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The discontinuity in the elastostatic displacement vector across a penny-shaped crack under arbitrary loads

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ABSTRACT

Consider an infinite elastic solid containing a penny-shaped crack. A familiar problem in linear elastostatics is the determination of the displacement in the solid when the crack is subjected to an arbitrarily prescribed loading; denote the corresponding boundary-value problem by P. We construct the exact Green's function for P, using a method due to Guidera and Lardner [4]. We then use our Green's function to solve P, by obtaining expressions for the discontinuity in the displacement vector across the crack. Finally, we compare our solutions for P with those obtained recently by Krenk [6].

1. Introduction

Consider a three-dimensional elastic solid of unbounded extent, containing a finite crack which occupies a surface γ, whose two surfaces are labelled γ_+ and γ_-. A familiar boundary-value problem in the linear theory of elastostatics is the determination of the displacement vector $\boldsymbol{u}$ in D, the region exterior to γ, when the surfaces of the crack are subjected to prescribed tractions. (We require the tractions to be equal and opposite at corresponding points of γ_+ and γ_-.) This problem may be formulated as follows.

Boundary-value problem P.

Determine $u_i(\boldsymbol{r})$, $\boldsymbol{r} \in D$, satisfying

P.1. elastostatic equations of equilibrium in the solid,

$$\frac{\partial}{\partial x_j} \tau_{ij}(\boldsymbol{r}) = 0 \qquad \boldsymbol{r} \in D;$$

P.2. prescribed tractions on the crack,

$$\tau_{ij} n_j = \tau_{ij}^{(i)} n_j, \qquad \boldsymbol{r} \in \gamma_\pm;$$

P.3. regularity conditions at infinity,

$$u_i(\boldsymbol{r}) = O(|\boldsymbol{r}|^{-1}) \quad \text{as} \quad |\boldsymbol{r}| \to \infty; \quad \text{and}$$

P.4. edge conditions,

$$u_i(\boldsymbol{r}) = O(1) \quad \text{as} \quad \rho \to 0, \quad \text{where } \rho \text{ is the distance of } \boldsymbol{r} \text{ from the crack edge.}$$

Here, τ_{ij} are the components of the stress tensor, which are related to the displacement vector by

$$\tau_{ij} = c_{ijkl} \frac{\partial}{\partial x_k} u_l, \tag{1.1}$$

c_{ijkl} are the material modulii, $\tau_{ij}^{(i)} n_j$ are the given boundary values of the traction and $\boldsymbol{n}$ is the unit normal vector, which is assumed to point into D. Henceforth, we shall always consider the elastic solid to be homogeneous and isotropic, whence the material modulii are given by

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{1.2}$$

Furthermore, we shall only consider flat, circular cracks ("penny-shaped" cracks); the region γ is then $z = 0$, $r < 1$, where (r, θ, z) are cylindrical polar coordinates.

In the first part of this paper (§§2–4), we shall construct the exact Green's function for P, which we denote by $G_{ij}(\boldsymbol{r}; \boldsymbol{r}_0)$. This represents the ith component of the displacement at $\boldsymbol{r}$ when the crack is opened by two equal and opposite concentrated forces, one at the point $\boldsymbol{r}_0$ on γ_+ acting in the jth direction, and one at the corresponding point of γ_-, acting in the opposite direction; in other words, G_{ij} represents the solutions of P for three particular loadings of the crack.

This work was motivated originally by a study of the corresponding elastodynamic problem i.e. the diffraction of time-harmonic elastic waves by a penny-shaped crack. Let us denote the corresponding boundary-value problem by P^*. Then, the exact Green's function for P, G_{ij}, can be used to construct an approximate Green's function for P^* and this may then be used to solve P^*, rigorously [7]; this work will be described elsewhere [8].

The exact Green's function may also be used in other ways. For example, it can be used to prove that the boundary-value problem P has at least one solution (it is already known that P has at most one solution [5]). In the second part of this paper, we shall use G_{ij} to solve P for arbitrary loadings of the crack. Actually, we shall only calculate the discontinuity in u_i across the crack, which we denote by $[u_i]$, i.e.

$$[u_i(\boldsymbol{r})] = \lim_{\boldsymbol{r}' \to \boldsymbol{r} \in \gamma_+} u_i(\boldsymbol{r}') - \lim_{\boldsymbol{r}' \to \boldsymbol{r} \in \gamma_-} u_i(\boldsymbol{r}'); \tag{1.3}$$

this is sufficient since any displacement field u_i has an integral representation in terms of $[u_i]$, given by

$$u_m(\boldsymbol{r}') = \int_\gamma [u_i(\boldsymbol{r})] \Sigma_{ijm}^F(\boldsymbol{r}; \boldsymbol{r}') n_j \, dS, \tag{1.4}$$

where

$$\Sigma_{ijm}^F(\boldsymbol{r}; \boldsymbol{r}') = c_{ijkl} \frac{\partial}{\partial x_k} G_{lm}^F(\boldsymbol{r}; \boldsymbol{r}'), \tag{1.5}$$

$$8\pi\mu G_{ij}^F(r; r') = \frac{2}{R} \delta_{ij} - \frac{1}{2(1-\nu)} \frac{\partial^2 R}{\partial x_i \partial x_j}, \tag{1.6}$$

$R = |\boldsymbol{r} - \boldsymbol{r}'|$ and ν is Poisson's ratio. (1.6) is known as Kelvin's point-load solution, whilst (1.4) is usually known as the Somigliana formula [4]. $[u_i]$ is given in terms of $[G_{ij}]$ in [7] as

$$[u_m(\boldsymbol{r}_0)] = -\int_\gamma \frac{1}{\mu} \tau_{ij}^{(i)}(\boldsymbol{r})[G_{jm}(\boldsymbol{r}; \boldsymbol{r}_0)] n_i \, dS. \tag{1.7}$$

There is an extensive literature on solutions of the boundary-value problem P, beginning with Sneddon's paper on the special case of a penny-shaped crack opened by an axisymmetric pressure [10]; much of the work prior to 1969 is surveyed in [11]. However, the general problem, of a penny-shaped crack opened by arbitrary equal and opposite tractions, has received rather less attention. Most of the work on this problem is based on Muki's solution for the arbitrary loading of a semi-infinite elastic solid [9]. He obtained the displacements and stresses in the solid as Hankel transforms of certain arbitrary functions. These arbitrary functions may be found, in principle, by applying boundary conditions on the surface of the solid; applying mixed boundary conditions leads to dual integral equations (see [12], where some simple problems of this type are formulated and solved).

J. C. Bell [1] has obtained a solution to P by solving some unusual systems of dual integral equations, which he derived from Muki's integral representations. S. Krenk [6], who also began with Muki's solution, has solved P by deriving a system of integral equations for the Fourier components of $[u_i]$; he was able to solve these integral equations by expanding all relevant quantities in infinite series of Gegenbauer polynomials (see §7).

