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Diffraction of elastic waves by a penny-shaped crack: analytical and numerical results

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The diffraction of time-harmonic stress waves by a penny-shaped crack in an infinite elastic solid is an important problem in fracture mechanics and in the theory of the ultrasonic inspection of materials. Martin (*Proc. R. Soc. Lond.* A **378**, 263 (1981)) has proved that the corresponding linear boundary-value problem has precisely one solution, and that this solution can be constructed by solving a two-dimensional Fredholm integral equation of the second kind. However, this integral equation has a complicated matrix kernel and the components of its vector solution are coupled. The main purpose of the present paper is to show how Martin's integral equation can be explicitly solved in terms of a sequence of functions, each of which satisfies a very simple scalar integral equation of the second kind; this simplification may be made for *any* incident wave. For an incident *plane* wave, further simplifications are possible. We show that the solution at an arbitrary angle of incidence can be derived from the solution at a particular angle of incidence, namely grazing incidence. The resulting computational procedure is especially attractive if only the stress-intensity factors or the far-field displacements are required. Finally, we present some numerical results for the scattering of a P-wave at normal incidence and an SV-wave at oblique incidence, and compare these with those of other authors.

1. INTRODUCTION

The diffraction of time-harmonic elastic waves by an internal crack in an otherwise unbounded elastic solid is an important problem in fracture mechanics and in the theory of the ultrasonic inspection of materials. When infinitesimal compressional (P) or shear (S) waves are incident upon the stress-free boundary of a homogeneous isotropic elastic solid, they are scattered, in general, as a combination of both modes. The scattered field is further complicated by the excitation of Rayleigh surface waves. These physical complexities are reflected in the difficulty of solving the corresponding mathematical boundary-value problem and so, in particular, there are very few exact solutions; see, for example, Eringen & Suhubi (1975) or Miklowitz (1978). Such solutions, however, play an important role in the validation of computational and heuristic methods, such as Kirchhoff theory or the geometrical theory of diffraction, and, to this extent, the problem considered here is canonical as well as being of relevance in its own right.

Given the paucity of explicit solutions to scattering problems in elastodynamics, a great deal of attention has been directed towards expressing the scattered field in terms of the solution of an integral equation which can be solved either numerically or by analytic approximations. When the scatterer is a crack, it can be shown that the solution may be expressed in terms of the 'crack opening displacement' (c.o.d.), i.e. the discontinuity in the displacement across the crack. Thus, Wickham (1981), by considering a simple two-dimensional problem, has proposed a new method for deriving Fredholm integral equations of the second kind for the c.o.d. This method is based on what is termed a 'crack Green function', and does not depend essentially on the crack geometry. Martin (1981) has shown how such a Green function can be constructed for the penny-shaped crack and has derived a Fredholm integral equation of the second kind that *uniquely determines the c.o.d. in this case*. Thus, he has established the existence of a solution to the original boundary-value problem (which we denote by S) for general incident waves.

Since the problem under consideration is essentially a three-dimensional one, the boundary integral equation derived by Martin (1981) is two-dimensional. Moreover, the components of its vector solution are coupled and the matrix kernel is extremely complicated. The main purpose of the present paper is to show how Martin's integral equation may be solved explicitly in terms of a sequence of functions, each of which satisfies a very simple scalar integral equation of the second kind in one space dimension. To be more specific, let us define cylindrical polar coordinates (r, θ, z) such that the penny-shaped crack γ occupies the region $z = 0$, $0 \leqslant r \leqslant 1$, $0 \leqslant \theta < 2\pi$, and let the displacement vector $\boldsymbol{u}$ have corresponding components (u_r, u_θ, u_z). Now consider, for example, an incident SV-wave (polarized in the plane $\theta = 0$) propagating at an angle ϕ to the z-axis. Then, the c.o.d. (defined by (2.12)) is given by

$$[u_z(r,\theta)] = \eta \sum_{n=0}^{\infty} w_n(r) \cos n\theta, \tag{1.1}$$

$$[u_r(r,\theta)] = \eta \sum_{n=0}^{\infty} u_n(r) \cos n\theta \quad \text{and} \quad [u_\theta(r,\theta)] = \eta \sum_{n=1}^{\infty} v_n(r) \sin n\theta, \tag{1.2}$$

where
$$w_n(r) = D_n \mathscr{A}_n\{\theta_n(x)\}, \quad n \geqslant 0,$$

$$u_n(r) = -\frac{C_n}{4r}\int_r^1 \left(\alpha_n(\rho)\left\{\left(\frac{\rho}{r}\right)^n + \left(\frac{r}{\rho}\right)^n\right\} + \beta_n(\rho)\left\{\left(\frac{\rho}{r}\right)^n - \left(\frac{r}{\rho}\right)^n\right\}\right)\rho\,\mathrm{d}\rho, \quad n \geqslant 1, \tag{1.3a}$$

$$v_n(r) = -\frac{C_n}{4r}\int_r^1 \left(\alpha_n(\rho)\left\{\left(\frac{\rho}{r}\right)^n - \left(\frac{r}{\rho}\right)^n\right\} + \beta_n(\rho)\left\{\left(\frac{\rho}{r}\right)^n + \left(\frac{r}{\rho}\right)^n\right\}\right)\rho\,\mathrm{d}\rho, \quad n \geqslant 1, \tag{1.3b}$$

$$\alpha_n(r) = 2K\sin\phi\, \mathscr{A}_n\{q_n(x)\} - S_n^+(r^n(1-r^2)^{-\frac{1}{2}} - \mathscr{A}_n\{p_n^+(x)\}),$$
$$\beta_n(r) = -S_n^-(r^n(1-r^2)^{-\frac{1}{2}} - \mathscr{A}_n\{p_n^-(x)\}),$$

and
$$u_0(r) = C_0 \mathscr{A}_1\{q_1(x)\}.$$

Here, $\theta_n(x;\, \phi)$ satisfies the one-dimensional Fredholm integral equation of the second kind,

$$\theta_n(x;\, \phi) + \int_0^1 \theta_n(y;\, \phi)\, N_n(x, y)\, \mathrm{d}y = x j_n(Kx \sin \phi), \tag{1.4}$$

where $N_n(x, y)$ is a symmetric kernel given by (3.16); $q_n(x;\, \phi)$ satisfies an equation of this form, but with a slightly different kernel (4.37); p_n^+ and p_n^- are independent of ϕ and also satisfy similar integral equations (4.28); $\mathscr{A}_n$ is an Abel-transform operator, defined by (3.9); K is the shear wavenumber; S_n^+ and S_n^- depend only on $K \sin \phi$, and are given as certain weighted integrals of $p_n^\pm$ and q_n (see §4.1); C_n and D_n are normalizing factors that depend only on ϕ, and are given by (A 2) and (A 1), respectively; $\eta = 4(1-\nu)/\pi$, and ν is Poisson's ratio. Thus, we are able to reduce the problem to the solution of some uncoupled one-dimensional integral equations of the form (1.4), and a few finite quadratures. This simplification may be made for *any* incident wave.

For an incident *plane* wave, further simplifications are possible. We show that the solution at an arbitrary angle of incidence, ϕ, can be derived from the solution at a particular angle of incidence, namely grazing incidence ($\phi = \frac{1}{2}\pi$). The resulting computational procedure is especially attractive if one only wishes to determine the dynamic stress-intensity factors and the far-field displacements (rather than the details of the complete displacement field). Our method is a generalization of a method due to Williams (1982), who studied the two-dimensional problem of diffraction of plane acoustic waves by a finite rigid strip. As Williams (1982, p. 107) remarks, the equations obtained are 'highly reminiscent of various formulae obtained by "invariant imbedding" methods in radiative transfer theory'; consequently, we shall label our method as an 'imbedding method' (even though it does not appear to fit into any of the known classes of imbedding methods, as described by, for example, Kagiwada & Kalaba (1974)).

As an example, we have used the imbedding method to solve S when the incident wave is an SV-wave. Numerical results are presented for the far-field displacements, and these are compared with those obtained by other authors.

The plan of the paper is as follows. In §2, we state the boundary-value problem S and then give a fairly complete review of the literature on its solution. Next, we review Martin's (1981) solution; we reduce S to two sub-problems (the normal problem and the shear problem) and state his (two-dimensional) integral equations for each. In §3, we consider the normal problem in detail; we introduce various transformations to show that each azimuthal harmonic satisfies a simple (one-dimensional) integral equation. If the incident waves are *plane*, these integral equations simplify further (§3.1). In §4, we describe some similar transformations for the shear problem; these lead to coupled pairs of integral equations with non-symmetric kernels. In §4.1, we show how these equations can be uncoupled and, again, further simplifications are exhibited for incident plane waves (§4.2).

In §5, we define the quantities of most physical interest, namely, the dynamic stress-intensity factors and the far-field displacements. We then show how these are related to the solutions of our integral equations.

In § 6, we describe the imbedding method, which is applicable for incident plane waves, and show how this leads to an efficient algorithm for evaluating our solution. Aspects of the numerical implementation of this algorithm and some results for the reflexion of an SV-wave at oblique incidence are given in § 7.

Notation

$h_n^{(1)}(z)$	spherical Hankel function, see § 7.1
$j_n(z)$	spherical Bessel function, see (3.2)
k	compressional wavenumber
K	shear wavenumber
α	$= \nu/(2-\nu)$
$\beta(\xi)$	$= (\xi^2 - K^2)^{\frac{1}{2}}$, see (2.3)
$\gamma(\xi)$	$= (\xi^2 - k^2)^{\frac{1}{2}}$, see (2.2)
ϵ_n	$\epsilon_0 = 1$; $\epsilon_n = 2$ for $n > 0$
η	$= 4(1-\nu)/\pi$
κ	$= k \sin \phi$ (incident P-wave)
	$= K \sin \phi$ (incident SV-wave)
κ_g	$= k$ (incident P-wave)
	$= K$ (incident SV-wave)
$\mathscr{K}$	$= k \sin \Phi$ (scattered P-wave)
	$= K \sin \Phi$ (scattered SV-wave)
λ, μ	Lamé constants
ν	Poisson's ratio
σ	$= k/K$
ϕ	angle of propagation of incident wave; $\phi = 0$ is normal incidence
Φ	angle of propagation of scattered wave, see § 5.1
$[u]$	discontinuity in u across the crack

2. The boundary-value problem, S

We consider a homogeneous, isotropic, elastic solid containing a penny-shaped crack, γ. Suppose that time-harmonic stress waves, of radian frequency ω, are incident on the crack. We wish to determine the scattered waves when the faces of the crack are free from applied tractions. We denote the scattered displacements and stresses by u_i and τ_{ij}, respectively, where a time-dependence of $e^{-i\omega t}$ will be suppressed throughout. Then, u_i is the solution of the following boundary-value problem.

Boundary-value problem S. Determine $u_i(\mathrm{P})$, $\mathrm{P} \in D$, the region exterior to γ, satisfying

(S1) elastodynamic equations of motion in the solid,

$$k^{-2} \operatorname{grad} \nabla \cdot \boldsymbol{u} - K^{-2} \nabla \times \nabla \times \boldsymbol{u} + \boldsymbol{u} = 0, \quad \mathrm{P} \in D; \tag{2.1}$$

(S2) boundary conditions on the crack faces,

$$\tau_{iz}(q) = -\tau_{iz}^{(i)}(q), \quad q \in \gamma,$$

where $u_i^{(i)}$ and $\tau_{ij}^{(i)}$ denote the incident displacements and stresses, respectively;

(S3) radiation conditions (see Martin 1981, p. 265); and

(S4) edge conditions, namely $u_i(\mathrm{P})$ is bounded in D.

We use a standard notation: capital letters P, Q denote points of D; small letters p, q denote points of γ; and $\boldsymbol{n}$ is the unit normal vector, which is assumed to point into D. The wavenumbers k and K are defined by

$$\rho_0 \omega^2 = (\lambda + 2\mu)\, k^2 = \mu K^2,$$

where ρ_0 is the mass density of the solid, and λ and μ are the Lamé constants, related to Poisson's ratio ν by

$$2\nu = \lambda/(\lambda+\mu) = (K^2 - 2k^2)/(K^2 - k^2).$$

The stress tensor τ_{ij} is related to u_i by

$$\tau_{ij}(\mathrm{P}) = \lambda \delta_{ij} \frac{\partial u_m}{\partial x_m} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

where an arbitrary point $\mathrm{P} \in D$ has cartesian coordinates $(x_1, x_2, x_3) = (x, y, z)$. We also define cylindrical polar coordinates (r, θ, z), where the region γ is taken to be $z = 0$, $r < 1$, $0 \leqslant \theta < 2\pi$ and an arbitrary point $q \in \gamma$ will always be assigned plane polar coordinates (r, θ).

Many solutions of S can be found in the literature. In the absence of an extensive survey, we shall attempt to give one here, before describing our solution. (Brief reviews have been made by Kraut (1976) and Datta (1978).) We begin by noting that almost all authors represent $\boldsymbol{u}$ by

$$\boldsymbol{u} = \operatorname{grad} \phi + \nabla \times (\psi \boldsymbol{z}) + \nabla \times \nabla \times (\chi \boldsymbol{z}),$$

where $\boldsymbol{z}$ is a unit vector in the z-direction, and ϕ, ψ and χ are scalar potentials that satisfy

$$(\nabla^2 + k^2)\phi = 0, \quad (\nabla^2 + K^2)\psi = 0 \quad \text{and} \quad (\nabla^2 + K^2)\chi = 0,$$

respectively.

2.1. *Axisymmetric problems*

The earliest solution was given by Filipczynski (1961). He considered the axisymmetric problem of the diffraction of compressional waves (P-waves) at normal incidence. After introducing oblate spheroidal coordinates, he obtained two equations for ϕ and χ ($\psi \equiv 0$), which he solved by the method of separation of variables. For low frequencies, he derived approximations for the far-field potentials.

Robertson (1967) solved the same axisymmetric problem; for $z \geqslant 0$, he used

Hankel transforms to solve (2.1) and obtained the following representations (in cylindrical polar coordinates) for u_r and u_z:

$$u_r(r,z) = \int_0^\infty \{A(\xi)\,(\xi/\gamma)\,\mathrm{e}^{-\gamma z} + C(\xi)\,(\beta/\xi)\,\mathrm{e}^{-\beta z}\}\,\xi J_1(\xi r)\,\mathrm{d}\xi,$$

and
$$u_z(r,z) = \int_0^\infty \{A(\xi)\,\mathrm{e}^{-\gamma z} + C(\xi)\,\mathrm{e}^{-\beta z}\}\,\xi J_0(\xi r)\,\mathrm{d}\xi,$$

where $\gamma^2 = \xi^2 - k^2$, $\beta^2 = \xi^2 - K^2$ and, in order to satisfy the radiation condition,

$$\gamma(\xi) = \begin{cases} (\xi^2-k^2)^{\frac{1}{2}}, & \xi > k \\ -\mathrm{i}(k^2-\xi^2)^{\frac{1}{2}}, & 0 \leqslant \xi \leqslant k \end{cases} \tag{2.2}$$

and
$$\beta(\xi) = \begin{cases} (\xi^2-K^2)^{\frac{1}{2}}, & \xi > K, \\ -\mathrm{i}(K^2-\xi^2)^{\frac{1}{2}}, & 0 \leqslant \xi \leqslant K. \end{cases} \tag{2.3}$$

Application of the boundary conditions then leads to a pair of dual integral equations for a single function related to A and C. These equations can be formally solved by reducing them to a Fredholm integral equation of the second kind, namely

$$\theta(x) + \int_0^1 \theta(y)\,N(x,y)\,\mathrm{d}y = x, \tag{2.4}$$

where the symmetric kernel is given by

$$N(x,y) = (xy)^{\frac{1}{2}} \int_0^\infty \xi H(\xi)\,J_{\frac{1}{2}}(\xi x)\,J_{\frac{1}{2}}(\xi y)\,\mathrm{d}\xi,$$

and
$$K^2 H(\xi) = -(1-\nu)\{(2\xi^2-K^2)^2/(\xi\gamma) - 4\xi\beta - 2(k^2-K^2)\}; \tag{2.5}$$

the normal displacement of the crack surface is given by

$$u_z(r,0) = \frac{2(1-\nu)}{\pi}\int_r^1 \frac{\theta(t)\,\mathrm{d}t}{(t^2-r^2)^{\frac{1}{2}}}, \quad 0 \leqslant r < 1. \tag{2.6}$$

Robertson reduces $N(x,y)$ to a finite integral (using a contour-integral method; see §7.1) and then derives an expansion for $u_z(r,0)$ as a power series in K. Actually, it turns out that our integral equation for this particular problem, namely (3.18), is identical to Robertson's integral equation, (2.4); see §3.1.

