $\phi_n(q)$. However, at present there does not appear to be a satisfactory criterion for choosing the (complete) set of functions $\{\phi_n(q)\}$; it is reasonable to hope that a judicious choice may lead to an efficient numerical method for solving the null-field equations. We shall not give any further discussion of the numerical aspects here, but simply remark that many successful computations, based on the null-field equations, have been reported in the literature since Waterman's first paper, in 1965.

6. Conclusions

Until recently, the exterior problems of acoustics were generally treated by solving an integral equation of the second kind over the boundary curve; it is well-known that the usual boundary integral equations are not uniquely solvable at the irregular values of k^2. An alternative method, which was first proposed by Waterman (**3, 7**), is to solve the infinite system of null-field equations. In this paper, we have shown that the null-field equations always have a unique solution—the unphysical irregular values do not occur. Moreover, this solution may be used to solve the original boundary-value problem. We have proved these results in two dimensions, for both the exterior Neumann problem and the exterior Dirichlet problem. Similar results can be proved in three dimensions.

In section 5, we described a simple exact method for reducing the null-field equations to an infinite system of linear algebraic equations. This method may be used to solve the null-field equations, numerically, by making two approximations: the infinite set of equations must first be truncated and then the unknown function must be approximated by a finite combination of the chosen basis functions. In the future, it is hoped to examine these approximations and to discuss other numerical aspects of the null-field method.

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