Guidera and Lardner [4] have used a different approach. They started with the Somigliana formula, (1.4), and then derived a system of integral equations which they solved; we shall give a description of their method in the next section.

In §§3 and 4, we use the method of Guidera and Lardner to obtain the exact Green's function for P. We find that all the Fourier components of $[G_{ij}(\boldsymbol{r}, \boldsymbol{r}_0)]$ may be expressed as linear combinations of the functions $g^m(r, r_0)$, defined by (3.3); these functions can be evaluated in terms of the incomplete elliptic integrals of the first and second kinds (see §3).

In §§5 and 6, we solve P, using (1.7) to derive expressions for the Fourier components of $[u_i]$ in terms of the Fourier components of the applied tractions. Because of the complicated nature of the problem, it is difficult to compare our solutions with those of Bell [1], Guidera and Lardner [4], and Krenk [6]. However, in §7, we give a detailed comparison of our solutions with those obtained by Krenk [6]; we find almost complete agreement. In an appendix, we evaluate certain interesting integrals of Gegenbauer polynomials, which are required in §7.

2. The method of Guidera and Lardner

The stress components in D are given by (1.1) and, by using (1.4), they may be expressed in terms of $[u_i]$. Consequently, by setting the three components of the traction on γ_+, say, equal to their prescribed values (given by P.2), a system of

integro-differential equations for $[u_i]$ is obtained, namely

$$\tau_{pq}^{(i)}(\boldsymbol{r}')n_q' = n_q' c_{pqlm}\left\{\frac{\partial}{\partial x_l'}\int_\gamma [u_i(\boldsymbol{r})]\Sigma_{ijm}^F(\boldsymbol{r};\boldsymbol{r}')n_j\,dS\right\}\Bigg|_{\boldsymbol{r}'\in\gamma} \tag{2.1}$$

where $n_i' = n_i(\boldsymbol{r}')$, $\boldsymbol{r}' \in \gamma$. For a planar crack, this system partially decouples and then problem P may be replaced by two separate problems:
Normal problem (symmetric about $z=0$)

$$\tau_{xz} = \tau_{yz} = 0, \qquad \tau_{zz} \neq 0 \quad \text{on} \quad \gamma;$$
$$[u_x] = [u_y] = 0, \qquad [u_z] \neq 0.$$

Shear problem (antisymmetric about $z=0$)

$$\tau_{xz}, \tau_{yz} \neq 0, \qquad \tau_{zz} = 0, \quad \text{on} \quad \gamma;$$
$$[u_x], [u_y] \neq 0 \qquad [u_z] = 0.$$

Guidera and Lardner [4] have used this approach to solve P, for a penny-shaped crack in a homogeneous isotropic elastic solid, and we shall now describe their method.

If we differentiate under the integral sign in (2.1) and then integrate by parts, we are then permitted to take the limit $\boldsymbol{r}' \to \gamma$; this yields a system of integral equations, which may be rewritten in terms of cylindrical polar coordinates as follows. (We write GL(2.9) to denote Eq. (2.9) of [4].)
Normal problem (GL(2.9)):

$$\tau_{zz}^{(i)}(r', \theta', 0) = -\eta\int_0^1\int_0^{2\pi}\left\{\frac{\partial}{\partial z}[u_z(r,\theta)]\frac{\partial}{\partial r}\left(\frac{1}{R}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}[u_z(r,\theta)]\frac{\partial}{\partial\theta}\left(\frac{1}{R}\right)\right\}d\theta r\,dr. \tag{2.2a}$$

Shear problem (GL(2.10, 2.11)):

$$\tau_{rz}^{(i)}(r', \theta', 0) = \eta\int_0^1\int_0^{2\pi}\left\{\alpha(r,\theta)\frac{\partial}{\partial r'}\left(\frac{1}{R}\right) + \beta(r,\theta)\frac{1}{r'}\frac{\partial}{\partial\theta'}\left(\frac{1}{R}\right)\right\}d\theta r\,dr, \tag{2.2b}$$

$$\tau_{\theta z}^{(i)}(r', \theta', 0) = \eta\int_0^1\int_0^{2\pi}\left\{\alpha(r,\theta)\frac{1}{r'}\frac{\partial}{\partial\theta'}\left(\frac{1}{R}\right) - \beta(r,\theta)\frac{\partial}{\partial r'}\left(\frac{1}{R}\right)\right\}d\theta r\,dr, \tag{2.2c}$$

where

$$\alpha(r,\theta) = \frac{\partial}{\partial r}[u_r] + \frac{1}{r}[u_r] + \frac{1}{r}\frac{\partial}{\partial\theta}[u_\theta], \tag{2.3a}$$

$$\beta(r,\theta) = (1-\nu)\left\{\frac{1}{r}\frac{\partial}{\partial\theta}[u_r] - \frac{\partial}{\partial r}[u_\theta] - \frac{1}{r}[u_\theta]\right\} \tag{2.3b}$$

and $4\pi(1-\nu)\eta = \mu$.

Next, we expand all quantities in their Fourier series and write

$$\tau_{kz}^{(i)}(r,\theta)=\tfrac{1}{2}\eta\sum_{n=0}^{\infty}\varepsilon_n(P_n^k(r)\cos n\theta+Q_n^k(r)\sin n\theta), \tag{2.4a}$$

$$[u_k(r,\theta)]=\tfrac{1}{2}\sum_{n=0}^{\infty}\varepsilon_n(f_n^k(r)\cos n\theta+g_n^k(r)\sin n\theta), \tag{2.4b}$$

$$\alpha(r,\theta)=\tfrac{1}{2}\sum_{n=0}^{\infty}\varepsilon_n(\alpha_n(r)\cos n\theta+\tilde{\alpha}_n(r)\sin n\theta),$$

$$\beta(r,\theta)=\tfrac{1}{2}\sum_{n=0}^{\infty}\varepsilon_n(\tilde{\beta}_n(r)\cos n\theta+\beta_n(r)\sin n\theta),$$

and

$$R^{-1}=\tfrac{1}{2}\sum_{n=0}^{\infty}\varepsilon_n I_n(r,r')\cos n(\theta-\theta'),$$

where $k=r$, θ or z and ε_n is the Neumann factor: $\varepsilon_0=1$, $\varepsilon_n=2$ for $n>0$. Substituting these expansions into (2.2) and integrating over θ, we obtain the following integral equations.