Mal (1968*a*) has presented a slight generalization of Robertson's work by considering general axisymmetric loading of the crack faces; two pairs of dual integral equations are derived which reduce to Robertson's in the special case of a normally-incident P-wave. In fact, Mal (1968*b*) has used Robertson's expansion for the kernel to derive a low-frequency approximation for the dynamic stress-intensity factor (see §5.2). Mal (1970) has also presented some numerical results.

Sih & Loeber (1969) have studied the same axisymmetric problem as Mal (1968*a*). However, their paper contains several errors; the corrected equations were given by Embley & Sih (1972) and are essentially those of Robertson for a normal P-wave.

Several other axisymmetric problems, involving a penny-shaped crack, have been investigated. For a normal P-wave, Paul (1969) has considered a crack in an infinite elastic plate, Shindo (1979) has considered an unbounded solid but has included the effect of an axisymmetric magnetic field, and Srivastava *et al.* (1979) have considered a crack at the interface between two bonded dissimilar elastic solids. In addition, Embley & Sih (1972) have examined the effect produced by a suddenly-applied tensile load, and Chen (1979) has considered the diffraction of a step-pulse by a crack situated inside a finite, concentric, circular cylinder.

2.2. *Asymmetric problems*

To solve asymmetric problems, it is natural to exploit the geometry further by replacing all quantities by their Fourier series in the azimuthal coordinate, θ. Thus, Mal (1968*c*) considered the diffraction of normally-incident shear waves (S-waves) polarized in a plane perpendicular to the crack, and wrote, e.g.

$$u_r(r,\theta,z) = U_r(r,z)\cos\theta \quad \text{and} \quad u_\theta(r,\theta,z) = U_\theta(r,z)\sin\theta.$$

Using his previous representations for ϕ, ψ and χ, he obtained two pairs of dual integral equations, from which he obtained low-frequency approximations for the dynamic stress-intensity factors.

Jain & Kanwal (1972) have studied the same asymmetric problem as Mal. They used different integral representations for the scalar potentials, leading to a pair of Fredholm integral equations of the first kind, which they formally converted into Fredholm equations of the second kind. After solving these equations by iteration, they obtained approximations for the far-field amplitudes and the dynamic stress-intensity factors; their results for the latter do not agree with those found by Mal (1968*c*). We have also studied this problem (see Appendix C) and are able to confirm that Jain & Kanwal's results are correct.

The general asymmetric problem has been studied by several authors. Garbin & Knopoff (1973) considered the diffraction of P-waves at oblique incidence to the crack. They used the representation

$$\phi(r,\theta,z) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}n\theta}\int_0^\infty \{P_n^1(\xi) \mp P_n^2(\xi)\}\, J_n(\xi r)\,\mathrm{e}^{-\gamma|z|}\,\mathrm{d}\xi, \quad z \gtrless 0,$$

with similar representations for ψ and χ, and hence obtained three pairs of dual integral equations for each value of n, which they solved for small values of K. However, their aim was to calculate the compressional modulus of a medium permeated by a random distribution of small circular cracks, and this did not require a complete solution of the boundary-value problem. Piau (1978, 1979) has also examined this problem, while Garbin & Knopoff (1975) have extended their method to determine the shear modulus of the same material; for references to related Russian work, see the review by Guz *et al.* (1978).

Visscher (1981) has proposed a novel scheme for solving S. He began by considering the crack to bisect a sphere of unit radius. In each hemisphere, he represented $\boldsymbol{u}$ as a series of regular partial-wave solutions of (2.1); outside the sphere, he represented $\boldsymbol{u}$ as a series of outgoing partial-wave solutions. (A partial-wave solution of (2.1) is obtained by separation of variables in spherical polar coordinates; 'regular' solutions are regular at the origin, and 'outgoing' solutions satisfy the radiation conditions at infinity.) The unknown coefficients in these three series were obtained by using the boundary conditions on the crack faces and on the spherical surface (continuity of displacements and tractions across the interface), and the far-field displacements were computed. It is clear that Visscher's (numerical) method will not yield a good approximation for $\boldsymbol{u}$ near the crack edge (he is aware of the deficiency, and proposes a modification which consists of introducing a toroidal region around the edge, in which the variation of $\boldsymbol{u}$ is supposed to be quasi-static), but this does not seem to affect the accuracy of his far-field results (for P-waves at oblique incidence).

Krenk & Schmidt (1982) began by writing (for $z \geqslant 0$)

$$\phi(r, \theta, z) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}n\theta} \int_0^\infty A_n(\xi)\, \mathrm{e}^{-\gamma z} J_n(\xi r)\, \mathrm{d}\xi,$$

with similar representations for ψ and χ, and then derived expressions for the stresses and displacements on $z = 0$. They then inverted these latter expressions to obtain integral representations for the unknown functions ($A_n(\xi)$, etc.) in terms of the (unknown) displacements of the crack surface. Substituting these into the expressions for the stresses, and using the boundary conditions, yields singular integral equations for the unknown displacements. Krenk & Schmidt reduced these to infinite systems of linear algebraic equations (using series of orthogonal polynomials to represent the unknown displacements), which they solved for various incident waves; we shall examine their numerical results in § 7.

2.3. *Approximate solutions*

In recent years, there has been much interest in finding approximate solutions to S, i.e. formulae that exhibit some of the characteristics of the exact solution, but that can be evaluated *without* solving any integral equations; it is necessary to validate such formulae by comparing them with some exact solutions. Achenbach and his co-workers have used Keller's geometrical theory of diffraction (GTD) to estimate the displacement field at high frequencies and/or at large distances from the crack edge: Gautesen *et al.* (1978) have considered P-waves at normal incidence to the crack and investigated the field near the z-axis, which is a caustic (GTD predicts that the displacement is singular there, and so the theory must be modified); Achenbach *et al.* (1979) have considered oblique P-waves, and obtained results which compared quite favourably with their experiments; and Achenbach *et al.* (1978) have studied the diffraction of waves produced by a point source, located at a finite distance from the crack. Other approximations have been presented by Domany *et al.* (1978), e.g. the 'quasi-static' approximation is obtained by replacing $\rho_i(q)$ by $[\tilde{u}_i(q)]$ in (2.7), below.

2.4. *Martin's solution*

Martin (1981) showed that the unique solution of S, $u_i(\mathrm{P})$, has an integral representation as an elastic double layer, whose density satisfies a two-dimensional Fredholm integral equation of the second kind, i.e.

$$u_k(\mathrm{P}) = \int_\gamma \rho_i(q)\, \Sigma^f_{ijk}(q;\, \mathrm{P})\, n_j\, \mathrm{d}s_q, \tag{2.7}$$

where $\rho_i(q)$ satisfies

$$\rho_k(\mathrm{p}) - \int_\gamma \rho_i(q)\, K_{ik}(q;\, \mathrm{p})\, \mathrm{d}s_q = [\tilde{u}_k(\mathrm{p})]. \tag{2.8}$$

The uniqueness theorem for S is then used to deduce that $+1$ is not an eigenvalue of (2.8).

In (2.7), Σ^f_{ijk} is the stress tensor given by

$$\Sigma^f_{ijk}(\mathrm{P};\, \mathrm{Q}) = \lambda \delta_{ij} \frac{\partial}{\partial x_m} G^f_{mk}(\mathrm{P};\, \mathrm{Q}) + \mu\left(\frac{\partial}{\partial x_i} G^f_{jk} + \frac{\partial}{\partial x_j} G^f_{ik}\right), \tag{2.9}$$

and G^f_{ij} is the fundamental Green function, given by

$$G^f_{ij}(\mathrm{P};\, \mathrm{Q}) = \frac{1}{4\pi\mu}\left\{\delta_{ij}\frac{\mathrm{e}^{\mathrm{i}KR}}{R} + \frac{1}{K^2}\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{\mathrm{e}^{\mathrm{i}KR}}{R} - \frac{\mathrm{e}^{\mathrm{i}kR}}{R}\right)\right\}, \tag{2.10}$$

where $R = |\boldsymbol{r}_\mathrm{P} - \boldsymbol{r}_\mathrm{Q}|$. Equation (2.7) has the property that

$$[u_k(\mathrm{p})] = \rho_k(\mathrm{p}), \tag{2.11}$$

where we use square brackets to denote the discontinuity across γ, i.e.

$$[u_k(\mathrm{q})] = [u_k(r, \theta)] = u_k(r, \theta, 0^+) - u_k(r, \theta, 0^-). \tag{2.12}$$

In (2.8), the kernel K_{ij} is continuous and is given explicitly in Appendix B of Martin (1981). The free term $[\tilde{u}_i]$ is precisely the displacement discontinuity that would be maintained by *static* stresses $-\tau^{(i)}_{iz}$ on the crack faces; formulae for evaluating $[\tilde{u}_i]$ have been given by Martin (1982).

Equation (2.8) (with (2.11)) is a system of three coupled Fredholm integral equations of the second kind for $[u_i(\mathrm{q})]$, $i = 1, 2, 3$. Actually, this system partially decouples (because the crack is flat), yielding two sub-problems:

normal problem (symmetric about $z = 0$)

$$[u_r] = [u_\theta] = 0,$$

$$[u_z(\mathrm{p})] - \int_\gamma [u_z(\mathrm{q})]\, K_{zz}(\mathrm{q};\, \mathrm{p})\, \mathrm{d}s_q = [\tilde{u}_z(\mathrm{p})]; \tag{2.13}$$

shear problem (antisymmetric about $z = 0$)

$$[u_z] = 0,$$

$$[u_r(\mathrm{p})] - \int_\gamma \{[u_r]\, K_{rr} + [u_\theta]\, K_{\theta r}\}\, \mathrm{d}s_q = [\tilde{u}_r(\mathrm{p})], \tag{2.14a}$$

$$[u_\theta(\mathrm{p})] - \int_\gamma \{[u_r]\, K_{r\theta} + [u_\theta]\, K_{\theta\theta}\}\, \mathrm{d}s_q = [\tilde{u}_\theta(\mathrm{p})]; \tag{2.14b}$$

this is a pair of coupled integral equations for $[u_r]$ and $[u_\theta]$.

In the present paper, we shall give a detailed study of the integral equations above and of their solutions.

3. The normal problem

Let $p_0 \in \gamma$ have plane polar coordinates (r_0, θ_0). From Martin) 1981, p. 284), we have

$$K_{zz}(q; p_0) = -\sum_{n=0}^{\infty} \epsilon_n \int_0^\infty \xi^2 H(\xi)\, M_n(\xi, r_0)\, J_n(\xi r)\, d\xi \cos n(\theta-\theta_0),$$

where
$$M_n(\xi, r_0) = \frac{r_0^n}{\pi^2}\int_{r_0}^1 \frac{j_n(\xi t)\, dt}{t^{n-1}(t^2-r_0^2)^{\frac{1}{2}}}; \tag{3.1}$$

$H(\xi)$ occurs in Robertson's (1967) solution and is defined by (2.5), $j_n(z)$ is the spherical Bessel function, related to $J_\nu(z)$ by

$$j_n(z) = (\pi/2z)^{\frac{1}{2}} J_{n+\frac{1}{2}}(z), \tag{3.2}$$

and ϵ_n is the Neumann factor, defined by $\epsilon_0 = 1$, $\epsilon_n = 2$ for $n > 0$.

Let us assume, for simplicity, that the loading of the crack is symmetric about $\theta = 0$ (the analysis for antisymmetric loading is similar). Thus, we can write

$$[u_z(r,\theta)] = \frac{4(1-\nu)}{\pi} \sum_{n=0}^{\infty} w_n(r) \cos n\theta, \tag{3.3}$$

with a corresponding expansion for $[\tilde{u}_z]$. It follows from (2.13) that w_n satisfies

$$w_n(r_0) + \int_0^1 w_n(r)\, K_n(r, r_0)\, r\, dr = \tilde{w}_n(r_0), \tag{3.4}$$

where
$$K_n(r, r_0) = 2\pi \int_0^\infty \xi^2 H(\xi)\, M_n(\xi, r_0)\, J_n(\xi r)\, d\xi. \tag{3.5}$$

From Martin (1982) we see that the free term in (3.4) is given by

$$\tilde{w}_n(r) = r^n \int_r^1 \frac{dt}{t^{2n}(t^2-r^2)^{\frac{1}{2}}} \int_0^t \frac{s^{n+1}\tau_n(s)}{(t^2-s^2)^{\frac{1}{2}}}\, ds, \tag{3.6}$$

where we have written

$$\tau_{zz}^{(i)}(r,\theta,0) = \mu \sum_{n=0}^{\infty} \tau_n(r) \cos n\theta. \tag{3.7}$$

Equation (3.4) is a Fredholm integral equation of the second kind for $w_n(r)$. We have thus reduced the two-dimensional integral equation (2.13) to an infinite system of uncoupled one-dimensional integral equations. However, the kernels of these integral equations, given by (2.5), (3.1) and (3.5), are still complicated. In order to obtain integral equations with simpler kernels, we introduce a new unknown function $\theta_n(t)$, related to $w_n(r)$ by

$$w_n(r) = D_n r^n \int_r^1 \frac{\theta_n(t)\, dt}{t^n(t^2-r^2)^{\frac{1}{2}}}, \tag{3.8}$$

i.e.
$$w_n(r) = D_n \mathscr{A}_n\{\theta_n(t); t \to r\}, \tag{3.9}$$

say; if there is no ambiguity, we shall write (3.9) more concisely as

$$w_n(r) = D_n \mathscr{A}_n \theta_n(t).$$

(Here, D_n is a normalizing factor which is independent of r.) Equation (3.8) is an Abel integral equation for θ_n, whence

$$D_n \theta_n(x) = \frac{-2x^n}{\pi} \frac{\mathrm{d}}{\mathrm{d}x} \int_x^1 \frac{w_n(r)\, r\, \mathrm{d}r}{r^n (r^2 - x^2)^{\frac{1}{2}}} \tag{3.10}$$

i.e.

$$D_n \theta_n(x) = \mathscr{A}_1^{-1}\{w_n(r); \quad r \to x\}, \tag{3.11}$$

say, which defines the inverse operator to $\mathscr{A}_n$.