Normal problem:

$$\int_0^1\left\{\frac{d}{dr}f_n^z(r)\frac{\partial}{\partial r}I_n(r,r')+\frac{n^2}{r^2}f_n^z(r)I_n(r,r')\right\}r\,dr=-\frac{1}{\pi}P_n^z(r'). \tag{2.5a}$$

Shear problem:

$$\int_0^1\left\{\alpha_n(r)\frac{\partial}{\partial r'}I_n(r,r')+\frac{n}{r'}\beta_n(r)I_n(r,r')\right\}r\,dr=\frac{1}{\pi}P_n^r(r'), \tag{2.5b}$$

$$\int_0^1\left\{\beta_n(r)\frac{\partial}{\partial r'}I_n(r,r')+\frac{n}{r'}\alpha_n(r)I_n(r,r')\right\}r\,dr=-\frac{1}{\pi}Q_n^\theta(r'). \tag{2.5c}$$

g_n^z, $\tilde{\alpha}_n$ and $\tilde{\beta}_n$ satisfy similar equations, obtained by means of the transformations $f_n^z\to g_n^z$, $P_n^z\to Q_n^z$, $\alpha_n\to\tilde{\alpha}_n$, $\beta_n\to-\tilde{\beta}_n$, $P_n^r\to Q_n^r$ and $Q_n^\theta\to-P_n^\theta$.

Next, Guidera and Lardner introduce the quantity L_n, defined by

$$L_n(r,r')=r^n\frac{\partial}{\partial r}(r^{-n}I_n(r,r')), \tag{2.6}$$

and the integral transform defined by

$$\bar{\phi}(r)=\frac{1}{r^{n+1}}\int_0^r\rho^{n+1}\phi(\rho)\,d\rho; \tag{2.7}$$

(2.5) then become the following integral equations.

Normal problem (GL(3.14)):

$$\int_0^1\left\{\frac{d}{dr}f_n^z(r)-\frac{n}{r}f_n^z(r)\right\}L_n(r,r')r\,dr=-\frac{1}{\pi}P_n^z(r'). \tag{2.8}$$

Shear problem (GL(3.15, 3.16)):

$$\int_0^1 \{(n+1)\bar{\alpha}_n - n\bar{\beta}_n - r\alpha_n\} L_n(r, r') r\, dr = P_n^*(r'), \tag{2.9a}$$

$$\int_0^1 \{(n+1)\bar{\beta}_n - n\bar{\alpha}_n - r\beta_n\} L_n(r, r') r\, dr = Q_n^*(r'), \tag{2.9b}$$

where

$$P_n^*(r') = \frac{r'}{\pi} P_n^r(r') - n(1+n\nu) H_n I_n(1, r'),$$

$$Q_n^*(r') = -\frac{r'}{\pi} Q_n^\theta(r') - n(1-\nu-n\nu) H_n I_n(1, r'),$$

and the constant H_n is given by

$$H_n = \int_0^1 \rho^n \{f_n^r(\rho) - g_n^\theta(\rho)\}\, d\rho. \tag{2.10}$$

The solution of (2.8) provides f_n^z directly, whereas the solution of (2.9) initially provides only α_n and β_n. However, it can be shown that f_n^r and g_n^θ are given by (GL(3.23))

$$f_n^r(r) = -\frac{1}{2r} \int_r^1 \left\{ \alpha_n(\rho) \left(\left(\frac{\rho}{r}\right)^n + \left(\frac{r}{\rho}\right)^n \right) - \frac{\beta_n(\rho)}{1-\nu} \left(\left(\frac{\rho}{r}\right)^n - \left(\frac{r}{\rho}\right)^n \right) \right\} \rho\, d\rho \tag{2.11a}$$

and

$$g_n^\theta(r) = -\frac{1}{2r} \int_r^1 \left\{ \alpha_n(\rho) \left(\left(\frac{\rho}{r}\right)^n - \left(\frac{r}{\rho}\right)^n \right) - \frac{\beta_n(\rho)}{1-\nu} \left(\left(\frac{\rho}{r}\right)^n + \left(\frac{r}{\rho}\right)^n \right) \right\} \rho\, d\rho. \tag{2.11b}$$

The integral equations (2.8) and (2.9) are all of the form

$$\int_0^1 \phi(r) L_n(r, r') r\, dr = \Phi(r'), \qquad 0 < r' < 1. \tag{2.12}$$

The solution of this equation is (GL, Appendix I)

$$\phi(r) = \frac{r^n}{\pi} \frac{\partial}{\partial r} \int_r^1 \frac{dt}{t^{2n}(t^2-r^2)^{1/2}} \int_0^t \frac{\Phi(s) s^{n+1}\, ds}{(t^2-s^2)^{1/2}}. \tag{2.13}$$

For the normal problem, we use this result to solve (2.8), and obtain an ordinary differential equation which may be integrated immediately to yield (GL(4.1))

$$f_n^z(r) = -\frac{r^n}{\pi^2} \int_r^1 \frac{dt}{t^{2n}(t^2-r^2)^{1/2}} \int_0^t \frac{P_n^z(s) s^{n+1}\, ds}{(t^2-s^2)^{1/2}}. \tag{2.14}$$

For the shear problem, we write

$$\psi(r) = r^{n+1}(\alpha_n + \beta_n), \qquad \chi(r) = r^{n+1}(\alpha_n - \beta_n), \tag{2.15}$$

and then use (2.13) to solve (2.9), yielding two uncoupled integral equations for ψ and χ. These equations are easily solved, giving (GL(5.8, 5.9))

$$\psi(r) = \Psi(r) + \int_0^r \Psi(\rho)\frac{d\rho}{\rho}, \tag{2.16a}$$

$$\chi(r) = X(r) + D_n r^{2n} - (2n+1)r^{2n}\int_r^1 X(\rho)\frac{d\rho}{\rho^{2n+1}}, \tag{2.16b}$$

where

$$\Psi(r) = F_n(r) + G_n(r) + (2-\nu)nH_nM_n(r), \tag{2.17a}$$

$$X(r) = F_n(r) - G_n(r) + n\nu(2n+1)H_nM_n(r), \tag{2.17b}$$

$$\begin{Bmatrix} F_n(r) \\ G_n(r) \end{Bmatrix} = -\frac{r^{2n}}{\pi^2}\frac{\partial}{\partial r}\int_r^1 \frac{dt}{t^{2n}(t^2-r^2)^{1/2}}\int_0^t \begin{Bmatrix} P_n^r(s) \\ Q_n^\theta(s) \end{Bmatrix}\frac{s^{n+2}\,ds}{(t^2-s^2)^{1/2}}, \tag{2.18}$$

$$M_n(r) = \frac{2}{\pi}r^{2n}\frac{\partial}{\partial r}f^n(r),$$

$$f^n(r) = \int_r^1 \frac{dt}{t^{2n}(t^2-r^2)^{1/2}}\int_0^t \frac{s^{2n}\,ds}{(1-s^2)^{1/2}}, \tag{2.19}$$

and the constant D_n is given by

$$D_n = -n\nu(2n+1)H_n. \tag{2.20}$$

Finally, it can be shown that H_n is given by (GL(5.11))

$$2(2-\nu)n\pi E_nH_n = -\int_0^1 \rho^n(1-\rho^2)^{1/2}\{P_n^r(\rho) - Q_n^\theta(\rho)\}\,d\rho,$$

where

$$E_n = \int_0^{\pi/2} \sin^{2n}\theta\,d\theta = \tfrac{1}{2}\pi^{1/2}\frac{\Gamma(n+\frac{1}{2})}{n!}. \tag{2.21}$$

This completes the formal solution of P, as given by Guidera and Lardner [4].

3. The exact Green's function – 1

To find the exact Green's function, we need to solve three separate boundary-value problems, corresponding to the following loadings of the crack.
Problem I: $\tau_{zz} = -\mu\delta(\boldsymbol{r}-\boldsymbol{r}_0)$, $\tau_{rz} = \tau_{\theta z} = 0$, $z = 0$;
Problem II: $\tau_{rz} = -\mu\delta(\boldsymbol{r}-\boldsymbol{r}_0)$, $\tau_{\theta z} = \tau_{zz} = 0$, $z = 0$; and
Problem III: $\tau_{\theta z} = -\mu\delta(\boldsymbol{r}-\boldsymbol{r}_0)$, $\tau_{zz} = \tau_{rz} = 0$, $z = 0$,
where the concentrated loads are applied at the point $(r_0, \theta_0, 0)$ on each side of the crack. We shall denote the corresponding discontinuities in the displacement vector by $[G_{ij}(\boldsymbol{r}; \boldsymbol{r}_0)]$, where $j = 1, 2, 3$ (or r, θ, z) correspond to the solutions of problems I, II, III, respectively.

Problem I may be solved immediately; using (2.4a) and (2.14), we have

$$[G_{zz}(\boldsymbol{r};\boldsymbol{r}_0)]=\tfrac{1}{2}\sum_{n=0}^{\infty}\varepsilon_n f_n^z(r,r_0)\cos n(\theta-\theta_0), \tag{3.1}$$

where

$$f_n^z(r,r_0)=4(1-\nu)(rr_0)^n g^n(r,r_0)/\pi^2 \tag{3.2}$$

and the function $g^n(r,s)$ is defined by

$$g^n(r,s)=\int_{\bar r}^{1}\frac{dt}{t^{2n}(t^2-r^2)^{1/2}(t^2-s^2)^{1/2}}, \tag{3.3}$$

with $\bar r=\max(r,s)$. In addition, we have

$$[G_{rz}]=[G_{\theta z}]=0.$$

$g^n(r,s)$ may be expressed in terms of elliptic integrals; we have ([2], §§ 216, 320)

$$\bar r g^0(r,s)=F(\psi,k),$$

$$(rs)^2 g^1(r,s)=\bar r\{F(\psi,k)-E(\psi,k)+k^2\bar r\sin\psi\},$$

and

$$(2n-1)(rs)^2 g^n=(2n-2)(r^2+s^2)g^{n-1}+(3-2n)g^{n-2}-(1-r^2)^{1/2}(1-s^2)^{1/2},$$

where the recurrence relation holds for $n>1$; if $\bar r=r$, then $k=s/r$; if $\bar r=s$, then $k=r/s$; $\sin\psi=(1-\bar r^2)^{1/2}(1-(k\bar r)^2)^{-1/2}$; and $F(\psi,k)$, $E(\psi,k)$ are the incomplete elliptic integrals of the first and second kind, respectively.

In the next section, we shall obtain the solutions to problems II and III. A suprising feature of these solutions is that they, too, may all be written in terms of the functions $g^m(r,r_0)$, powers of r and r_0, and trigonometric functions of $\theta-\theta_0$.

4. The exact Green's function – 2

Rather than solve problems II and III, it turns out to be more convenient to solve two related problems, corresponding to concentrated forces acting in the x and y directions, given by

Problem II′: $\tau_{xz}=-\mu\delta(\boldsymbol{r}-\boldsymbol{r}_0)$, $\tau_{yz}=\tau_{zz}=0$, $z=0$; and

Problem III′: $\tau_{yz}=-\mu\delta(\boldsymbol{r}-\boldsymbol{r}_0)$, $\tau_{zz}=\tau_{xz}=0$, $z=0$,

where $x=r\cos\theta$ and $y=r\sin\theta$. Let us consider Problem II′; the solution of III′ may be found by suitably transforming the solution of II′. We write the displacement discontinuitites as

$$[G_{\beta x}(\boldsymbol{r};\boldsymbol{r}_0)]=\tfrac{1}{2}\sum_{n=0}^{\infty}\varepsilon_n(f_n^\beta(r,\boldsymbol{r}_0)\cos n\theta+g_n^\beta(r,\boldsymbol{r}_0)\sin n\theta), \tag{4.1}$$

where $\beta = r$ or θ, and $[G_{zx}] = 0$. From (2.11) and (2.15), we have

$$2(1-\nu)(f_n^r + g_n^\theta) = \frac{1}{r^{n+1}} \int_r^1 (\nu\psi(\rho) + (\nu-2)\chi(\rho))\, d\rho, \tag{4.2a}$$

$$2(1-\nu)(f_n^r - g_n^\theta) = r^{n-1} \int_r^1 \frac{1}{\rho^{2n}} ((\nu-2)\psi(\rho) + \nu\chi(\rho))\, d\rho, \tag{4.2b}$$

where ψ and χ, which are given by (2.16), are related in a complicated way to the Fourier components of the applied stresses; these are obtained from (2.4a) as

$$\begin{Bmatrix} P_n^r(r) \\ Q_n^\theta(r) \end{Bmatrix} = -4(1-\nu)\frac{\delta(r-r_0)}{r} \begin{Bmatrix} \cos\theta_0 \cos n\theta_0 \\ -\sin\theta_0 \sin n\theta_0 \end{Bmatrix},$$

with similar expressions for P_n^θ and Q_n^r. Having found these, the next step is to compute Ψ and X, as given by (2.17); the result is

$$\Psi(r) = r^{2n}\frac{\partial}{\partial r} U^1(r), \qquad X(r) = r^{2n}\frac{\partial}{\partial r} U^2(r), \tag{4.3}$$

where

$$U^i(r) = A_n^i g^n(r, r_0) + B_n^i f^n(r), \qquad i = 1, 2, \tag{4.4}$$

$$A_n^1 = A_n \cos(n-1)\theta_0, \qquad A_n^2 = A_n \cos(n+1)\theta_0, \tag{4.5}$$

$$B_n^1 = (2-\nu)B_n, \qquad B_n^2 = (2n+1)\nu B_n, \tag{4.6}$$

$$\pi^2 A_n = 4(1-\nu)r_0^{n+1} \tag{4.7}$$

and

$$\pi^2 B_n = 2n\pi H_n = \frac{4(1-\nu)}{(2-\nu)E_n} r_0^{n-1}(1-r_0^2)^{1/2} \cos(n-1)\theta_0; \tag{4.8}$$

$f^n(r)$ and $g^n(r, r_0)$ are given by (2.19) and (3.3), respectively.