Using our operator notation, we see that (3.5) and (3.6) may be written as

$$K_n(r, r_0) = \mathscr{A}_n \left\{ t \int_0^\infty f(\xi)\, J_n(\xi r)\, j_n(\xi t)\, \mathrm{d}\xi; \quad t \to r_0 \right\}$$

and

$$\tilde{w}_n(r_0) = \mathscr{A}_n \left\{ t^{-n} \int_0^t s^{n+1} (t^2 - s^2)^{-\frac{1}{2}}\, \tau_n(s)\, \mathrm{d}s; \quad t \to r_0 \right\} \tag{3.12}$$

respectively, where $f(\xi) = (2/\pi)\, \xi^2 H(\xi)$. Also, using (3.8), we obtain

$$\int_0^1 w_n(r)\, J_n(\xi r)\, r\, \mathrm{d}r = D_n \int_0^1 y \theta_n(y)\, j_n(\xi y)\, \mathrm{d}y, \tag{3.13}$$

where we have interchanged the order of integration and used equation 8.5 (33) from Erdélyi *et al.* (1954):

$$\int_0^a r^{\nu+1} J_\nu(\xi r)\, (a^2 - r^2)^\mu\, \mathrm{d}r = 2^\mu \Gamma(\mu + 1)\, \xi^{-\mu-1} a^{\nu+\mu+1} J_{\nu+\mu+1}(a\xi), \tag{3.14}$$

which is valid for $\mathrm{Re}\,(\nu) > -1$ and $\mathrm{Re}\,(\mu) > -1$; hence

$$\int_0^1 w_n(r)\, K_n(r, r_0)\, r\, \mathrm{d}r = D_n \mathscr{A}_n \left\{ t \int_0^\infty f(\xi)\, j_n(\xi t) \int_0^1 y \theta_n(y)\, j_n(\xi y)\, \mathrm{d}y\, \mathrm{d}\xi; \quad t \to r_0 \right\}.$$

If we now apply the inverse operator $\mathscr{A}_n^{-1}$ to the integral equation (3.4), we obtain

$$\theta_n(x) + \int_0^1 \theta_n(y)\, N_n(x, y)\, \mathrm{d}y = D_n^{-1} x^{-n} \int_0^x s^{n+1} (x^2 - s^2)^{-\frac{1}{2}}\, \tau_n(s)\, \mathrm{d}s, \tag{3.15}$$

where the symmetric kernel $N_n(x, y)$ is given by

$$N_n(x, y) = xy \int_0^\infty f(\xi)\, j_n(\xi x)\, j_n(\xi y)\, \mathrm{d}\xi. \tag{3.16}$$

Equation (3.15) is a Fredholm integral equation of the second kind for $\theta_n(x)$. Once θ_n has been determined, w_n is given by (3.8), and the displacement anywhere in D is given by (2.7), (2.11) and (3.3).

3.1. *Plane-wave data*

So far, we have considered an arbitrary incident wave. Further simplifications are possible if we suppose that the incident wave is a plane wave, propagating at an angle ϕ to the z-axis. Then (see Appendix A),

$$\tau_n(r) = D_n(\phi)\, J_n(\kappa r), \tag{3.17}$$

where $\kappa = k\sin\phi$ (for an incident P-wave) or $\kappa = K\sin\phi$ (for an incident SV-wave), and the normalizing factor $D_n(\phi)$ is a known function of ϕ; we shall call (3.17) our 'plane-wave data'. If we now use (3.14), we can evaluate the free term in (3.15) to give

$$\theta_n(x;\kappa)+\int_0^1 \theta_n(y;\kappa)\,N_n(x,y)\,\mathrm{d}y = xj_n(\kappa x), \tag{3.18}$$

where we have indicated explicitly the dependence of θ_n on κ.

If we set $n = 0$ and let $\phi \to 0$, (3.18) becomes identical to the integral equation (2.4) obtained by Robertson (1967), for a plane P-wave at normal incidence to the crack. Indeed, our choice of representation for $w_n(r)$, equation (3.8), was originally motivated by a comparison of our integral equation (3.4) with Robertson's equation (cf. (2.6)).

4. The shear problem

Let us now consider the pair of coupled integral equations (2.14), corresponding to the shear problem. From Martin (1981, p. 285),

$$K_{rr}(\mathrm{q};\mathrm{p}_0) = \int_0^\infty (h_1-h_2)\,J_1(\xi r)\,P_0^+(\xi;r_0)\,\mathrm{d}\xi - \sum_{n=1}^\infty \int_0^\infty \{A_n^+ J_{n+1}(\xi r)+B_n^+ J_{n-1}(\xi r)\}\,\mathrm{d}\xi\cos n(\theta-\theta_0),$$

$$K_{\theta r}(\mathrm{q};\mathrm{p}_0) = -\sum_{n=1}^\infty \int_0^\infty \{A_n^+ J_{n+1}(\xi r)-B_n^+ J_{n-1}(\xi r)\}\,\mathrm{d}\xi\sin n(\theta-\theta_0),$$

$$K_{r\theta}(\mathrm{q};\mathrm{p}_0) = \sum_{n=1}^\infty \int_0^\infty \{A_n^- J_{n+1}(\xi r)+B_n^- J_{n-1}(\xi r)\}\,\mathrm{d}\xi\sin n(\theta-\theta_0),$$

$$K_{\theta\theta}(\mathrm{q};\mathrm{p}_\theta) = -\int_0^\infty (h_1+h_2)\,J_1(\xi r)\,P_0^-(\xi;r_0)\,\mathrm{d}\xi - \sum_{n=1}^\infty \int_0^\infty \{A_n^- J_{n+1}(\xi r)-B_n^- J_{n-1}(\xi r)\}\,\mathrm{d}\xi\cos n(\theta-\theta_0),$$

where

$$K^2h_1(\xi) = \xi^3(4\xi^2-3K^2-4\gamma\beta)/\beta+\xi^2(K^2-2k^2), \tag{4.1}$$

$$h_2(\xi) = -h_1(\xi)+2\xi(\beta-\xi), \tag{4.2}$$

$$A_n^\pm(\xi;r_0) = h_2P_n^\pm+h_1Q_n^\pm,\quad B_n^\pm(\xi;r_0) = h_1P_n^\pm+h_2Q_n^\pm,$$

with

$$Q_n^+ + Q_n^- = \alpha(P_n^+ + P_n^-) = \nu\Omega_n^+, \tag{4.3}$$

$$Q_n^+ - Q_n^- = \alpha(P_n^+ - P_n^-)+2(1-\alpha)\,\Omega_n^-, \tag{4.4}$$

$$P_n^+ - P_n^- = 2n\nu\Omega_n^- - (2n+1)\,\nu\Omega_n^+, \tag{4.5}$$

$$\Omega_n^\pm(\xi;r_0) = \tfrac{1}{2}\pi^{-2}\mathscr{A}_{n\pm1}\{tj_{n\pm1}(\xi t);\quad t\to r_0\}, \tag{4.6}$$

and $\alpha = \nu/(2-\nu)$.

Let us again assume, for simplicity, that the loading of the crack is symmetric about $\theta = 0$. Thus, we can replace $[u_r]$ and $[u_\theta]$ by (1.2), with corresponding expansions for $[\tilde{u}_r]$ and $[\tilde{u}_\theta]$.

Substituting into (2.13), integrating over θ, and rearranging yields

$$u_n(r_0) + \pi \int_0^1 (u_n(r) + v_n(r)) \int_0^\infty (h_2 P_n^+ + h_1 Q_n^+) J_{n+1}(\xi r)\, \mathrm{d}\xi\, r\, \mathrm{d}r$$
$$+ \pi \int_0^1 (u_n(r) - v_n(r)) \int_0^\infty (h_1 P_n^+ + h_2 Q_n^+) J_{n-1}(\xi r)\, \mathrm{d}\xi\, r\, \mathrm{d}r = \tilde{u}_n(r_0), \quad (4.7)$$

$$v_n(r_0) + \pi \int_0^1 (u_n(r) + v_n(r)) \int_0^\infty (h_2 P_n^- + h_1 Q_n^-) J_{n+1}(\xi r)\, \mathrm{d}\xi\, r\, \mathrm{d}r$$
$$+ \pi \int_0^1 (u_n(r) - v_n(r)) \int_0^\infty (h_1 P_n^- + h_2 Q_n^-) J_{n-1}(\xi r)\, \mathrm{d}\xi\, r\, \mathrm{d}r = \tilde{v}_n(r_0). \quad (4.8)$$

Adding and subtracting (4.7) and (4.8), we obtain

$$u_n(r_0) \pm v_n(r_0) + \pi \int_0^\infty U_n^+(\xi) \{h_2(P_n^+ \pm P_n^-) + h_1(Q_n^+ \pm Q_n^-)\}\, \mathrm{d}\xi$$
$$+ \pi \int_0^\infty U_n^-(\xi) \{h_1(P_n^+ \pm P_n^-) + h_2(Q_n^+ \pm Q_n^-)\}\, \mathrm{d}\xi = \tilde{u}_n(r_0) \pm \tilde{v}_n(r_0), \quad (4.9^\pm)$$

where
$$U_n^\pm(\xi) = \int_0^1 \{u_n(r) \pm v_n(r)\} J_{n\pm1}(\xi r)\, r\, \mathrm{d}r, \quad n > 0. \quad (4.10)$$

For each $n > 0$, (4.9) yields two coupled one-dimensional integral equations for the Fourier components of $[u_r(q)]$ and $[u_\theta(q)]$. For $n = 0$, we have a single integral equation for $u_0(r)$, namely

$$u_0(r_0) + 2\pi \int_0^\infty U_0^+(\xi)\, (h_2 - h_1)\, P_0^+(\xi; r_0)\, \mathrm{d}\xi = \tilde{u}_0(r_0), \quad (4.11)$$

where
$$U_0^+(\xi) = \int_0^1 u_0(r) J_1(\xi r)\, r\, \mathrm{d}r.$$

From (Martin 1982), we see that the free terms in $(4.9^\pm)$ and (4.11) are given by

$$\tilde{u}_0(r_0) = \mathscr{A}_1 R_0(t),$$
$$\tilde{u}_n(r_0) + \tilde{v}_n(r_0) = \mathscr{A}_{n+1} R_n(t),$$

and
$$\tilde{u}_n(r_0) - \tilde{v}_n(r_0) = \mathscr{A}_{n-1}\{\hat{R}_n(t) + 2n\alpha R_n(t)\} - (2n+1)\, \alpha \mathscr{A}_{n+1} R_n(t),$$

where

$$R_0(t) = \frac{1}{t} \int_0^t \frac{r^2 t_0(r)}{(t^2 - r^2)^{\frac{1}{2}}}\, \mathrm{d}r,$$

$$R_n(t) = \frac{(2-\nu)}{2(1-\nu)\, t^{n+1}} \int_0^t \frac{r^n}{(t^2 - r^2)^{\frac{1}{2}}} \{r^2(t_n + s_n) - \alpha(t_n - s_n)\, ((2n+1)\, r^2 - 2nt^2)\}\, \mathrm{d}r,$$

$$\hat{R}_n(t) = \frac{2}{(2-\nu)\, t^{n-1}} \int_0^t \frac{r^n(t_n - s_n)}{(t^2 - r^2)^{\frac{1}{2}}}\, \mathrm{d}r, \quad n \geqslant 1,$$

and (for symmetric loading)

$$\tau_{rz}^{(i)}(r,\theta,0) = \mu \sum_{m=0}^{\infty} t_m(r)\cos m\theta, \quad \tau_{\theta z}^{(i)}(r,\theta,0) = \mu \sum_{m=1}^{\infty} s_m(r)\sin m\theta. \tag{4.12}$$

Since $u_n(r) \pm v_n(r)$ occurs in the combination (4.10), this suggests that we introduce new unknown functions $\theta_n^{\pm}(x)$, where

$$u_n(r) \pm v_n(r) = C_n \mathscr{A}_{n\pm1}\theta_n^{\pm}(t). \tag{4.13}$$

After some manipulations along the lines described in §3, we eventually obtain the coupled integral equations

$$\theta_n^+(x) + \int_0^1 \{\theta_n^+(y)\,K_n^1(x,y) + \theta_n^-(y)\,K_n^2(x,y)\}\,\mathrm{d}y = f_n^+(x), \tag{4.14$^+$}$$

$$\theta_n^-(x) + \int_0^1 \{\theta_n^+(y)\,K_n^3(x,y) + \theta_n^-(y)\,K_n^4(x,y)\}\,\mathrm{d}y = f_n^-(x), \tag{4.14$^-$}$$

where the kernels are given by

$$\left.\begin{aligned} K_n^1(x,y) &= A_{n+1}^1(x,y), \quad K_n^2(x,y) = B_n^2(y;x), \\ K_n^3(x,y) &= B_n^2(x;y) - \alpha x^n\{B_n^1(1;y) + A_{n+1}^1(1,y)\}, \\ K_n^4(x,y) &= A_{n-1}^1(x,y) - \alpha x^n\{A_{n-1}^2(1,y) + B_n^2(y;1)\}, \\ A_n^i(x,y) &= xy\int_0^\infty m_i(\xi)\,j_n(\xi x)\,j_n(\xi y)\,\mathrm{d}\xi, \\ B_n^i(x;y) &= xy\int_0^\infty m_i(\xi)\,j_{n-1}(\xi x)\,j_{n+1}(\xi y)\,\mathrm{d}\xi, \end{aligned}\right\} \tag{4.15}$$

with $$2\pi m_1(\xi) = \nu h_1 + (2-\nu)h_2 \quad \text{and} \quad 2\pi m_2(\xi) = (2-\nu)h_1 + \nu h_2; \tag{4.16}$$

the free terms are given by

$$f_n^+(x) = C_n^{-1}R_n(x) \tag{4.17$^+$}$$

and $$f_n^-(x) = C_n^{-1}\{\hat{R}_n(x) - \alpha R_n(x) + (2n+1)\,\alpha x^n \int_x^1 t^{-n-1}R_n(t)\,\mathrm{d}t\}. \tag{4.17$^-$}$$

Similarly, we transform the single integral equation for $n = 0$, (4.11); writing

$$u_0(r) = C_0\mathscr{A}_1\theta_0^+(t), \tag{4.18}$$

we find that $$\theta_0^+(x) + \int_0^1 \theta_0^+(y)\{A_1^1(x,y) - A_1^2(x,y)\}\,\mathrm{d}y = f_0^+(x). \tag{4.19}$$

4.1. *Decoupling and symmetrization*

It is clearly desirable to seek some further simplification of equations (4.14) as not only are they coupled, but their kernels are not all symmetric in x and y (this latter property would prevent the immediate application of the imbedding procedure, to be described in §6). However, a close inspection of the K_n^i reveals that they are all simply related, via the differential operators

$$x^n\frac{\mathrm{d}}{\mathrm{d}x}x^{-n} \quad \text{and} \quad x^{-(n+1)}\frac{\mathrm{d}}{\mathrm{d}x}x^{n+1},$$

to the symmetric kernels $A_n^i(x,y)$, defined by (4.15). Thus, by introducing the new dependent variables $\phi_n^{\pm}$, given by

$$\theta_n^+(x) = x^{-(n+1)}\left\{\theta_n^+(1) - \int_x^1 \phi_n^+(y)\, y^{n+1}\, \mathrm{d}y\right\} \tag{4.20$^+$}$$

and

$$\theta_n^-(x) = x^{n}\left\{\theta_n^-(1) - \int_x^1 \phi_n^-(y)\, y^{-n}\, \mathrm{d}y\right\}, \tag{4.20$^-$}$$

we are able to reduce (4.14) to

$$\Phi_n^{\pm}(x) + \int_0^1 \Phi_n^{\pm}(y)\, K_n^{\pm}(x,y)\, \mathrm{d}y = F_n^{\pm}(x) + S_n^{\pm}\, K_n^{\pm}(x,1), \tag{4.21$^\pm$}$$

where

$$\Phi_n^{\pm}(x) = \phi_n^+(x) \pm \phi_n^-(x), \tag{4.22}$$

$$K_n^{\pm}(x,y) = A_n^1(x,y) \mp A_n^2(x,y), \tag{4.23}$$

$$S_n^{\pm} = \theta_n^+(1) \pm \theta_n^-(1) \tag{4.24}$$

and

$$F_n^{\pm}(x) = x^{-(n+1)}\frac{\mathrm{d}}{\mathrm{d}x}\{x^{n+1}f_n^+(x)\} \pm x^n \frac{\mathrm{d}}{\mathrm{d}x}\{x^{-n}f_n^-(x)\}. \tag{4.25}$$

Since (4.21) are linear, we may write

$$\Phi_n^{\pm}(x) = q_n^{\pm}(x) + S_n^{\pm}\, p_n^{\pm}(x), \tag{4.26}$$

where $q_n^{\pm}$ and $p_n^{\pm}$ satisfy

$$q_n^{\pm}(x) + \int_0^1 q_n^{\pm}(y)\, K_n^{\pm}(x,y)\, \mathrm{d}y = F_n^{\pm}(x), \tag{4.27}$$

and

$$p_n^{\pm}(x) + \int_0^1 p_n^{\pm}(y)\, K_n^{\pm}(x,y)\, \mathrm{d}y = K_n^{\pm}(x,1), \tag{4.28}$$

respectively. Note that $p_n^{\pm}(x)$ are *independent of the incident wave*, and the kernels $K_n^{\pm}(x,y)$ are symmetric.

Examining (4.21), we see that their free terms contain $S_n^{\pm}$, which are presently unknown. However, they may be determined by substituting (4.20) into the integral equations for $\theta_n^{\pm}(x)$. From (4.14$^+$) we obtain an equation which, after multiplying by x^{n+1} and letting $x \to 0$, yields

$$\theta_n^+(1) = \int_0^1 \phi_n^+(y)\, y^{n+1}\, \mathrm{d}y. \tag{4.29}$$

Similarly, if we use (4.20) in (4.14$^-$), and set $x = 1$, we obtain

$$\theta_n^-(1)\{1 + M_n^2(1)\} - \theta_n^+(1)\, M_n^1(1) + \int_0^1 \{\phi_n^+(s)\, M_n^1(s) - \phi_n^-(s)\, M_n^2(s)\}\, s\, \mathrm{d}s = f_n^-(1), \tag{4.30}$$

where

$$M_n^1(s) = \int_0^\infty \xi^{-1} j_n(\xi s)\{(m_2 - \alpha m_1)\, j_{n-1}(\xi) - \alpha m_1 j_{n+1}(\xi)\}\, \mathrm{d}\xi,$$

and M_n^2 is defined similarly with m_1 and m_2 interchanged. If we now use (4.22)–(4.26), (4.29) and (4.30) become

$$\left.\begin{aligned} c_n^+ S_n^+ + c_n^- S_n^- &= F_n, \\ d_n^+ S_n^+ + d_n^- S_n^- &= G_n, \end{aligned}\right\} \tag{4.31}$$

and

where
$$\left.\begin{aligned} c_n^\pm &= 1 - \int_0^1 x^{n+1} p_n^\pm(x)\,\mathrm{d}x, \quad F_n = \int_0^1 x^{n+1}\{q_n^+(x) + q_n^-(x)\}\,\mathrm{d}x, \\ d_n^\pm &= \pm 1 - M_n^\pm(1) + \int_0^1 M_n^\pm(x)\, p_n^\pm(x)\,\mathrm{d}x, \end{aligned}\right\} \tag{4.32}$$

$$G_n = 2f_n^-(1) - \int_0^1 \{M_n^+(x)\, q_n^+(x) + M_n^-(x)\, q_n^-(x)\}\,\mathrm{d}x \tag{4.33}$$

and
$$M_n^\pm(x) = x\{M_n^1(x) \mp M_n^2(x)\}.$$

Solving (4.31), we obtain

$$S_n^+ = (d_n^- F_n - c_n^- G_n)/\Delta_n, \tag{4.34$^+$}$$

and
$$S_n^- = (-d_n^+ F_n + c_n^+ G_n)/\Delta_n, \tag{4.34$^-$}$$

where
$$\Delta_n = c_n^+ d_n^- - d_n^+ c_n^-.$$

We can give alternative expressions for $d_n^\pm$ and G_n: write

$$\begin{aligned} L_n^\pm(x) &= x \int_0^\infty \xi^{-2}\{m_1 \mp m_2\}\, j_n(\xi x)\, j_n(\xi)\,\mathrm{d}\xi \\ &= x^{-n} \int_0^1 s^n \int_0^x y^n K_{n-1}^\pm(s, y)\,\mathrm{d}y\,\mathrm{d}s, \end{aligned} \tag{4.35}$$

whence
$$M_n^\pm(x) = \pm \int_0^1 s^{n+1} K_n^\pm(x, s)\,\mathrm{d}s \mp (2n+1)(1 \pm \alpha)\, L_n^\pm(x).$$

Hence, using the integral equations satisfied by $p_n^\pm$ and $q_n^\pm$, we find that

$$d_n^\pm = \pm\{c_n^\pm + (2n+1)(1 \pm \alpha)\, e_n^\pm\},$$

and
$$\begin{aligned} G_n = \int_0^1 x^{n+1}\{q_n^+(x) - q_n^-(x)\}\,\mathrm{d}x + 2(2n+1) \int_0^1 x^n f_n^-(x)\,\mathrm{d}x \\ + (2n+1) \int_0^1 \{(1+\alpha)\, L_n^+(x)\, q_n^+(x) - (1-\alpha)\, L_n^-(x)\, q_n^-(x)\}\,\mathrm{d}x, \end{aligned} \tag{4.36}$$

where
$$e_n^\pm = L_n^\pm(1) - \int_0^1 L_n^\pm(x)\, p_n^\pm(x)\,\mathrm{d}x.$$

4.2. *Plane-wave data*

For an incident plane wave propagating at an angle ϕ to the z-axis, we have (see Appendix A)

$$t_n(r) \pm s_n(r) = \pm C_n(\phi)\, J_{n+1}(\kappa r),$$

and
$$t_0(r) = C_0(\phi)\, J_1(\kappa r).$$

For this particular incident wave, many of the preceding formulae simplify. Using (3.14), we obtain

$$R_n(t) = C_n(\phi)\, t j_{n+1}(\kappa t), \quad n \geqslant 0,$$
$$\hat{R}_n(t) = -(1+\alpha)\, C_n(\phi)\, t j_{n-1}(\kappa t), \quad n \geqslant 1,$$
$$f_n^+(x) = x j_{n+1}(\kappa x), \quad n \geqslant 0,$$
$$f_n^-(x) = -x j_{n-1}(\kappa x) - \alpha x^n \{j_{n-1}(\kappa) + j_{n+1}(\kappa)\}, \quad n \geqslant 1,$$
$$F_n^+(x) = 2\kappa x j_n(\kappa x), \quad n \geqslant 1,$$

and
$$F_n^-(x) = 0, \quad n \geqslant 1.$$

Hence, from (4.27), we obtain $q_n^-(x) \equiv 0$ and,

$$q_n^+(x) = 2\kappa q_n(x;\kappa),$$

say, where $q_n(x;\kappa)$ satisfies

$$q_n(x;\kappa) + \int_0^1 q_n(y;\kappa)\, K_n^+(x,y)\,\mathrm{d}y = x j_n(\kappa x), \tag{4.37}$$

an integral equation of precisely the same form as that satisfied by $\theta_n(x;\kappa)$, namely (3.18).

The $S_n^\pm(\kappa)$ are given by (4.34), where, from (4.32) and (4.36),

$$F_n(\kappa) = 2\kappa \int_0^1 x^{n+1} q_n(x;\kappa)\,\mathrm{d}x, \tag{4.38}$$

and

$$G_n(\kappa) = F_n(\kappa) - 2(1+\alpha)\left\{j_{n-1}(\kappa) + j_{n+1}(\kappa) - (2n+1)\,\kappa \int_0^1 L_n^+(x)\, q_n(x;\kappa)\,\mathrm{d}x\right\}. \tag{4.39}$$

It remains to consider the integral equation (4.19) for θ_0^+. Comparing it with (4.37), we see that

$$\theta_0^+(x;\kappa) = q_1(x;\kappa), \tag{4.40}$$

i.e. solving (4.37) for $n \geqslant 1$ also provides the solution of (4.19).

4.3. *Summary*

Let us summarize the results of this section. The Fourier components of the discontinuity in $\boldsymbol{u}$ across the crack, $u_n \pm v_n$, are given in terms of $\theta_n^\pm$ by (4.13); $\theta_n^\pm$ are given in terms of $S_n^\pm$ and $\Phi_n^\pm$ by (4.20), (4.22) and (4.24); $S_n^\pm$ and $\Phi_n^\pm$ are given in terms of $p_n^\pm$ and $q_n^\pm$ by (4.26) and (4.34); and $p_n^\pm$ and $q_n^\pm$ are obtained by solving the uncoupled integral equations, (4.28) and (4.27), respectively. The functions $p_n^\pm(x)$ are independent of the incident wave. For plane-wave data, $q_n^- \equiv 0$ and $q_n^+ = 2\kappa\, q_n$, where q_n, is the solution of the simple integral equation (4.37).

Explicitly, from (4.13) and (4.20), we obtain

$$\left.\begin{aligned} u_n(r) - v_n(r) &= C_n r^{n-1}\left\{\theta_n^-(1)\,(1-r^2)^{\frac{1}{2}} - \int_r^1 y^{-n}(y^2-r^2)^{\frac{1}{2}}\,\phi_n^-(y)\,\mathrm{d}y\right\} \\ \text{and}\quad u_n(r) + v_n(r) &= C_n r^{-(n+1)}\left\{\theta_n^+(1)\,B_n(r) - \int_r^1 y^{n+1} B_n(r/y)\,\phi_n^+(y)\,\mathrm{d}y\right\}, \end{aligned}\right\} \tag{4.41}$$

where

$$B_n(r/y) = r^{n+2}\int_r^y \frac{\mathrm{d}t}{t^{2n+2}(t^2-r^2)^{\frac{1}{2}}};$$

we have $$B_n(\rho) = \int_{\arcsin\rho}^{\frac{1}{2}\pi} (\sin x)^{2n+1}\,\mathrm{d}x = (1-\rho^2)^{\frac{1}{2}} \sum_{j=0}^{n} a_j^{(n)} \rho^{2n-2j},$$

where the coefficients are given by ($n > 0$)

$$a_j^{(n)} = \frac{2^j n(n-1)\dots(n-j+1)}{(2n+1)(2n-1)\dots(2n-2j+1)}, \tag{4.42}$$

for $j > 0$, and $a_0^{(n)} = (2n+1)^{-1}$. Hence,

$$u_n(r)+v_n(r) = C_n r^{n-1} \sum_{j=0}^{n} a_j^{(n)} r^{-2j} \left\{\theta_n^+(1)(1-r^2)^{\frac{1}{2}} - \int_r^1 y^{2j-n}(y^2-r^2)^{\frac{1}{2}}\phi_n^+(y)\,\mathrm{d}y\right\}. \tag{4.43}$$

We note that u_n and v_n may also be expressed as (1.3a) and (1.3b), respectively, by introducing the functions α_n and β_n, defined by

$$\alpha_n(r)+\beta_n(r) = r^{-(n+1)}(\mathrm{d}/\mathrm{d}r)\{r^{n+1}(u_n+v_n)\}$$

and $$\alpha_n(r)-\beta_n(r) = r^{n-1}(\mathrm{d}/\mathrm{d}r)\{r^{-(n-1)}(u_n-v_n)\}.$$

We have thus shown that Martin's integral equation may be reduced to a system of simple one-dimensional integral equations with simple symmetric kernels. These equations are identical to those derived by Robertson (1967) and Mal (1968a) in the case of axisymmetric loading. It seems to us, however, that the structure of the transformations just described is so complicated that it is not obvious how our equations for general loading could have been obtained by the more direct dual integral equation method.

5. The quantities of physical interest

In this section, we show how the scattered displacement field at large distances from the crack, and the asymptotic stress field near the crack edge can be simply related to the solutions of one-dimensional integral equations (3.15), (4.27) and (4.28).

5.1. *The far field*

Let $\mathrm{P} \in D$ have spherical polar coordinates (R, Θ, Φ), where

$$x = R\sin\Phi\cos\Theta, \quad y = R\sin\Phi\sin\Theta \quad \text{and} \quad z = R\cos\Phi.$$

Furthermore, let

$$\boldsymbol{u}(P) = (u_R, u_\theta, u_\phi),$$

in this coordinate system where $\boldsymbol{u}$ is given by (2.7) and (2.11); estimating this quantity for large R (see, for example, Martin 1981, p. 268), we obtain

$$u_R(\mathrm{P}) = -\tfrac{1}{2}\mathrm{i}k\{\sigma^2 \sin 2\Phi I_c(k\sin\Phi) + (1-2\sigma^2\sin^2\Phi)\, I_z(k\sin\Phi)\} \times \mathrm{e}^{\mathrm{i}kR}/R + O(R^{-2}), \quad (5.1)$$

$$u_\theta(\mathrm{P}) = \tfrac{1}{2}\mathrm{i}K\cos\Phi I_s(K\sin\Phi)\,\mathrm{e}^{\mathrm{i}KR}/R + O(R^{-2}), \quad (5.2)$$

$$u_\phi(\mathrm{P}) = -\tfrac{1}{2}\mathrm{i}K\{\cos 2\Phi I_c(K\sin\Phi) - \sin 2\Phi I_z(K\sin\Phi)\}\,\mathrm{e}^{\mathrm{i}KR}/R + O(R^{-2}), \quad (5.3)$$

as $R\to\infty$, where $\sigma = k/K$,

$$2\pi I_z(\lambda;\Theta) = \int_\gamma [u_z(r,\theta)]\,\mathrm{e}^{-\mathrm{i}\lambda r\cos\chi}\, r\,\mathrm{d}r\,\mathrm{d}\theta, \quad (5.4)$$

$$2\pi I_c(\lambda;\Theta) = \int_\gamma \{[u_r(r,\theta)]\cos\chi + [u_\theta(r,\theta)]\sin\chi\}\,\mathrm{e}^{-\mathrm{i}\lambda r\cos\chi}\, r\,\mathrm{d}r\,\mathrm{d}\theta, \quad (5.5)$$

$$2\pi I_s(\lambda;\Theta) = \int_\gamma \{[u_r(r,\theta)]\sin\chi - [u_\theta(r,\theta)]\cos\chi\}\,\mathrm{e}^{-\mathrm{i}\lambda r\cos\chi}\, r\,\mathrm{d}r\,\mathrm{d}\theta, \quad (5.6)$$

and $\chi = \Theta - \theta$.

If we now assume that the loading is symmetric about $\theta = 0$, and use (1.1), (1.2), (3.9), (3.14), (4.13), (4.18) and the Fourier cosine series (equation 7.2.4 (27) of Erdélyi *et al.* 1953),

$$\exp(\pm\mathrm{i}z\cos\theta) = \sum_{n=0}^{\infty} \epsilon_n(\pm\mathrm{i})^n J_n(z)\cos n\theta, \quad (5.7)$$

we find that

$$I_z(\lambda;\Theta) = \eta\sum_{n=0}^{\infty}(-\mathrm{i})^n D_n I_n^\theta(\lambda)\cos n\Theta, \quad (5.8)$$

$$I_c(\lambda;\Theta) = \tfrac{1}{2}\eta\sum_{n=0}^{\infty}(-\mathrm{i})^{n+1} C_n I_n^+(\lambda)\cos n\Theta, \quad (5.9)$$

and

$$I_s(\lambda;\Theta) = -\tfrac{1}{2}\eta\sum_{n=1}^{\infty}(-\mathrm{i})^{n+1} C_n I_n^-(\lambda)\sin n\Theta, \quad (5.10)$$

where $\eta = 4(1-\nu)/\pi$,

$$I_n^\theta(\lambda) = \int_0^1 x\theta_n(x)\, j_n(\lambda x)\,\mathrm{d}x, \quad (5.11)$$

$$I_0^+(\lambda) = 2\int_0^1 x\theta_0^+(x)\, j_1(\lambda x)\,\mathrm{d}x, \quad (5.12)$$

$$\begin{aligned} I_n^\pm(\lambda) &= \int_0^1 x\{\theta_n^+(x)\, j_{n+1}(\lambda x) \mp \theta_n^-(x)\, j_{n-1}(\lambda x)\}\,\mathrm{d}x \\ &= \lambda^{-1}\left\{-j_n(\lambda)\, S_n^\pm + \int_0^1 x\Phi_n^\pm(x)\, j_n(\lambda x)\right\}\mathrm{d}x, \end{aligned} \quad (5.13)$$

for $n \geqslant 1$. To obtain the last formula, we have used the decomposition given in §4.1. Note that, for $\lambda = 0$, we have

$$I_0^\theta(0) = \int_0^1 x\theta_0(x)\,\mathrm{d}x,$$

$$I_1^\pm(0) = \frac{1}{3}\int_0^1 x^2 q_1^\pm(x)\,\mathrm{d}x - \tfrac{1}{3}c_1^\pm S_1^\pm,$$

and all other terms vanish.