We now wish to evaluate the integrals in (4.2). For example, using (2.16a) and (4.3), we have

$$\int_r^1 \psi(\rho)\, d\rho = \int_0^1 \Psi(\rho)\frac{d\rho}{\rho} - r\int_0^r \Psi(\rho)\frac{d\rho}{\rho}$$

$$= -r^{2n}U^1(r) + (2n-1)r\int_0^r \rho^{2n-2}U^1(\rho)\, d\rho - (2n-1)\int_0^1 \rho^{2n-2}U^1(\rho)\, d\rho;$$

the remaining integrals in (4.2) can be reduced in a similar manner. Collecting these

results together, we obtain

$$2(1-\nu)(g_n^\theta+f_n^r)=r^{n-1}\{(2-\nu)U^2-\nu U^1\}-(2n-1)\frac{\nu}{r^n}\left\{I_n+C_n\left(\frac{1}{r}-1\right)\right\}$$
$$-(2-\nu)r^n\left\{J_n+\frac{D_n}{2n+1}\left(\left(\frac{1}{r}\right)^{2n+1}-1\right)\right\}, \tag{4.9a}$$

$$2(1-\nu)(g_n^\theta-f_n^r)=r^{n-1}\{\nu U^2-(2-\nu)U^1\}+\frac{(2-\nu)}{r^n}\{I_n+C_n(r^{2n-1}-1)\}$$
$$-(2n+1)\nu r^n\left\{J_n+\frac{D_n}{2n+1}\left(\frac{1}{r}-1\right)\right\}, \tag{4.9b}$$

where D_n is given by (2.20) and (4.8),

$$I_n(r)=\int_r^1 \rho^{2n-2}U^1(\rho)\,d\rho, \qquad J_n(r)=\int_r^1 \frac{1}{\rho^2}U^2(\rho)\,d\rho$$

and $C_n=I_n(0)$; these integrals may be evaluated explicitly (the rather lengthy details are omitted; they are given in [7], Appendix A). We have

$$I_n(r)=A_n^1 r^{2n-1}(g^n-g^{n-1}/r_0^2)+B_n^1\frac{1}{r}\left\{\frac{\pi}{4n(2n-1)}(2nr-r^{2n}-(2n-1))-E_nS_{n-1}(r)\right\},$$

$$J_n(r)=A_n^2\frac{1}{r}(g^n-r^2g^{n+1})$$
$$+B_n^2\frac{1}{r}\left\{\frac{\pi}{4n(2n+1)}(2n+1-2nr-r^{-2n})+r^{-2n}E_n(S_n(r)-S_{n-1}(r))\right\},$$

$$C_n=\frac{(2-\nu)\pi B_n}{2(2n-1)} \quad \text{and} \quad D_n=-\tfrac{1}{2}(2n+1)\nu\pi B_n,$$

where

$$S_m(r)=-\int_r^1 \frac{s^{2m+1}\,ds}{(1-s^2)^{1/2}},$$

and E_n is given by (2.21).

We now substitute these expressions into (4.9). After some manipulation, we see that the coefficients multiplying $f^n(r)$ are both zero. The Fourier components take on the following simple forms.

$$2(1-\nu)(g_n^\theta+f_n^r)=A_n^* r^{n+1}g^{n+1}+2n\nu A_n^1 r^{n+1}g^n/r_0^2, \tag{4.10a}$$

$$2(1-\nu)(g_n^\theta-f_n^r)=\alpha A_n^*\{(2n+1)r^{n+1}g^{n+1}-2nr^{n-1}g^n\}$$
$$+\alpha A_n^1\frac{2n\nu}{r_0^2}\{(2n+1)r^{n+1}g^n-2nr^{n-1}g^{n-1}\}-\frac{4(1-\nu)}{(2-\nu)r_0^2}A_n^1 r^{n-1}g^{n-1}, \tag{4.10b}$$

where $\alpha=\nu/(2-\nu)$ and $A_n^*=(2-\nu)A_n^2-(2n+1)\nu A_n^1$; these expressions are valid for $n>0$.

For $n=0$, we proceed somewhat differently. From (2.11a), we have

$$f_0^r(r)=\bar{\alpha}_0(r). \tag{4.11}$$

Also, from (2.13), we see that the solution, $\phi(r)$, of (2.12), for $n=0$, is given by

$$\int_r^1 \phi(\rho)\,d\rho=-\frac{1}{\pi}\int_r^1 \frac{dt}{(t^2-r^2)^{1/2}}\int_0^t \frac{\Phi(s)s\,ds}{(t^2-s^2)^{1/2}}. \tag{4.12}$$

Since α_0 satisfies (2.9a), with $n=0$, we can use (4.12) to give

$$\int_r^1 (\bar{\alpha}_0(\rho)-\rho\alpha_0(\rho))\,d\rho=A_0^1 g^0(r,r_0).$$

Hence, using (4.11), we have

$$rf_0^r(r)+\int_r^1 f_0^r(\rho)\,d\rho=A_0^1 g^0(r,r_0)$$

which may be solved to yield

$$f_0^r(r)=-A_0^1\int_r^1 \frac{1}{\rho}\frac{\partial}{\partial\rho}\,g^0(\rho,r_0)\,d\rho=A_0 r g^1(r,r_0)\cos\theta_0. \tag{4.13}$$

Similarly, we have

$$(1-\nu)f_0^\theta(r)=-A_0 r g^1(r,r_0)\sin\theta_0.$$

In order to obtain the coefficients g_n^r and f_n^θ, make the transformations

$$f_n^r\to g_n^r,\qquad g_n^\theta\to -f_n^\theta \quad\text{and}\quad \cos(n\pm1)\theta_0\to\sin(n\pm1)\theta_0.$$

For Problem III′, write

$$[G_{ky}(\boldsymbol{r};\boldsymbol{r}_0)]=\tfrac{1}{2}\sum_{n=0}^{\infty}\varepsilon_n(F_n^k(r)\cos n\theta+G_n^k(r)\sin n\theta),$$

where $k=r$ or θ, and $[G_{zy}]=0$. It is straightforward to show that we can calculate the Fourier components for III′ from those for II′ by making the transformations

$$f_n^k\to F_n^k,\qquad g_n^k\to G_n^k,$$

$$\cos(n\pm1)\theta_0\to\pm\sin(n\pm1)\theta_0 \quad\text{and}\quad \sin(n\pm1)\theta_0\to\mp\cos(n\pm1)\theta_0.$$

To obtain the solutions to Problems II and III, we combine the solutions of II′ and III′ in a suitable manner; we have

$$[G_{\alpha r}]=[G_{\alpha x}]\cos\theta_0+[G_{\alpha y}]\sin\theta_0,$$

$$[G_{\alpha\theta}]=-[G_{\alpha x}]\sin\theta_0+[G_{\alpha y}]\cos\theta_0,$$

and $[G_{z\alpha}]=0$, with $\alpha=r$ or θ. Let us write

$$f_n^r(r)\equiv f_n(\cos(n+1)\theta_0;\cos(n-1)\theta_0)$$

and

$$g_n^\theta(r)\equiv g_n(\cos(n+1)\theta_0;\cos(n-1)\theta_0),$$

in order to exhibit the dependence on θ_0. Then, we obtain the following solutions to II and III.