5.2. *Dynamic stress–intensity factors*

There are three dynamic stress–intensity factors which, following Krenk (1979), we define by

$$k_1(\theta) = (1/\mu) \lim_{r\to 1+} \{(2(r-1))^{\frac{1}{2}}\tau_{zz}(r,\theta,0)\} = [1/2(1-\nu)] \lim_{r\to 1-} \{[u_z(r,\theta)]/(2(1-r))^{\frac{1}{2}}\}, \quad (5.14)$$

$$k_2(\theta) = (1/\mu) \lim_{r\to 1+} \{(2(r-1))^{\frac{1}{2}}\tau_{rz}(r,\theta,0)\} = [1/2(1-\nu)] \lim_{r\to 1-} \{[u_r(r,\theta)]/(2(1-r))^{\frac{1}{2}}\}, \quad (5.15)$$

and

$$k_3(\theta) = (1/\mu) \lim_{r\to 1+} \{(2(r-1))^{\frac{1}{2}}\tau_{\theta z}(r,\theta,0)\} = \tfrac{1}{2} \lim_{r\to 1+} \{[u_\theta(r,\theta)]/(2(1-r))^{\frac{1}{2}}\}. \quad (5.16)$$

From (3.3) and (3.8), we obtain

$$\pi k_1(\theta) = 2\sum_{n=0}^{\infty} D_n \theta_n(1) \cos n\theta, \quad (5.17)$$

while, from (1.2), (4.41) and (4.43), we obtain

$$\pi k_2(\theta) = 2C_0\theta_0^+(1) + \sum_{n=1}^{\infty} C_n S_n^+ \cos n\theta, \quad (5.18)$$

and

$$\pi k_3(\theta) = (1-\nu)\sum_{n=1}^{\infty} C_n S_n^- \sin n\theta, \quad (5.19)$$

where we have also used (proof by induction)

$$\sum_{j=0}^{n} a_j^{(n)} = 1.$$

5.3. *Plane-wave data*

Recall that for incident plane waves, we introduced (in §3.1) the parameter κ ($= k\sin\phi$ or $K\sin\phi$); we now introduce a second parameter $\mathscr{K}$, where $\mathscr{K} = k\sin\Phi$ or $K\sin\Phi$ (these are the values of λ required in §5.1). Thus, κ and $\mathscr{K}$ correspond to the incident wave and scattered wave, respectively.

Exhibiting the dependence on κ, we rewrite (5.11)–(5.13) as

$$I_n^\theta(\mathscr{K};\kappa) = \int_0^1 x\theta_n(x;\kappa)\,j_n(\mathscr{K}x)\,\mathrm{d}x, \tag{5.20}$$

$$\mathscr{K}I_n^+(\mathscr{K};\kappa) = S_n^+(\kappa)\left\{-j_n(\mathscr{K})+\int_0^1 xp_n^+(x)\,j_n(\mathscr{K}x)\,\mathrm{d}x\right\}+2\kappa I_n^q(\mathscr{K};\kappa), \tag{5.21$^+$}$$

$$\mathscr{K}I_n^-(\mathscr{K};\kappa) = S_n^-(\kappa)\{-j_n(\mathscr{K})+\int_0^1 xp_n^-(x)\,j_n(\mathscr{K}x)\,\mathrm{d}x\}, \tag{5.21$^-$}$$

and
$$I_0^+(\mathscr{K};\kappa) = 2I_1^q(\mathscr{K};\kappa), \tag{5.22}$$

where
$$I_n^q(\mathscr{K};\kappa) = \int_0^1 xq_n(x;\kappa)\,j_n(\mathscr{K}x)\,\mathrm{d}x, \tag{5.23}$$

and we have used the simplifications given in §4.2.

The fundamental quantities $I_n^\theta(n \geqslant 0)$ and $I_n^q(n \geqslant 1)$ satisfy *reciprocity relations*, given by the following.

LEMMA 1.
$$I_n^\theta(\mathscr{K};\kappa) = I_n^\theta(\kappa;\mathscr{K}) \tag{5.24}$$

and
$$I_n^q(\mathscr{K};\kappa) = I_n^q(\kappa;\mathscr{K}). \tag{5.25}$$

Proof. Multiply (3.18) by $\theta_n(x;\mathscr{K})$ and integrate over $0 \leqslant x \leqslant 1$ to give

$$\int_0^1 \theta_n(x;\kappa)\,\theta_n(x;\mathscr{K})\,\mathrm{d}x+\int_0^1 \theta_n(y;\kappa)\int_0^1 \theta_n(x;\mathscr{K})\,N_n(x,y)\,\mathrm{d}x\,\mathrm{d}y = I_n^\theta(\kappa;\mathscr{K}).$$

However, since the kernel $N_n(x,y)$ is symmetric, we have from (3.18) that

$$\int_0^1 \theta_n(x;\mathscr{K})\,N_n(x,y)\,\mathrm{d}x = y\,j_n(\mathscr{K}y)-\theta_n(y;\mathscr{K}),$$

and the result (5.24) follows. Similarly, the result (5.25) is obtained using the integral equation for $q_n(x;\kappa)$, namely (4.37).

Suppose, for example, that we wished to calculate $I_n^\theta(\kappa,\mathscr{K})$ for several values of κ and $\mathscr{K}$; for each κ, we must solve the integral equation (3.18) and for each pair $(\kappa,\mathscr{K})$, we must evaluate the integral (5.20). (Actually, this process may be simplified slightly, by noting that the kernel of (3.18) is independent of κ.) Thus, for $(N+1)$ values of n $(n = 0, 1, \ldots, N)$, M_1 values of κ, and M_2 values of $\mathscr{K}$, we must solve $(N+1)M_1$ integral equations and then evaluate $(N+1)M_1M_2$ integrals. In the next section, we shall describe a more efficient procedure for computing $I_n^\theta(\kappa,\mathscr{K})$; this reduces the computation to the solution of only $(N+M_1)$ integral equations and the evaluation of only $N(M_1+M_2)+M_1(M_2+1)$ integrals. Similar remarks apply to the calculations of $I_n^q(\kappa,\mathscr{K})$.

6. The imbedding method

In this section, we shall concentrate on plane-wave data. We obtain further simplification by exploiting the structure of the kernels and free terms of the integral equations for $\theta_n(x;\kappa)$ and $q_n(x;\kappa)$. We begin with a detailed study of the normal problem.

6.1. *The normal problem*

Consider the linear integral equation (3.18), which we rewrite here for convenience:

$$\theta_n(x;\kappa)+\int_0^1 \theta_n(y;\kappa)\, N_n(x,y)\,\mathrm{d}y = xj_n(\kappa x). \tag{6.1}$$

Some properties of the kernel and free term of this equation are contained in the following two lemmas.

Lemma 2. (Equations 7.2.8 (50), (51) from Erdélyi *et al.* 1953)

$$x^{-(n+1)}(\mathrm{d}/\mathrm{d}x)\{x^{n+1}j_n(\xi x)\} = \xi j_{n-1}(\xi x), \tag{6.2}$$

$$x^{n}(\mathrm{d}/\mathrm{d}x)\{x^{-n}j_n(\xi x)\} = -\xi j_{n+1}(\xi x). \tag{6.3}$$

Lemma 3.

$$x^{-n}(\mathrm{d}/\mathrm{d}x)\{x^{n}N_n(x,y)\} = -y^{n}(\mathrm{d}/\mathrm{d}y)\{y^{-n}N_{n-1}(x,y)\}, \tag{6.4}$$

$$y^{-n}(\mathrm{d}/\mathrm{d}y)\{y^{n}N_n(x,y)\} = -x^{n}(\mathrm{d}/\mathrm{d}x)\{x^{-n}N_{n-1}(x,y)\}. \tag{6.5}$$

Proof.

$$\begin{aligned}
x^{-n}\frac{\mathrm{d}}{\mathrm{d}x}\{x^nN_n(x,y)\} &= yx^{-n}\frac{\mathrm{d}}{\mathrm{d}x}\left\{x^{n+1}\int_0^\infty f(\xi)\,j_n(\xi x)\,j_n(\xi y)\,\mathrm{d}\xi\right\} \\
&= xy\int_0^\infty \xi f(\xi)\,j_{n-1}(\xi x)\,j_n(\xi y)\,\mathrm{d}\xi, \qquad \text{from (6.2)}, \\
&= -\,xy\int_0^\infty f(\xi)\,j_{n-1}(\xi x)\,y^{n-1}\frac{\mathrm{d}}{\mathrm{d}y}\{y^{-(n-1)}j_{n-1}(\xi y)\}\,\mathrm{d}\xi, \\
&\qquad\qquad \text{from (6.3)} \\
&= -\,y^n\frac{\mathrm{d}}{\mathrm{d}y}\{y^{-n}N_{n-1}(x,y)\}.
\end{aligned}$$

Equation (6.5) follows by interchanging x and y, and noting that $N_n(x,y)$ is symmetric.

Let us define $\alpha_n(x;\kappa)$ by

$$\alpha_n(x;\kappa) = \theta_n(x;\kappa)+\lambda_n^\theta(\kappa)\,\theta_n(x;\kappa_{\mathrm{g}}), \tag{6.6}$$

where $\lambda_n^\theta(\kappa)$ is unspecified at present, and κ_{g} is that value of κ which corresponds to grazing incidence ($\phi = \tfrac{1}{2}\pi$), i.e. $\kappa_{\mathrm{g}} = k$ if $\kappa = k\sin\phi$ and $\kappa_{\mathrm{g}} = K$ if $\kappa = K\sin\phi$. From (6.1), α_{n+1} satisfies

$$\alpha_{n+1}(x;\kappa)+\int_0^1 \alpha_{n+1}(y;\kappa)\,N_{n+1}(x,y)\,\mathrm{d}y = x\{j_{n+1}(\kappa x)+\lambda_{n+1}^\theta(\kappa)\,j_{n+1}(\kappa_{\mathrm{g}}x)\}.$$

Noting (6.2) and (6.4), we apply the differential operator $x^{-(n+1)}(\mathrm{d}/\mathrm{d}x)x^{n+1}$ to this integral equation and obtain

$$\beta_n(x;\kappa) - \int_0^1 \alpha_{n+1}(y;\kappa)\, y^{n+1} \frac{\mathrm{d}}{\mathrm{d}y}\{y^{-(n+1)} N_n(x,y)\}\,\mathrm{d}y = x\{\kappa j_n(\kappa x) + \kappa_\mathrm{g}\,\lambda^\theta_{n+1}(\kappa)\, j_n(\kappa_\mathrm{g} x)\},$$

where $\beta_n(x;\kappa)$ is defined by

$$\beta_n(x;\kappa) = x^{-(n+1)}(\mathrm{d}/\mathrm{d}x)\{x^{n+1}\alpha_{n+1}(x;\kappa)\}. \tag{6.7}$$

If we choose $\lambda^\theta_n(\kappa)$ such that

$$\alpha_n(1;\kappa) = 0, \tag{6.8}$$

then an integration by parts gives

$$\beta_n(x;\kappa) + \int_0^1 \beta_n(y;\kappa)\, N_n(x,y)\,\mathrm{d}y = x\{\kappa j_n(\kappa x) + \kappa_\mathrm{g}\lambda^\theta_{n+1}(\kappa)\, j_n(\kappa_\mathrm{g} x)\}. \tag{6.9}$$

Comparing (6.9) with (6.1), we find by linear superposition that

$$\beta_n(x;\kappa) = \kappa\theta_n(x;\kappa) + \kappa_\mathrm{g}\lambda^\theta_{n+1}(\kappa)\,\theta_n(x;\kappa_\mathrm{g}). \tag{6.10}$$

Integrating this equation, using (6.7) and substituting from (6.6) then gives

$$\begin{aligned}\alpha_{n+1}(x;\kappa) &= \theta_{n+1}(x;\kappa) + \lambda^\theta_{n+1}(\kappa)\,\theta_{n+1}(x;\kappa_\mathrm{g}) \\ &= x^{-(n+1)} \int_0^x y^{n+1}\{\kappa\theta_n(y;\kappa) + \kappa_\mathrm{g}\lambda^\theta_{n+1}(\kappa)\,\theta_n(y;\kappa_\mathrm{g})\}\,\mathrm{d}y.\end{aligned} \tag{6.11}$$

This is a formula for $\theta_{n+1}(x;\kappa)$ in terms of $\theta_{n+1}(x;\kappa_\mathrm{g})$ (the solution at grazing incidence), $\theta_n(y;\kappa)$ and $\theta_n(y;\kappa_\mathrm{g})$ for $0 \leqslant y \leqslant x$, and $\lambda^\theta_{n+1}(\kappa)$; it only remains to determine the function $\lambda^\theta_n(\kappa)$.

Multiply (6.10) by $xj_n(\mathscr{K}x)$ and integrate over $0 \leqslant x \leqslant 1$ to give

$$\begin{aligned}\kappa I^\theta_n(\kappa,\mathscr{K}) + \kappa_\mathrm{g}\lambda^\theta_{n+1}(\kappa)\, I^\theta_n(\kappa_\mathrm{g},\mathscr{K}) &= \int_0^1 \beta_n(x;\kappa)\, xj_n(\mathscr{K}x)\,\mathrm{d}x \\ &= -\int_0^1 x^{n+1}\frac{\mathrm{d}}{\mathrm{d}x}\{x^{-n}j_n(\mathscr{K}x)\}\,\alpha_{n+1}(x;\kappa)\,\mathrm{d}x \\ &= \mathscr{K}\{I^\theta_{n+1}(\kappa,\mathscr{K}) + \lambda^\theta_{n+1}(\kappa)\, I^\theta_{n+1}(\kappa_\mathrm{g},\mathscr{K})\},\end{aligned}$$

where we have used (5.20), (6.3) and integrated by parts. Thus

$$\mathscr{K} I^\theta_{n+1}(\kappa,\mathscr{K}) = \kappa I^\theta_n(\kappa,\mathscr{K}) + \lambda^\theta_{n+1}(\kappa)\{\kappa_\mathrm{g} I^\theta_n(\kappa_\mathrm{g},\mathscr{K}) - \mathscr{K} I^\theta_{n+1}(\kappa_\mathrm{g},\mathscr{K})\}. \tag{6.12}$$

To determine $\lambda^\theta_n(\kappa)$, we set $\mathscr{K} = \kappa_\mathrm{g}$ in (6.12) to give

$$\lambda^\theta_{n+1}(\kappa) = \frac{I^\theta_{n+1}(\kappa,\kappa_\mathrm{g}) - \sin\phi\, I^\theta_n(\kappa,\kappa_\mathrm{g})}{I^\theta_n(\kappa_\mathrm{g},\kappa_\mathrm{g}) - I^\theta_{n+1}(\kappa_\mathrm{g},\kappa_\mathrm{g})}. \tag{6.13}$$

We note that $\lambda^\theta_{n+1}(\kappa_\mathrm{g}) = -1$, whence, from (6.6), $\alpha_{n+1}(x;\kappa_\mathrm{g}) = 0$.

Let us now describe the computation procedure. Suppose that we know $I_n^\theta(\kappa, \mathscr{K})$ for M_1 values of κ (including κ_g) and M_2 values of $\mathscr{K}$. In order to obtain $I_{n+1}^\theta(\kappa, \mathscr{K})$, we

(i) determine $\theta_{n+1}(x; \kappa_g)$ by solving the integral equation

$$\theta_{n+1}(x; \kappa_g) + \int_0^1 \theta_{n+1}(y; \kappa_g)\, N_n(x, y)\, dy = x j_{n+1}(\kappa_g x);$$

(ii) calculate

$$I_{n+1}^\theta(\kappa_g, \mathscr{K}) = \int_0^1 x\theta_{n+1}(x; \kappa_g)\, j_{n+1}(\mathscr{K}x)\, dx,$$

for the M_2 values of $\mathscr{K}$;

(iii) calculate

$$I_{n+1}^\theta(\kappa, \kappa_g) = \int_0^1 x\theta_{n+1}(x; \kappa_g)\, j_{n+1}(\kappa x)\, dx,$$

for the M_1 values of κ;

(iv) calculate $\lambda_{n+1}^\theta(\kappa)$ from (6.13); and

(v) calculate $I_{n+1}^\theta(\kappa, \mathscr{K})$ from (6.12).†

This procedure can now be repeated; for starting values, we require $\theta_0(x; \kappa)$, $I_0^\theta(\kappa, \mathscr{K})$ and $I_0^\theta(\kappa, \kappa_g)$. Counting up, for $(N+1)$ values of n ($n = 0, 1, \ldots, N$), we see that we must solve $N + M_1$ integral equations and evaluate $N(M_1 + M_2) + M_1(M_2 + 1)$ finite integrals. This is a considerable saving when compared with the direct method (outlined at the end of § 5) which requires the solution of a further $N(M_1 - 1)$ integral equations and the evaluation of a further $N(M_1 M_2 - M_1 - M_2) - M_1$ integrals. Moreover, from (6.6) and (6.8), we have

$$\theta_n(1; \kappa) = -\lambda_n^\theta(\kappa)\, \theta_n(1; \kappa_g),$$

and so the dynamic stress-intensity factor $k_1(\theta; \kappa)$, defined by (5.17), can also be computed efficiently. Also, if the value of $\theta_{n+1}(x; \kappa)$ is required, it may be calculated from (6.11), after step (iv) above.

6.2. *The shear problem*

In the shear problem, the function that corresponds to θ_n is q_n; q_n satisfies the integral equation (4.37), and this equation is immediately amenable to the imbedding procedure. Let

$$\alpha_n^q(x; \kappa) = q_n(x; \kappa) + \lambda_n^q(\kappa)\, q_n^*(x; \kappa_g), \tag{6.14}$$

and

$$\beta_n^+(x; \kappa) = x^{-(n+1)}\, (d/dx)\, \{x^{n+1} \alpha_{n+1}^q(x; \kappa)\}, \tag{6.15}$$

then, if $\lambda_n^q(\kappa)$ is chosen such that

$$\alpha_n^q(1; \kappa) = 0, \tag{6.16}$$

† If $\kappa > \mathscr{K}$, it may be necessary to reverse this procedure, i.e. use backwards recursion and the asymptotic estimates of I_n^θ for large n given in § 7.2.

we find that

$$\beta_n^+(x;\kappa) = \kappa q_n(x;\kappa) + \kappa_g \lambda_{n+1}^q(\kappa)\, q_n(x;\kappa_g), \tag{6.17}$$

whence

$$\alpha_{n+1}^q(x;\kappa) = x^{-(n+1)} \int_0^x y^{n+1} \{\kappa q_n(y;\kappa) + \kappa_g \lambda_{n+1}^q(\kappa)\, q_n(y;\kappa_g)\}\, \mathrm{d}y. \tag{6.18}$$

It follows that I_n^q satisfies

$$\mathscr{K} I_{n+1}^q(\kappa, \mathscr{K}) = \kappa I_n^q(\kappa, \mathscr{K}) + \lambda_{n+1}^q(\kappa)\{\kappa_g I_n^q(\kappa_g, \mathscr{K}) - \mathscr{K} I_{n+1}^q(\kappa_g, \mathscr{K})\} \tag{6.19}$$

and λ_n^q is given by

$$\lambda_{n+1}^q(\kappa) = \frac{I_{n+1}^q(\kappa, \kappa_g) - \sin\phi\, I_n^q(\kappa, \kappa_g)}{I_n^q(\kappa_g, \kappa_g) - I_{n+1}^q(\kappa_g, \kappa_g)}. \tag{6.20}$$

The computation procedure for determining I_n^q is identical to that for I_n^θ, except, for starting values, we require $q_1(x;\kappa)$, $I_1^q(\kappa, \mathscr{K})$ and $I_1^q(\kappa, \kappa_g)$.

To complete the computation of the far-field displacements and the stress–intensity factors, we must determine $S_n^\pm(\kappa)$. Thus, from (4.34), we must determine $F_n(\kappa)$ and $G_n(\kappa)$, defined by (4.38) and (4.39), respectively; these can be computed efficiently using the imbedding procedure, as follows.

Set $x = 1$ in (6.18); from (4.38) and (6.16), we obtain

$$F_n(\kappa) = -\lambda_{n+1}^q(\kappa)\, F_n(\kappa_g). \tag{6.21}$$

Thus, we can compute $F_n(\kappa)$ from $F_n(\kappa_g)$.

In order to obtain a corresponding formula for $G_n(\kappa)$, we consider the function β_n^-, defined by

$$\beta_n^-(x;\kappa) = x^n(\mathrm{d}/\mathrm{d}x)\{x^{-n}\alpha_{n-1}^q(x;\kappa)\}; \tag{6.22}$$

an analogous imbedding procedure yields

$$\beta_n^-(x;\kappa) = -\kappa q_n(x;\kappa) - \kappa_g \lambda_{n-1}^q(\kappa)\, q_n(x;\kappa_g). \tag{6.23}$$

Now, from (4.39), we have

$$\begin{aligned} G_n(\kappa) + \lambda_{n-1}^q(\kappa)\, G_n(\kappa_g) &= F_n(\kappa) + \lambda_{n-1}^q(\kappa)\, F_n(\kappa_g) \\ &\quad - 2(1+\alpha)(2n+1)\left\{ j_n(\kappa)/\kappa + \lambda_{n-1}^q(\kappa) j_n(\kappa_g)/\kappa_g + \int_0^1 L_n^+(x)\, \beta_n^-(x;\kappa)\, \mathrm{d}x \right\}. \end{aligned} \tag{6.24}$$

To evaluate the integral, we define

$$\mathscr{L}_n(x,t) = xt \int_0^\infty \xi^{-2}\{m_1 - m_2\}\, j_n(\xi x)\, j_n(\xi t)\, \mathrm{d}\xi,$$

and

$$\Lambda_n(t;\kappa) = \int_0^1 \mathscr{L}_n(x,t)\, \beta_n^-(x;\kappa)\, \mathrm{d}x,$$

whence (see (4.35))

$$L_n^+(x) = \mathscr{L}_n(x,1),$$

and

$$\Lambda_n(1;\kappa) = \int_0^1 L_n^+(x)\, \beta_n^-(x;\kappa)\, \mathrm{d}x. \tag{6.25}$$

Integrating by parts, and using (6.16) and (6.22), gives

$$A_n(t;\kappa) = -\int_0^1 \alpha^q_{n-1}(x;\kappa)\, x^{-n}\frac{\mathrm{d}}{\mathrm{d}x}\{x^n \mathscr{L}_n(x,t)\}\,\mathrm{d}x.$$

Now, from (6.2), we have

$$t^{-n}\frac{\mathrm{d}}{\mathrm{d}t}\left[t^n x^{-n}\frac{\mathrm{d}}{\mathrm{d}x}\{x^n\mathscr{L}_n(x,t)\}\right] = K^+_{n-1}(x,t),$$

whence

$$\begin{aligned} t^{-n}\frac{\mathrm{d}}{\mathrm{d}t}\{t^n A_n(t;\kappa)\} &= -\int_0^1 \alpha^q_{n-1}(x;\kappa)\, K^+_{n-1}(x,t)\,\mathrm{d}x \\ &= \alpha^q_{n-1}(t;\kappa) - t\{j_{n-1}(\kappa t) + \lambda^q_{n-1}(\kappa) j_{n-1}(\kappa_{\mathrm{g}} t)\}, \end{aligned}$$

where we have used the integral equation satisfied by α^q_{n-1}. Integrating, we find that

$$A_n(t;\kappa) = t^{-n}\int_0^t x^n\{\alpha^q_{n-1}(x;\kappa) - xj_{n-1}(\kappa x) - \lambda^q_{n-1}(\kappa)\, xj_{n-1}(\kappa_{\mathrm{g}} x)\}\,\mathrm{d}x,$$

whence $\quad A_n(1;\kappa) = \{\tfrac{1}{2}F_{n-1}(\kappa) - j_n(\kappa)\}/\kappa + \lambda^q_{n-1}(\kappa)\{\tfrac{1}{2}F_{n-1}(\kappa_{\mathrm{g}}) - j_n(\kappa_{\mathrm{g}})\}/\kappa_{\mathrm{g}}.$

Using (6.25) and substituting back into (6.24), we obtain

$$G_n(\kappa) = \mathscr{F}_n(\kappa) + \lambda^q_{n-1}(\kappa)\{\mathscr{F}_n(\kappa_{\mathrm{g}}) - G_n(\kappa_{\mathrm{g}})\}, \tag{6.26}$$

where

$$\mathscr{F}_n(\kappa) = F_n(\kappa) - (2n+1)(1+\alpha)F_{n-1}(\kappa)/\kappa. \tag{6.27}$$

Suppose, now, that we use (6.21) and (6.26) to compute $F_n(\kappa)$ and $G_n(\kappa)$, given $F_n(\kappa_{\mathrm{g}})$ and $G_n(\kappa_{\mathrm{g}})$, for $2 \leqslant n \leqslant N$, say. Then, we require λ^q_1 (for G_2) and λ^q_{N+1} (for F_N): λ^q_1 is given by (6.14) and (6.16) as

$$\lambda^q_1(\kappa) = -q_1(1;\kappa)/q_1(1;\kappa_{\mathrm{g}})$$

(for starting values, we have already computed $q_1(x;\kappa)$ for all values of κ); λ^q_{N+1} is not known, but, for N sufficiently large, it can be replaced by its asymptotic approximation (see §7.2), i.e.

$$\lambda^q_{N+1}(\kappa) \sim -(\sin\phi)^{N+1}.$$

7. Numerical solutions

In this section, we shall describe some further simplifications which enable us to give numerical solutions of S for the particular case of an obliquely-incident SV-wave.

7.1. *Reduction of the kernels*

The kernels of our integral equations are given as infinite integrals. These can be reduced to finite integrals using a contour-integral method, as described by, for example, Noble (1962, p. 343): write $g(\xi) = g(\xi;\beta;\gamma)$ to show the dependence on β and γ. If $g(\xi)\to 0$ as $\xi\to\infty$ and $\xi^2 g(\xi)\to 0$ as $\xi\to 0$, then

$$\begin{aligned}\int_0^\infty \xi^2 g(\xi)\, j_n(\xi x)\, j_n(\xi y)\,\mathrm{d}\xi \\ &= \frac{1}{2}\int_0^k \xi^2 h_n^{(1)}(\xi x_>)\, j_n(\xi x_<)\{g(\xi;\,-\mathrm{i}\beta';\,-\mathrm{i}\gamma')-g(\xi;\mathrm{i}\beta';\mathrm{i}\gamma')\}\,\mathrm{d}\xi \\ &\quad+\frac{1}{2}\int_k^K \xi^2 h_n^{(1)}(\xi x_>)\, j_n(\xi x_<)\{g(\xi;\,-\mathrm{i}\beta';\gamma)-g(\xi;\mathrm{i}\beta';\gamma)\}\,\mathrm{d}\xi, \end{aligned} \tag{7.1}$$

where $\beta'(\xi) = (K^2-\xi^2)^{\frac{1}{2}}$, $\gamma'(\xi) = (k^2-\xi^2)^{\frac{1}{2}}$,

$$x_> = \max(x,y), \quad x_< = \min(x,y) \quad \text{and} \quad h_n^{(1)}(z) = (\pi/2z)^{\frac{1}{2}} H^{(1)}_{n+\frac{1}{2}}(z).$$

Thus, the infinite integral has been replaced by two finite integrals along the branch cuts of β and γ. It follows that terms in g that are continuous across these cuts will make no contribution.

Using (7.1), we quickly find that

$$N_n(x,y) = -\frac{2}{\pi}\mathrm{i}xy\int_0^K \xi m(\xi)\, H_n(\xi x,\xi y)\,\mathrm{d}\xi, \tag{7.2}$$

and

$$K_n^{\pm}(x,y) = -\frac{2}{\pi}\mathrm{i}xy\int_0^K \xi m^{\pm}(\xi)\, H_n(\xi x,\xi y)\,\mathrm{d}\xi, \tag{7.3}$$

where

$$\begin{aligned} K^2 m(\xi) &= (1-\nu)\{(2\xi^2-K^2)^2(k^2-\xi^2)^{-\frac{1}{2}}H(k-\xi)+4\xi^2(K^2-\xi^2)^{\frac{1}{2}}\},\\ K^2 m^{+}(\xi) &= (1-\nu)\{(2\xi^2-K^2)^2(K^2-\xi^2)^{-\frac{1}{2}}+4\xi^2(k^2-\xi^2)^{\frac{1}{2}}H(k-\xi)\},\\ m^{-}(\xi) &= (K^2-\xi^2)^{\frac{1}{2}}. \end{aligned}$$

Here, $H(x)$ is the Heaviside step function, and

$$H_n(\xi x,\xi y) = h_n^{(1)}(\xi x_>)\, j_n(\xi x_<). \tag{7.4}$$

Also, from (4.35), we obtain

$$L_n^{\pm}(x) = -\frac{2}{\pi}\mathrm{i}x\int_0^K \frac{m^{\pm}(\xi)}{\xi^2}\left\{\frac{\mathrm{i}x^n}{2n+1}+\xi H_n(\xi x,\xi)\right\}\mathrm{d}\xi, \tag{7.5}$$

by integrating (7.3).

Equations (7.2), (7.3) and (7.5) are suitable for the numerical evaluation of N_n, $K_n^{\pm}$ and $L_n^{\pm}$, respectively. They can also be used to obtain low-frequency asymptotic solutions of the integral equations, for we can expand $H_n(\xi x,\xi y)$ as an ascending power series in ξ; see Appendix B for these expansions. In Appendix C, we derive low-frequency approximations for the diffraction of a normally-incident SV-wave, and compare our results with those obtained by Mal (1968*c*) and Jain & Kanwal (1972); we find complete agreement with those of Jain & Kanwal.

7.2. *Asymptotic behaviour for large n*

In this section, we shall obtain the asymptotic behaviour of all quantities as $n\to\infty$, for fixed K (recall that the crack has unit radius). Our procedure is to note that, for K fixed, each of the kernels N_n and $K_n^{\pm}$ tends to zero as $n \to \infty$, and hence the associated integral equations can be solved by iteration for sufficiently large n. Since the original integral equation (2.8) has a small kernel in the limit $K \to 0$ (Martin 1981), it is clear that the estimation that we are about to provide also gives an approximation to the general solution of (2.8) for sufficiently small K.

We begin with $\theta_n(x;\kappa)$, which satisfies (3.18). From (7.4), we have (see Appendix B)

$$H_n(\xi x, \xi y) = \frac{-\mathrm{i}x_<^n}{(2n+1)\,\xi x_>^{n+1}}\{1+O(1/n)\} \quad \text{as} \quad n \to \infty,$$

whence an approximation to $N_n(x,y)$ follows from (7.2). Hence, solving the integral equation (3.18), we obtain the estimate

$$\theta_n(x;\kappa) \sim x j_n(\kappa x)\{1+A[(2n+1)^{-1}-x^2(2n+3)^{-1}]\},$$

as $n \to \infty$, where

$$\begin{aligned} A &= \frac{1}{\pi}\int_0^K m(\xi)\,\mathrm{d}\xi \\ &= \tfrac{1}{4}(1-\nu)\,K^2(3-4\sigma^2+3\sigma^4), \end{aligned}$$

and $\sigma^2 = k^2/K^2 = \frac{1}{2}(1-2\nu)/(1-\nu)$. Similarly,

$$q_n(x;\kappa) \sim x j_n(\kappa x)\{1+A^+[(2n+1)^{-1}-x^2(2n+3)^{-1}]\}$$

and
$$p_n^{\pm}(x) \sim -2x^{n+1}A^{\pm}/(2n+1),$$

where
$$A^{\pm} = \frac{1}{\pi}\int_0^K m^{\pm}(\xi)\,\mathrm{d}\xi,$$

i.e.
$$A^+ = \tfrac{1}{4}(1-\nu)\,K^2(1+\sigma^4)$$

and
$$A^- = \tfrac{1}{4}K^2.$$

From equation 7.14.1 (9) of Erdélyi *et al.* (1953), we have

$$I_n^{\theta}(\kappa,\mathscr{K}) \sim \int_0^1 x^2 j_n(\kappa x)\,j_n(\mathscr{K}x)\,\mathrm{d}x \sim (2n+3)\,(\kappa\mathscr{K})^{-1} j_{n+1}(\kappa)\,j_{n+1}(\mathscr{K}), \tag{7.6}$$

whence, from (6.13), we obtain

$$\lambda_n^{\theta}(\kappa) \sim -(\sin\phi)^n; \tag{7.7}$$

identical estimates can be derived for I_n^q and λ_n^q, respectively. Also, from (6.2), we have

$$\int_0^1 x p_n^{\pm}(x)\,j_n(\mathscr{K}x)\,\mathrm{d}x \sim -2A^{\pm} j_{n+1}(\mathscr{K})/(2n+1). \tag{7.8}$$

Let us now estimate $S_n^{\pm}(\kappa)$. Using the definitions given in §4.1, we find that

$$c_n^{\pm} = 1+O(n^{-2}),$$

and
$$e_n^{\pm} = O(n^{-3}),$$

whence $$d_n^{\pm} = \pm 1 + O(n^{-2}),$$

and $$\Delta_n = -2 + O(n^{-2})$$

as $n \to \infty$. Also, $$F_n(\kappa) \sim 2j_{n+1}(\kappa),$$

and $$G_n(\kappa) \sim -2(1+\alpha) j_{n-1}(\kappa),$$

whence $$S_n^{\pm}(\kappa) \sim \mp (1+\alpha) j_{n-1}(\kappa).$$

The above calculations allow us to estimate, for a fixed value of K, the number of Fourier components required to compute the stress–intensity factors or the far-field displacements, to a specified accuracy. For example, with $K = 10$, we can compute the far-field displacements to about 5 decimal places by taking the first 15 Fourier components; to compute the stress–intensity factors to the same accuracy would require about 20 Fourier components.