$$[G_{rr}]=\tfrac{1}{2}\sum_{n=0}^{\infty}\varepsilon_n f_n^+\cos n(\theta-\theta_0),\qquad [G_{\theta r}]=\sum_{n=1}^{\infty} g_n^+\sin n(\theta-\theta_0), \tag{4.14a}$$

$$[G_{r\theta}]=-\sum_{n=1}^{\infty} f_n^-\sin n(\theta-\theta_0),\qquad [G_{\theta\theta}]=\tfrac{1}{2}\sum_{n=0}^{\infty}\varepsilon_n g_n^-\cos n(\theta-\theta_0), \tag{4.14b}$$

where

$$f_n^{\pm}(r, r_0)\cos n\theta_0=f_n(\cos n\theta_0;\pm\cos n\theta_0) \tag{4.15a}$$

and

$$g_n^{\pm}(r, r_0)\cos n\theta_0=g_n(\cos n\theta_0;\pm\cos n\theta_0). \tag{4.15b}$$

This completes the construction of the exact Green's function, for the penny-shaped crack. Although the analysis is formal, it can be shown that G_{ij} satisfies P.1, P.3, P.4 and exhibits the correct "point-force behaviour", near $\boldsymbol{r}=\boldsymbol{r}_0$; the necessary proofs are given in [7]. Instead of giving a detailed examination of G_{ij}, we shall simply use it to solve P; this will be done in the next two sections.

5. Penny-shaped crack subjected to arbitrary normal loading

Consider an arbitrary normal loading of a penny-shaped crack. Then, the only non-zero component of $[u_i]$ is $[u_z]$, and this is given by (1.7), (3.1) and (3.2) as

$$[u_z(\boldsymbol{r}_0)]=-\int_\gamma \frac{1}{\mu}\tau_{zz}^{(i)}(\boldsymbol{r})[G_{zz}(\boldsymbol{r};\boldsymbol{r}_0)]\,dS, \tag{5.1}$$

where

$$[G_{zz}]=\frac{2(1-\nu)}{\pi^2}\sum_{n=0}^{\infty}\varepsilon_n(rr_0)^n g^n(r, r_0)\cos n(\theta-\theta_0). \tag{5.2}$$

Suppose, for simplicity, that the loading is symmetric about $\theta=0$; hence, we may write

$$\frac{1}{\mu}\tau_{zz}^{(i)}(\boldsymbol{r})=\sum_{m=0}^{\infty}\tau_m(r)\cos m\theta. \tag{5.3}$$

Substituting (5.2) and (5.3) into (5.1), and integrating over θ, we obtain

$$[u_z(\boldsymbol{r}_0)]=\sum_{n=0}^{\infty} w_n(r_0)\cos n\theta_0,$$

where

$$\begin{aligned}\pi w_n(s)&=-4(1-\nu)s^n\int_0^1 r^{n+1}\tau_n(r)g^n(r, s)\,dr\\&=-4(1-\nu)s^n\int_s^1\frac{dt}{t^{2n}(t^2-s^2)^{1/2}}\int_0^t\frac{r^{n+1}\tau_n(r)\,dr}{(t^2-r^2)^{1/2}},\end{aligned} \tag{5.4}$$

and we have used (3.3); thus, we have recovered (2.14), the formula given by Guidera and Lardner. (5.4) may also be derived, in a straightforward manner, from the solution obtained by Bell [1].

6. Penny-shaped crack subjected to arbitrary shear loading

Consider an arbitrary shear loading of a penny-shaped crack. Then, $[u_z]=0$ and, from (1.7),

$$[u_\beta(\boldsymbol{r}_0)]=-\frac{1}{\mu}\int_\gamma \{\tau_{rz}^{(i)}(\boldsymbol{r})[G_{r\beta}(\boldsymbol{r};\boldsymbol{r}_0)]+\tau_{\theta z}^{(i)}(\boldsymbol{r})[G_{\theta\beta}(\boldsymbol{r};\boldsymbol{r}_0)]\}\,dS, \tag{6.1}$$

where $\beta=r$ or θ and $[G_{\alpha\beta}]$ is given by (4.14). Let us again suppose that the loading is symmetric about $\theta=0$, and so we may write

$$\frac{1}{\mu}\tau_{rz}^{(i)}(\boldsymbol{r})=\sum_{m=0}^{\infty} t_m(r)\cos m\theta, \qquad \frac{1}{\mu}\tau_{\theta z}^{(i)}(\boldsymbol{r})=\sum_{m=1}^{\infty} s_m(r)\sin m\theta.$$

Substituting these expressions into (6.1), and integrating over θ, we obtain

$$[u_r(\boldsymbol{r}_0)]=\sum_{n=0}^{\infty} u_n(r_0)\cos n\theta_0, \qquad [u_\theta(\boldsymbol{r}_0)]=\sum_{n=1}^{\infty} v_n(r_0)\sin n\theta_0, \tag{6.2}$$

where, for $n>0$,

$$u_n(r_0)\pm v_n(r_0)=-\pi\int_0^1 \{t_n(f_n^+\pm f_n^-)+s_n(g_n^+\pm g_n^-)\}r\,dr, \tag{6.3}$$

and $f_n^\pm$, $g_n^\pm$ are defined by (4.15). Using (4.10) and (4.15) in (6.3), and interchanging the order of integration, we eventually arrive at the following formulae for $u_n\pm v_n$.

$$u_n(s)+v_n(s)=-s^{n+1}\int_s^1 \frac{R_n(t)\,dt}{t^{n+1}(t^2-s^2)^{1/2}}, \tag{6.4}$$

$$u_n(s)-v_n(s)=-\alpha s^{n-1}\int_s^1 \frac{(2nt^2-(2n+1)s^2)}{t^{n+1}(t^2-s^2)^{1/2}}R_n(t)\,dt-s^{n-1}\int_s^1 \frac{R_n^*(t)\,dt}{t^{n-1}(t^2-s^2)^{1/2}}, \tag{6.5}$$

where

$$R_n(t)=\frac{2}{\pi t^{n+1}}\int_0^t \frac{r^n}{(t^2-r^2)^{1/2}}\{(2-\nu)r^2(t_n+s_n)-\nu(t_n-s_n)((2n+1)r^2-2nt^2)\}\,dr, \tag{6.6}$$

$$R_n^*(t)=\frac{8(1-\nu)}{\pi(2-\nu)t^{n-1}}\int_0^t \frac{r^n(t_n-s_n)}{(t^2-r^2)^{1/2}}\,dr, \tag{6.7}$$

and $\alpha=\nu/(2-\nu)$. For $n=0$, we have

$$\pi u_0(s)=-4(1-\nu)s\int_s^1 \frac{dt}{t^2(t^2-s^2)^{1/2}}\int_0^t \frac{t_0(r)r^2\,dr}{(t^2-r^2)^{1/2}}. \tag{6.8}$$

This last formula may also be derived directly from [4]; alternatively it may be derived using the method of dual integral equations for axially symmetric stress distributions (see e.g. [11]).