7.3. *The scattering cross section for* P-*waves at normal incidence*

Consider a plane compressional wave that is incident upon the crack. We suppose that this wave propagates in the plane $\theta = 0$, at an angle ϕ to the z-axis. The radial component of the scattered displacement, u_R, in the plane $\theta = 0$, has the form (see (5.1))
$$u_R(R, 0, \Phi) \sim F(\Phi)\, \mathrm{e}^{\mathrm{i}kR}/R,$$
as $R \to \infty$, and the scattering cross section, Σ_{P} is defined by (Barratt & Collins 1965)
$$\Sigma_{\mathrm{P}} = 4\pi\sigma^{-2}\, \mathrm{Re}\,\{F(\phi)\}.$$

We have computed Σ_{P} for the special case of a P-wave at normal incidence ($\phi = 0$), and compared our results with those of Keogh (1983) and Krenk & Schmidt (1982). This is the simplest possible case, because the problem is axisymmetric and is governed by a single integral equation, namely Robertson's equation, (2.4). This equation was solved using a simple quadrature method (see, for example, Baker 1977, §4.3), yielding an approximation to $\theta_0(x)$ at a set of M equally spaced points in $[0, 1]$, including the end points (recall that the stress-intensity factor, k_1, is proportional to $\theta_0(1)$; see (5.17)). This can then be used to obtain an approximation to $F(0)$.

In the third column of table 1, we present our computed values of Σ_{P}/π for $K = 1.0\,(1.0)\,10.0$ and $\nu = 0.25$. In the first column, we reproduce the corresponding results obtained by Krenk & Schmidt (1982; their table 1) while the second contains those computed by Keogh (1983) from a rigorous short-wave ($K \to \infty$) asymptotic analysis. We see that the present results (column 3) agree with those of Krenk & Schmidt for $K \lesssim 3.0$ and are confirmed by Keogh's analysis for larger values of K. This suggests that the numerical scheme used by Krenk & Schmidt is inaccurate at high frequencies, possibly because their approximating polynomial for w_0 is not of sufficiently high order. Accordingly, we may expect similar inaccuracies to arise in their treatment of asymmetric problems.

TABLE 1. THE NORMALIZED SCATTERING CROSS SECTION, Σ_P/π, FOR A P-WAVE AT NORMAL INCIDENCE WITH $\nu = 0.25$, AS COMPUTED BY KRENK & SCHMIDT (1982), KEOGH (1983) AND THE PRESENT METHOD

K	Krenk & Schmidt	Keogh	present method
0.5	0.009	—	0.009
1.0	0.214	—	0.214
2.0	2.894	3.066	2.895
3.0	1.910	1.836	1.910
4.0	1.655	1.617	1.600
5.0	2.106	2.364	2.314
6.0	1.801	1.851	1.877
7.0	1.987	1.770	1.831
8.0	1.941	2.212	2.208
9.0	—	1.901	1.896
10.0	—	1.942	1.925

As K increases, the kernel of the integral equation (2.4) oscillates more rapidly, and so it is necessary to compute $\theta_0(x)$ at more points in $[0, 1]$. For example, in table 2 we give our computed values of Σ_P/π for several values of M, at $K = 8$ and $\nu = 0.25$; an accuracy of four significant figures is obtained with $M = 30$.

TABLE 2. THE NUMERICAL CONVERGENCE OF Σ_P/π WITH INCREASING M, FOR A P-WAVE AT NORMAL INCIDENCE WITH $\nu = 0.25$, WHERE M IS THE NUMBER OF POINTS IN $[0, 1]$ AT WHICH THE APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION (2.4) IS COMPUTED

M	Σ_P/π
10	2.202
15	2.212
20	2.210
25	2.209
30	2.208
35	2.208

7.4. *The scattering of an* SV*-wave at oblique incidence*

We have used our method to compute the far-field scattered displacement when the crack is insonified by a plane shear wave polarized in the plane $\theta = 0$ and propagating at an angle ϕ to the z-axis. The component of the scattered displacement u_ϕ in the plane $\theta = 0$ has the form (see (5.3))

$$u_\phi(R, 0, \Phi) \sim G(\Phi)\, \mathrm{e}^{\mathrm{i}KR}/R,$$

as $R \to \infty$, and the scattering cross-section, Σ_S, is defined by

$$\Sigma_S = 4\pi \,\mathrm{Re}\,\{G(\phi)\}.$$

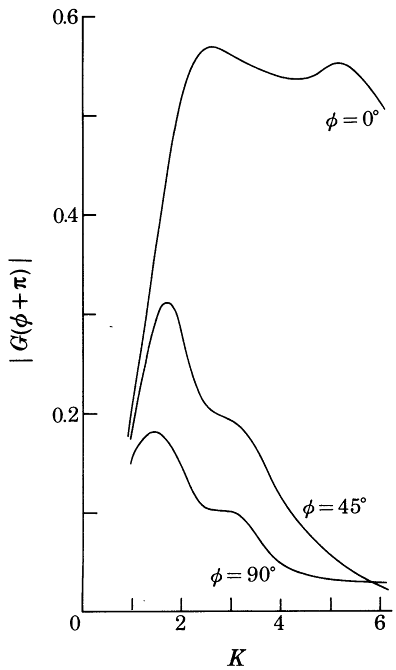

FIGURE 1. The amplitude of the back-scattered SV-wave, $|G(\phi+\pi)|$, for an incident SV-wave, as a function of the shear wavenumber K, for 3 angles of incidence ($\phi = 0°$, 45° and 90°) and Poisson's ratio $\nu = 0.25$.

TABLE 3. THE NORMALIZED SCATTERING CROSS SECTION, Σ_S/π, FOR AN SV-WAVE AT VARIOUS ANGLES OF INCIDENCE, $1 \leqslant K \leqslant 10$, AND $\nu = 0.25$

(The numbers in parentheses (for $1 \leqslant K \leqslant 8$) are the corresponding results of Krenk & Schmidt (1982).)

K	ϕ... 0°	15°	30°	45°	60°	75°	90°
1.0	0.068	0.081	0.105	0.112	0.095	0.069	0.056
	(0.068)	(0.081)	(0.105)	(0.112)	(0.095)	(0.069)	(0.056)
2.0	1.145	1.220	1.305	1.199	0.928	0.669	0.567
	(1.145)	(1.220)	(1.304)	(1.199)	(0.928)	(0.669)	(0.567)
3.0	1.970	1.700	1.521	1.627	1.414	0.958	0.737
	(1.972)	(1.700)	(1.519)	(1.619)	(1.403)	(0.950)	(0.733)
4.0	2.036	1.683	1.425	1.665	1.573	1.098	0.851
	(2.048)	(1.690)	(1.419)	(1.639)	(1.571)	(1.160)	(0.944)
5.0	2.136	1.768	1.299	1.667	1.757	1.235	0.941
	(2.187)	(1.779)	(1.203)	(1.327)	(1.405)	(1.073)	(0.872)
6.0	2.034	2.032	1.382	1.534	1.891	1.392	1.048
	(1.973)	(1.975)	(1.234)	(1.157)	(1.494)	(1.194)	(0.955)
7.0	1.834	2.040	1.421	1.388	1.863	1.488	1.140
	(1.820)	(2.012)	(1.280)	(0.981)	(1.207)	(1.004)	(0.811)
8.0	1.943	1.947	1.533	1.271	1.773	1.553	1.254
	(1.977)	(1.872)	(1.182)	(0.868)	(1.128)	(0.990)	(0.811)
9.0	2.041	1.954	1.574	1.265	1.660	1.636	1.363
10.0	2.044	1.933	1.564	1.306	1.516	1.703	1.464

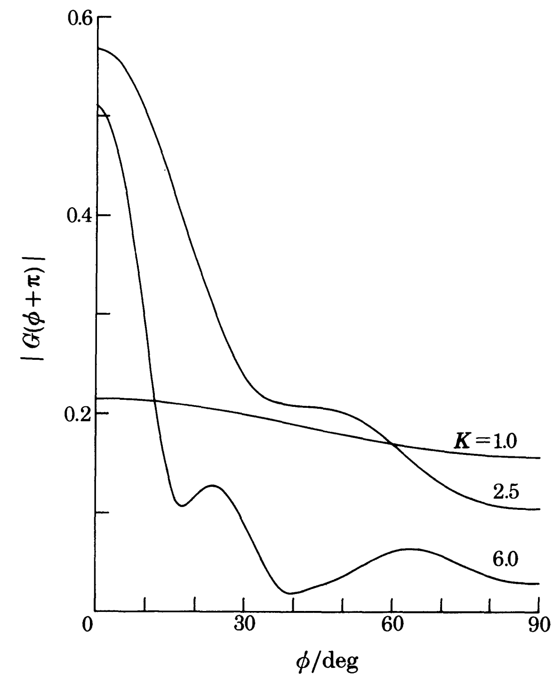

FIGURE 2. The amplitude of the back-scattered SV-wave, $|G(\phi+\pi)|$, for an incident SV-wave, as a function of the angle of incidence ϕ, for 3 values of the shear wavenumber ($K = 1.0$, 2.5 and 6.0) and Poisson's ratio $\nu = 0.25$.

Figure 1 shows the back-scattered amplitude, $|G(\phi+\pi)|$, as a function of K, for $\nu = 0.25$ and three values of ϕ, namely 0° (normal incidence), 45° and 90° (grazing incidence); in the second case, the incident wave induces only normal stresses on the crack faces. These graphs show that the resonant frequency for normal incidence corresponds to $K \approx 2.6$. Figure 2 shows the back-scattered amplitude as a function of ϕ, for $\nu = 0.25$ and three values of K, namely 1.0, 2.5 and 6.0.

In table 3, we present our computed values of Σ_S/π for $K = 1.0\,(1.0)\,10.0$, $\phi = 0°\,(15°)\,90°$ and $\nu = 0.25$. For comparison, we also reproduce the corresponding results of Krenk & Schmidt (1982; their table 2). This comparison shows discrepancies which increase as K increases (as in the P-wave case; see §7.3) and also as the angle of incidence, ϕ, increases. For example, when $K = 8$ and $\phi = 90°$, Krenk & Schmidt give a result that differs from our result by some 35%. The probable cause of the increase with ϕ is that Krenk & Schmidt only use 8 azimuthal harmonics to compute their estimates whereas it may readily be shown, using the asymptotic formulae in §7.2, that this is insufficient. Indeed, when $K = 8$ and $\phi = 90°$, we found that to obtain convergence to four significant figures, 11 harmonics were required.

In this paper, we have obtained the exact solution of the boundary-value problem S. It is of interest to compare this solution with certain well known 'approximate' solutions. Thus, figure 3 shows the back-scattered amplitude, $|G(\phi+\pi)|$, as a

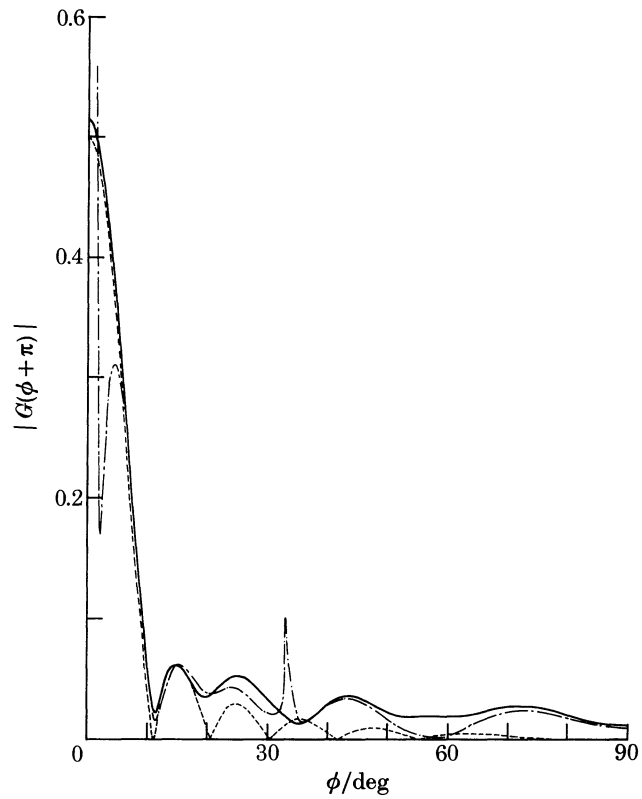

FIGURE 3. The amplitude of the back-scattered SV-wave, $|G(\phi+\pi)|$, for an incident SV-wave, as a function of the angle of incidence ϕ, for $K = 10.0$ and $\nu = 0.29$. The solid line is the exact solution, obtained by the present method. The dashed lines are two approximations to $|G(\phi+\pi)|$: - - - -, the Kirchhoff approximation; — · — ·, the ray-theory approximation (Chapman 1983).

function of ϕ for $K = 10$ and $\nu = 0.29$ (corresponding to steel) obtained by (i) the Kirchhoff approximation, (ii) Keller's ray theory (GTD) and (iii) the present method. The Kirchhoff approximation is derived as follows: replace $\rho_i(q) = [u_i(q)]$ in the integral representation (2.7) by the scattered displacement that would be induced at q if the crack were replaced by an infinite, plane, traction-free surface (with the same incident wave). This yields the following approximation:

$$G(\phi+\pi) \approx -\cos\phi\, J_1(2K\sin\phi)/2K\sin\phi.$$

(Note that this formula is independent of ν.) This approximation appears to be accurate near normal incidence, but is clearly only reliable up to the first side-lobe ($\phi \approx 18°$). The ray-theory approximation shown in figure 3 was computed by Chapman (1983). It is apparent that this approximation is good, except near normal incidence, near the 'critical' angle arcsin $(k/K) \approx 33°$ and in the interval $50° \lesssim \phi \lesssim 70°$. At normal incidence, it is well known that GTD fails because the z-axis is a caustic of the diffracted field. Keogh (1983) has shown (for the two-dimensional problem of scattering by a Griffith crack) that the discrepancy at the critical angle is due to multiply-diffracted body waves (which are neglected in GTD). It is possible that the discrepancy in the third range of ϕ may be due to a similar effect.