This completes our solution of the boundary-value problem P, for a penny-shaped crack subjected to arbitrary loadings. To check our solutions, we could solve some particular problems, with known solutions [11] (in [7], four such problems are solved). Instead, we shall compare our solutions for arbitrary loadings with those obtained by Krenk [6]; we do this in the next section.

7. Comparison with Krenk's solution

Let us begin by considering the normal problem, as described in §5. We have already noted that our solution for this problem agrees with those of Guidera and Lardner [4] and Bell [1] (see §5). Krenk [6] solves this problem by deriving some integral equations for w_n, which he solves by expanding τ_n and w_n as follows.

$$\tau_n(r) = r^n \sum_{j=0}^{\infty} S_{2j+1}^n \frac{\Gamma(n+\frac{1}{2})\Gamma(j+\frac{3}{2})}{(n+j)!} \frac{C_{2j+1}^{n+1/2}((1-r^2)^{1/2})}{(1-r^2)^{1/2}}, \tag{7.1}$$

$$w_n(s) = s^n \sum_{j=0}^{\infty} W_{2j+1}^n \frac{\Gamma(n+\frac{1}{2})j!}{\Gamma(n+j+\frac{3}{2})} C_{2j+1}^{n+1/2}((1-s^2)^{1/2}), \tag{7.2}$$

where the coefficients S_{2j+1}^n are assumed to be known and $C_m^\lambda(x)$ is a Gegenbauer polynomial of degree m with index λ ([3], §10.9); these polynomials are orthogonal and satisfy

$$\int_0^1 \frac{r^{2m+1}}{(1-r^2)^{1/2}} C_{2j+1}^{m+1/2}((1-r^2)^{1/2}) C_{2k+1}^{m+1/2}((1-r^2)^{1/2})\, dr = h_j^m \delta_{jk}, \tag{7.3}$$

where

$$h_j^m = \frac{\pi(m+j)!\Gamma(m+j+\frac{3}{2})}{(2m+4j+3)j!\Gamma(j+\frac{3}{2})\{\Gamma(m+\frac{1}{2})\}^2}.$$

Using the expansions (7.1) and (7.2), and the orthogonality relation (7.3), Krenk obtained a formula relating W_{2j+1}^n to S_{2j+1}^n. Suppose, now, that we substitute Krenk's expansions, (7.1) and (7.2), into our formula for w_n, (5.4), and then use (7.3); we get

$$\frac{W_{2j+1}^n}{(2n+4j+3)} = -(1-\nu)\left\{\frac{2}{\pi}\Gamma(n+\tfrac{1}{2})\right\}^2 \frac{\Gamma(j+\frac{3}{2})}{(n+j)!} \sum_{k=0}^{\infty} S_{2k+1}^n \frac{\Gamma(k+\frac{3}{2})}{(n+k)!} I_{jk}^n \tag{7.4}$$

where

$$I_{jk}^m = \int_0^1 K_j^m(t) K_k^m(t)\, dt \tag{7.5}$$

and

$$t^m K_j^m(t) = \int_0^t \frac{r^{2m+1} C_{2j+1}^{m+1/2}((1-r^2)^{1/2})}{(t^2-r^2)^{1/2}(1-r^2)^{1/2}}\, dr. \tag{7.6}$$

In the Appendix, we prove that

$$K_j^m(t) = \frac{\pi(m+j)!\,t^{m+1}}{2\Gamma(m+\frac{1}{2})\Gamma(j+\frac{3}{2})} P_j^{(m+1/2,\,0)}(1-2t^2), \tag{7.7}$$

where $P_j^{(\alpha,\beta)}(x)$ is a Jacobi polynomial ([3], §10.8). Moreover, we show there that K_j^m are also orthogonal, and satisfy

$$I_{jk}^m = \frac{\delta_{jk}}{(2m+4j+3)} \left(\frac{\pi(m+j)!}{2\Gamma(m+\frac{1}{2})\Gamma(j+\frac{3}{2})}\right)^2. \tag{7.8}$$

Using this result in (7.4), we see that

$$W_{2j+1}^n = -(1-\nu)S_{2j+1}^n,$$

a result that is in agreement with Krenk (K(33) i.e. Eq. (33) of [6]).

Let us now consider the shear problem, as described in §6. For $n>0$, Krenk writes

$$t_n(r)+s_n(r) = r^{n+1} \sum_{j=0}^{\infty} T_{2j+1}^n \frac{\Gamma(n+\frac{3}{2})\Gamma(j+\frac{3}{2})}{(n+j+1)!} \frac{C_{2j+1}^{n+3/2}((1-r^2)^{1/2})}{(1-r^2)^{1/2}}, \tag{7.9a}$$

$$t_n(r)-s_n(r) = r^{n-1} \sum_{j=0}^{\infty} S_{2j+1}^n \frac{\Gamma(n-\frac{1}{2})\Gamma(j+\frac{3}{2})}{(n+j-1)!} \frac{C_{2j+1}^{n-1/2}((1-r^2)^{1/2})}{(1-r^2)^{1/2}}, \tag{7.9b}$$

$$u_n(s)+v_n(s) = s^{n+1} \sum_{j=0}^{\infty} U_{2j+1}^n \frac{\Gamma(n+\frac{3}{2})j!}{\Gamma(n+j+\frac{5}{2})} C_{2j+1}^{n+3/2}((1-s^2)^{1/2}), \tag{7.10}$$

$$u_n(s)-v_n(s) = s^{n-1} \sum_{j=0}^{\infty} V_{2j+1}^n \frac{\Gamma(n-\frac{1}{2})j!}{\Gamma(n+j+\frac{1}{2})} C_{2j+1}^{n-1/2}((1-s^2)^{1/2}), \tag{7.11}$$

where T_{2j+1}^n and S_{2j+1}^n are assumed to be known. Our expressions for $u_n \pm v_n$ involve the functions R_n and R_n^*, defined by (6.6) and (6.7), respectively. Substituting (7.9) into (6.6), we get

$$\begin{aligned}\tfrac{1}{2}\pi R_m(t) = (2-\nu) &\sum_{j=0}^{\infty} T_{2j+1}^m \frac{\Gamma(m+\frac{3}{2})\Gamma(j+\frac{3}{2})}{(m+j+1)!} K_j^{m+1}(t) \\ &- \nu \sum_{j=0}^{\infty} S_{2j+1}^m \frac{\Gamma(m-\frac{1}{2})\Gamma(j+\frac{3}{2})}{(m+j-1)!} L_j^m(t),\end{aligned} \tag{7.12}$$

where $K_j^m(t)$ is defined by (7.7) and

$$t^{m+1}L_j^m(t) = \int_0^t \frac{r^{m-1}\{(2m+1)r^2-2mt^2\}}{(t^2-r^2)^{1/2}(1-r^2)^{1/2}} C_{2j+1}^{m-1/2}((1-r^2)^{1/2})\,dr. \tag{7.13}$$

In the Appendix, we show that

$$L_0^m(t) = 0 \quad \text{and} \quad L_j^m(t) = -\frac{(m-\frac{1}{2})(m+\frac{1}{2})}{(j+\frac{1}{2})(m+j)} K_{j-1}^{m+1}(t), \tag{7.14}$$

for $j>0$. Substitution into (7.12) gives

$$R_m(t) = \frac{2}{\pi} \sum_{j=0}^{\infty} \{(2-\nu)T_{2j+1}^m + \nu S_{2j+3}^m\} \frac{\Gamma(m+\frac{3}{2})\Gamma(j+\frac{3}{2})}{(m+j+1)!} K_j^{m+1}(t). \tag{7.15}$$

We can evaluate R_m^* in a similar fashion; the result is

$$R_m^*(t)=\frac{8(1-\nu)}{\pi(2-\nu)}\sum_{j=0}^{\infty} S_{2j+1}^m \frac{\Gamma(m-\frac{1}{2})\Gamma(j+\frac{3}{2})}{(m+j-1)!} K_j^{m-1}(t). \tag{7.16}$$

Consider first the expression for u_n+v_n. If we use (7.10) and (7.15) in (6.4), and then use the orthogonality relations, (7.3) and (7.8), we find that

$$U_{2k+1}^n=-\tfrac{1}{2}\{(2-\nu)T_{2k+1}^n+\nu S_{2k+3}^n\},$$

for $n>0$, $k\geqslant 0$, in agreement with Krenk (K(38)).

Consider now the expression for u_n-v_n. If we use (7.11) in (6.5), and then use the orthogonality relation (7.3), we obtain

$$\frac{\pi(n+k-1)!}{\Gamma(k+\frac{3}{2})\Gamma(n-\frac{1}{2})}\frac{V_{2k+1}^n}{(2n+4k+1)}=\int_0^1 \{\alpha R_n(t)L_k^n(t)-R_n^*(t)K_k^{n-1}(t)\}\,dt. \tag{7.17}$$

We can now use (7.14), (7.15), (7.16) and the orthogonality relation (7.8): for $k=0$, $L_0^n=0$ and so (7.17) yields

$$V_1^n=\frac{-2(1-\nu)}{(2-\nu)}S_1^n, \qquad n>0;$$

for $k>0$, we find that

$$V_{2k+1}^n=-\tfrac{1}{2}\{(2-\nu)S_{2k+1}^n+\nu T_{2k+1}^n\}, \qquad n>0;$$

these results are also in agreement with Krenk (K(37, 39)).

It only remains to consider what happens when $n=0$; in this case, the functions $s_0(r)$ and $v_0(r)$ are redundant. Krenk supposes that $s_0\equiv t_0$ and $v_0\equiv u_0$, puts $S_{2j+1}^0=V_{2j+1}^0=0$, and then obtains (K(34))

$$U_{2j+1}^0=\frac{-2(1-\nu)}{(2-\nu)}T_{2j+1}^n, \qquad j>0, \tag{7.18}$$

where, from (7.9a) and (7.10),

$$2t_0(r)=r\sum_{j=0}^{\infty} T_{2j+1}^0 \frac{\Gamma(\frac{3}{2})\Gamma(j+\frac{3}{2})}{(j+1)!}\frac{C_{2j+1}^{3/2}((1-r^2)^{1/2})}{(1-r^2)^{1/2}}, \tag{7.19a}$$

and

$$2u_0(s)=s\sum_{j=0}^{\infty} U_{2j+1}^0 \frac{\Gamma(\frac{3}{2})j!}{\Gamma(j+\frac{5}{2})} C_{2j+1}^{3/2}((1-s^2)^{1/2}). \tag{7.19b}$$

If we now use Krenk's expansion, (7.19), in our formula (6.8) (which was derived without making any assumptions about s_0 and v_0), we find that

$$U_{2j+1}^0=-(1-\nu)T_{2j+1}^0, \qquad j>0, \tag{7.20}$$

in contrast with (7.18).

So, with the exception of (7.18), we see that we have agreement with Krenk's solution.

Note added in proof (March 10, 1982): In a private communication, Dr. Krenk has confirmed Eq. (7.20).

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Appendix

In this appendix, we prove some results required in §7. Substituting from ([3], 10.9 (22)) into (7.6), we get

$$t^m K_j^m(t) = \frac{(2m+1)_{2j+1}}{(2j+1)!} \int_0^t \frac{r^{2m+1}}{(t^2-r^2)^{1/2}} \, {}_2F_1(-j, j+m+\tfrac{3}{2}; m+1; r^2)\, dr, \tag{A.1}$$

where $(\lambda)_k = \Gamma(\lambda+k)/\Gamma(\lambda)$ and ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function. Since

$$\int_0^t \frac{r^{2m+1}}{(t^2-r^2)^{1/2}}\, dr = \frac{\pi^{1/2} m!\, t^{2m+1}}{2\Gamma(m+\frac{3}{2})} = t^{2m+1} F_{m+1}, \text{ say}, \tag{A.2}$$

we can integrate (A.1) term-by-term. Rearranging, we obtain

$$K_j^m(t) = \frac{\pi^{1/2} m!\, t^{m+1} (2m+1)_{2j+1}}{2(2j+1)!\, \Gamma(m+\frac{3}{2})} \, {}_2F_1(-j, j+m+\tfrac{3}{2}; m+\tfrac{3}{2}; t^2).$$

If we now use ([3], 10.8(16)), we obtain the desired expression for $K_j^m(t)$, given by (7.7).

The Jacobi polynomials satisfy the following orthogonality relation.

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_j^{\alpha,\beta)}(x)(1-x)^\alpha (1+x)^\beta\, dx = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)\delta_{jn}}{n!\,\Gamma(\alpha+\beta+n+1)(\alpha+\beta+2n+1)}.$$

After some simple transformations, this may be rewritten as

$$(2m+4j+3) \int_0^1 P_j^{(m+1/2,0)}(1-2t^2) P_k^{(m+1/2,0)}(1-2t^2) t^{2m+2}\, dt = \delta_{jk}$$

and hence, we can immediately prove (7.8).

Using ([3], 10.9(12)), (7.13) gives

$$L_0^m(t) = (2m-1)t^m\{(2m+1)F_{m+1} - 2mF_m\} = 0, \quad \text{by} \quad \text{(A.2)}.$$

For $j>0$, we use ([3], 10.9(36)) in (7.13), to give

$$L_j^m(t) = -2mK_j^{m-1}(t) + \frac{(4m^2-1)}{(2m+4j+1)t}\{K_j^m(t) - K_{j-1}^m(t)\}.$$

We now reduce the index of the first term to m, by using ([3], 10.8(35)) and (7.7). If we then use ([3], 10.8(32)), and make some rearrangements, we eventually obtain (7.14), as required.

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