8. Conclusions

The diffraction of stress waves by a penny-shaped crack in an elastic solid is a canonical problem in the theory of three-dimensional elastodynamics. In a previous paper (Martin 1981), we have proved that the corresponding linear boundary-value problem is uniquely solvable: the solution ($\boldsymbol{u}$, say) has a representation as an elastic double layer whose (vector) density satisfies a two-dimensional Fredholm integral equation of the second kind (see § 2.4). In the present paper, we have expressed $\boldsymbol{u}$ in a form which renders its computation straightforward, i.e. we have uncoupled the components of $\boldsymbol{u}$ and expressed them in terms of the solutions of certain simple one-dimensional Fredholm integral equations of the second kind. The principal computational task is the routine calculation of the kernels of the integral equations.

In § 6, we have derived some imbedding formulae which simplify the computation of the quantities of physical interest. Moreover, the method of derivation of these formulae is applicable to the corresponding problems in acoustics and electromagnetism.

We have computed the far-field displacements for a P-wave at normal incidence and for an SV-wave at various angles of incidence, and compared our results with the high-frequency asymptotic estimates of Keogh (1983) and the numerical results of Krenk & Schmidt (1982). We found good agreement with Keogh's formula. We conclude that Krenk & Schmidt's scheme is inaccurate both at high frequencies and near grazing incidence, although this may merely be due to premature truncation of their infinite systems of equations. However, it should be noted that their formulation does not afford *a priori* estimates of the error incurred by truncation at a certain point, whereas we have provided asymptotic estimates (for large n and fixed K; see § 7.2).

In the future we hope to present our computations of the dynamic stress–intensity factors for various incident waves. Another problem of interest is the scattering of waves generated by a point source, or by a source of finite extent. Also, further work is required on high-frequency scattering, where the present approach is computationally inefficient; some progress with the axisymmetric problem has been made recently by Keogh (1983).

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Appendix A. The incident plane waves

Consider an incident plane wave, with displacement vector

$$\boldsymbol{u}^{(i)}(\boldsymbol{r}) = \boldsymbol{A}\mathrm{e}^{\mathrm{i}K\boldsymbol{N}\cdot\boldsymbol{r}} + B\boldsymbol{N}\mathrm{e}^{\mathrm{i}k\boldsymbol{N}\cdot\boldsymbol{r}},$$

where $\boldsymbol{r} = (x, y, z)$, $\boldsymbol{N}\cdot\boldsymbol{A} = 0$ and $\boldsymbol{N}$ is a unit vector in the direction of propagation. Write

$$\boldsymbol{N} = (\sin\phi, 0, \cos\phi) \quad \text{and} \quad \boldsymbol{A} = A(\cos\phi, 0, -\sin\phi),$$

i.e. the wave is propagating at an angle ϕ to the z-axis, with the motion restricted to the xz-plane. Differentiating, we find that

$$\tau_{xz}^{(i)}(\boldsymbol{r}) = \mu\mathrm{i}KA\cos 2\phi\mathrm{e}^{\mathrm{i}K\boldsymbol{N}\cdot\boldsymbol{r}} + \mu\mathrm{i}kB\sin 2\phi\mathrm{e}^{\mathrm{i}k\boldsymbol{N}\cdot\boldsymbol{r}},$$

$$\tau_{zz}^{(i)}(\boldsymbol{r}) = -\mu\mathrm{i}KA\sin 2\phi\mathrm{e}^{\mathrm{i}K\boldsymbol{N}\cdot\boldsymbol{r}} + \mu(\mathrm{i}kB/\sigma^2)(1-2\sigma^2\sin^2\phi)\,\mathrm{e}^{\mathrm{i}k\boldsymbol{N}\cdot\boldsymbol{r}},$$

and

$$\tau_{yz}^{(i)}(\boldsymbol{r}) = 0,$$

where $\sigma^2 = k^2/K^2$.

P-*wave at oblique incidence*. Take $A = 0$ and $\mathrm{i}kB = \sigma^2$, whence, on $z = 0$,

$$\tau_{xz}^{(i)}(r,\theta,0) = \mu\sigma^2\sin 2\phi\mathrm{e}^{\mathrm{i}kx\sin\phi}$$

and

$$\tau_{zz}^{(i)}(r,\theta,0) = \mu(1-2\sigma^2\sin^2\phi)\,\mathrm{e}^{\mathrm{i}kx\sin\phi}.$$

We must now decompose these expressions into their Fourier series in θ, as required in §§3 and 4. Since $x = r\cos\theta$, we may use (5.7) immediately to give

$$\tau_n(r) = D_n(\phi)\,J_n(kr\sin\phi),$$

with

$$D_n(\phi) = \epsilon_n\mathrm{i}^n(1-2\sigma^2\sin^2\phi).$$

For the shear problem, we have

$$\tau_{rz}^{(i)}(r,\theta,0) = \tau_{xz}^{(i)}(r,\theta,0)\cos\theta \quad \text{and} \quad \tau_{\theta z}^{(i)}(r,\theta,0) = -\tau_{xz}^{(i)}(r,\theta,0)\sin\theta.$$

Thus, since

$$\cos\theta\mathrm{e}^{\mathrm{i}\lambda\cos\theta} = \tfrac{1}{2}\sum_{n=0}^{\infty}\epsilon_n\mathrm{i}^{n+1}\{J_{n+1}(\lambda) - J_{n-1}(\lambda)\}\cos n\theta$$

and

$$-\sin\theta\mathrm{e}^{\mathrm{i}\lambda\cos\theta} = \sum_{n=1}^{\infty}\mathrm{i}^{n+1}\{J_{n+1}(\lambda) + J_{n-1}(\lambda)\}\sin n\theta,$$

we have

$$t_0(r) = C_0(\phi)\,J_1(kr\sin\phi)$$

and

$$t_n(r) \pm s_n(r) = \pm C_n(\phi)\,J_{n\pm1}(kr\sin\phi),$$

with

$$C_n(\phi) = \epsilon_n\mathrm{i}^{n+1}\sigma^2\sin 2\phi.$$

SV-*wave at oblique incidence*. Take $B = 0$ and $\mathrm{i}KA = 1$, whence, on $z = 0$,

$$\tau_{xz}^{(i)}(r,\theta,0) = \mu\cos 2\phi\mathrm{e}^{\mathrm{i}Kx\sin\phi}$$

and

$$\tau_{zz}^{(i)}(r,\theta,0) = -\mu\sin 2\phi\mathrm{e}^{\mathrm{i}Kx\sin\phi}.$$

Hence,

$$\tau_n(r) = D_n(\phi)\,J_n(Kr\sin\phi),$$

$$t_0(r) = C_0(\phi)\,J_1(Kr\sin\phi)$$

and

$$t_n(r) \pm s_n(r) = \pm C_n(\phi)\,J_{n+1}(Kr\sin\phi),$$

with

$$D_n(\phi) = -\epsilon_n\,\mathrm{i}^n \sin 2\phi \tag{A 1}$$

and

$$C_n(\phi) = \epsilon_n\,\mathrm{i}^{n+1}\cos 2\phi. \tag{A 2}$$

Appendix B. Power-series expansions

The kernels $K_n^{\pm}$ and N_n are given by (7.3) and (7.2), respectively, where H_n is defined by (7.4). We have

$$H_n(\xi x, \xi y) = \sum_{m=0}^{\infty} (\mathrm{i}\xi)^{m-1} P_m^n(x,y), \quad n \geqslant 0, \tag{B 1}$$

where $P_m^n(x,y)$ can be found by using equations 7.11 (3) and 7.2.1. (2) from Erdélyi *et al.* (1953), namely

$$h_n^{(1)}(\xi x) = 2\mathrm{e}^{\mathrm{i}\xi x} \sum_{m=0}^{n} \frac{(n+m)!\,\mathrm{i}^{m-n-1}}{m!(n-m)!\,(2\xi x)^{m+1}},$$

and

$$j_n(\xi y) = \tfrac{1}{2}\pi^{\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{1}{2}\xi y)^{n+2m}}{m!\,\Gamma(m+n+\frac{3}{2})}.$$

We note that $P_m^n(x,y)$ are real; in particular,

$$P_0^n(x,y) = x_<^n x_>^{-(n+1)}/(2n+1). \tag{B 2}$$

Substituting (B 1) into (7.2), we obtain

$$N_n(x,y) = xyK^2 \sum_{m=0}^{\infty} (\mathrm{i}K)^m P_m^n(x,y)\,l_m,$$

where

$$\pi l_m = -2(1-\nu)\int_0^{\frac{1}{2}\pi} \{(\sigma\sin\theta)^m (2\sigma^2\sin^2\theta - 1)^2 + 4\cos^2\theta\,(\sin\theta)^{m+2}\}\,\mathrm{d}\theta$$

and $\sigma = k/K$. The coefficient l_m, which depends only on Poisson's ratio, can be evaluated explicitly: we have

$$\pi l_m = -8(1-\nu)\{(\sigma^{m+4}-1)\,E_m + (1-\sigma^{m+2})\,E_{m+2} + \tfrac{1}{4}\sigma^m E_m)\},$$

where

$$E_m = \int_0^{\frac{1}{2}\pi} \sin^m\theta\,\mathrm{d}\theta,$$

i.e. $\quad E_{2n} = \Gamma(\frac{3}{2})\,\Gamma(n+\frac{1}{2})/n! \quad$ and $\quad E_{2n-1} = \Gamma(\frac{3}{2})\,(n-1)!/\Gamma(n+\frac{1}{2}).$

Similarly, we have

$$K_n^{\pm}(x,y) = xy\,K^2 \sum_{m=0}^{\infty} (\mathrm{i}K)^m P_m^n(x,y)\,l_m^{\pm}, \tag{B 3}$$

and

$$L_n^{\pm}(x) = -xK^2 \sum_{m=0}^{\infty} (\mathrm{i}K)^m P_{m+2}^n(x,y)\, l_m^{\pm}, \tag{B 4}$$

where

$$\pi l_m^{+} = 8(1-\nu)\{(\sigma^{m+4}-1)(E_{m+4}-E_{m+2})-\tfrac{1}{4}E_m\}, \tag{B 5}$$

$$\pi l_m^{-} = 2(E_{m+2}-E_m), \tag{B 6}$$

and we have noted that $P_1^n(x,y) = 0$ for $n \geqslant 1$.

Appendix C. Diffraction of an SV-wave at normal incidence

For an SV-wave, polarized in the xz-plane ($\theta = 0$) and at normal incidence ($\phi = 0$), we have (see Appendix A)

$$t_1 = -s_1 = 1, \quad C_1 = -2 \quad \text{and} \quad C_n = 0 \quad \text{for} \quad n \neq 1.$$

Taking $\kappa = K\sin\phi$, letting $\phi \to 0$ and putting $n = 1$, we find that

$$q_1 = F_1 = I_1^q = 0$$

and

$$G_1 = -2(1+\alpha).$$

Now, from (B 3),

$$\begin{aligned} K_1^{\pm}(x,y) &= xyK^2P_0^1(x,y)\, l_0^{\pm} + O(K^3) \\ &= \tfrac{1}{3}x_<^2 K^2 l_0^{\pm}/x_< + O(K^3), \end{aligned}$$

as $K \to 0$, and (from (4.35) or (B 4))

$$L_1^{\pm}(x) = K^2 l_0^{\pm}\, x^2(5-x^2)/30 + O(K^3).$$

Hence,

$$\begin{aligned} p_1^{\pm}(x) &= K_1^{\pm}(x,1) + O(K^4) = \tfrac{1}{3}x^2K^2 l_0^{\pm} + O(K^3), \\ c_1^{\pm} &= 1 - \int_0^1 x^2 p_1^{\pm}(x)\,\mathrm{d}x = 1 - K^2 l_0^{\pm}/15 + O(K^3), \\ e_1^{\pm} &= L_1^{\pm}(1) + O(K^4) = 2K^2 l_0^{\pm}/15 + O(K^3), \\ d_1^{\pm} &= \pm\{c_1^{\pm} + 3(1\pm\alpha)\, e_1^{\pm}\} = \pm\{1 + K^2 l_0^{\pm}(5\pm 6\alpha)/15\} + O(K^3) \end{aligned}$$

and

$$\begin{aligned} \Delta_1 &= c_1^+ d_1^- - d_1^+ c_1^- \\ &= -2 + 2K^2\{(3\alpha-2)\, l_0^- - (3\alpha+2)\, l_0^+\}/15 + O(K^3), \end{aligned}$$

as $K \to 0$.

Stress–intensity factors. From (5.18) and (5.19), we have

$$k_2(\theta) = -(2/\pi)\, S_1^+ \cos\theta \quad \text{and} \quad k_3(\theta) = -(2/\pi)(1-\nu)\, S_1^- \sin\theta,$$

where

$$S_1^+ = -c_1^- G_1/\Delta_1 \quad \text{and} \quad S_1^- = c_1^+ G_1/\Delta_1.$$

Thus,

$$S_1^{\pm} = \mp(1+\alpha)\{1 + K^2 A_1^{\pm}/15 + O(K^3)\},$$

where

$$A_1^+ = 3(\alpha-1)\, l_0^- - (3\alpha+2)\, l_0^+,$$

and

$$A_1^- = (3\alpha-2)\, l_0^- - 3(\alpha+1)\, l_0^+.$$

From (B 5) and (B 6), we have $l_0^- = -\frac{1}{2}$ and $l_0^+ = -\frac{1}{2}(1-\nu)(1+\sigma^4)$, with $\sigma^2 = k^2/K^2$. Thus, since $\alpha = \nu/(2-\nu)$ and $2\nu = (1-2\sigma^2)/(1-\sigma^2)$, we obtain

$$A_1^+ = \frac{21-22\sigma^2+9\sigma^4-10\sigma^6}{4(3-2\sigma^2)(1-\sigma^2)},$$

and

$$A_1^- = \frac{9+2\sigma^2+6\sigma^4}{2(3-2\sigma^2)},$$

and this completes our low-frequency expansions for the dynamic stress-intensity factors. Similar expansions have been obtained by Jain & Kanwal (1972) and by Mal (1968*c*). Our results are in complete agreement with those given by Jain & Kanwal, and thus confirm their statement that Mal's results are incorrect.

Far-field displacements. From (5.21), we have

$$\begin{aligned}\mathscr{K} I_1^\pm &= S_1^\pm\{-j_1(\mathscr{K}) + \int_0^1 x p_1^\pm(x) j_1(\mathscr{K}x)\,dx\} \\ &= S_1^\pm\{-j_1(\mathscr{K}) + \tfrac{1}{3}K^2 l_0^\pm j_2(\mathscr{K})/\mathscr{K} + O(K^3\mathscr{K})\},\end{aligned}$$

where we have used (6.2). Thus

$$I_1^\pm = -\tfrac{1}{3}S_1^\pm\{1+K^2 l_0^\pm/15 - \mathscr{K}^2/10\} + O(K^3).$$

Substituting into (5.9) and (5.10), we obtain

$$I_c(\mathscr{K};\Theta) = \frac{8(1-\nu)}{3\pi(2-\nu)}\{1+\tfrac{1}{15}K^2(A_1^\pm - l_0^+) - \tfrac{1}{10}\mathscr{K}^2 + O(K^3)\}\cos\Theta$$

and

$$I_s(\mathscr{K};\Theta) = \frac{8(1-\nu)}{3\pi(2-\nu)}\{1+\tfrac{1}{15}K^2(A_1^- - l_0^-) - \tfrac{1}{10}\mathscr{K}^2 + O(K^3)\}\sin\Theta$$

Substituting for $A_1^\pm$ and $l_0^\pm$, we see that

$$\begin{aligned}A_1^+ - l_0^+ = A_1^- - l_0^- &= 3\{(\alpha-1)l_0^- - (\alpha+1)l_0^+\} \\ &= 3(2+\sigma^4)/(3-2\sigma^2).\end{aligned}$$

Jain & Kanwal (1972) have also obtained low-frequency asymptotic expansions for I_c and I_s. We find complete agreement with their results (apart from a typographical error in their equation (100): the second term inside the first pair of square brackets should be $-(1+\tau^4)/(1-\tau^2)$). It is noteworthy that $I_c(\mathscr{K};\Theta)\sin\Theta = I_s(\mathscr{K};\Theta)\cos\Theta$, correct to $O(K^2)$.