INTEGRAL-EQUATION METHODS FOR MULTIPLE-SCATTERING PROBLEMS I. ACOUSTICS

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SUMMARY

Integral-equation methods are often used to treat the exterior problems of acoustics. It is known that the simplest equations fail to be uniquely solvable at certain frequencies (the *irregular* frequencies). For a single smooth scatterer, D. S. Jones has shown how any given irregular frequency can be removed by using a fundamental solution which has a finite number of additional singularities inside the scatterer. This approach is extended here to treat the two-dimensional exterior Neumann problem for a pair of scatterers, using a fundamental solution which has additional singularities inside each of them. A partial generalization of Jones's result is obtained, involving fundamental solutions with an *infinite* number of singularities inside one scatterer and a finite number inside the other. Similar results can be obtained for the Dirichlet problem, and in three dimensions.

1. Introduction

Two rigid cylinders, with their generators vertical, are immersed in water, and a surface wave is incident upon them. If the water is of uniform depth and the cylinders reach down to the bottom, the corresponding three-dimensional linear boundary-value problem can be reduced to a two-dimensional problem by separating out the dependence on the vertical coordinate (**1**). It is this plane problem that we shall study here.

To be more specific, let D denote the infinite region exterior to two simple, closed, disjoint, Lyapunov curves, ∂D^1 and ∂D^2. Let $\partial D = \partial D^1 \cup \partial D^2$. Then we wish to find a function u which satisfies the two-dimensional Helmholtz equation

$$(\nabla^2 + k^2)u = 0$$

in D, together with a Neumann boundary condition on ∂D and a radiation condition at infinity (see section 4). This boundary-value problem (labelled $\mathcal{P}_2$ below) also arises in other contexts, for example in acoustic scattering by a pair of sound-hard cylinders. There are also obvious generalizations to more scatterers, and to three dimensions.

Several methods have been devised for treating $\mathcal{P}_2$. In section 2 we give a brief review, concentrating on exact methods (which, at least formally, could yield the exact solution) rather than approximate methods (which can, at best, only yield approximations to the solution). We shall use integral-equation methods to treat $\mathcal{P}_2$. It is straightforward to obtain integral equations for $\mathscr{P}_2$, using Green's theorem or source distributions, but the simplest of these fail to be uniquely solvable at certain values of k^2 (called *irregular* values). We shall obtain an integral equation which is *always* uniquely solvable. This is an extension to scattering by two cylinders of some work by Jones (**2**) and Ursell (**3**) on scattering by one cylinder; their work is described in section 3. We give the proof of our result in section 4; we use a modified fundamental solution which has additional singularities inside both scatterers.

2. A brief survey

The literature on multiple-scattering problems is vast, and was first surveyed by Twersky (**4**) in 1960; an addendum was published four years later (**5**). We shall begin our survey by describing some of this early work (section 2.1); the main emphasis is on exact formulations and analytical approximations. More recently, the emphasis has shifted towards numerical solutions; this work will be described in section 2.2.

2.1. *Early researches*

In his review, Twersky (**4**) identified three different methods: 'One may seek to solve the boundary-value problem for the "compound body"; one may use a self-consistent procedure based on the known response of the isolated elements, such that each object is considered as excited by the primary wave plus the resultant of the initially unknown total scattered fields of the other objects; or onc may use an iterative procedure corresponding to the "successive scatterings" of the primary wave.'

The first method usually reduces to finding an integral equation over ∂D, using the free-space wave source G_0 (defined by (3.4) below). For two *circular* cylinders this equation can be replaced by an infinite system of linear algebraic equations; see, for example, (**6**). For two arbitrary cylinders this is not possible in general, and only approximate solutions can be found.

The second and third methods differ from the first in that they explicitly use the solution of the single-body problem. For a circular cylinder this solution can be found by separation of variables; it is in the form of an infinite series of the cylindrical wave-functions $H_n^{(1)}(kr)e^{in\theta}$, where $H_n^{(1)}(z)$ is a Hankel function and (r, θ) are circular polar coordinates with origin at the centre of the circle. For two circular cylinders the two methods are as follows. In the second method the scattered wave is written as the sum of two infinite series of cylindrical wave-functions, corresponding to the waves scattered by each cylinder. The unknown coefficients are determined by applying the boundary condition on each cylinder and using the addition theorem for Hankel functions; this leads to the same system of linear algebraic equations as obtained using the integral-equation method, described above. This method was devised by Záviška in 1913 (**7**). (Actually, he treated the corresponding transmission problem for a finite number of circular cylinders.) As we shall see, it is very useful for numerical computations but less so for analytical purposes.

In the third method, which was pioneered by Twersky (**8**), the solution is constructed in a step-by-step fashion as follows. Calculate the wave scattered by each cylinder *separately* when it is excited by the incident (or primary) wave; this is called the 'first order of scattering'. The wave scattered by the first (second) cylinder is then rescattered by the second (first) cylinder; this is the 'second order of scattering'. This procedure is then repeated, yielding an infinite series of 'orders of scattering' (each one of which is itself a nested infinite series). The advantage of this method is that there are no equations to solve, that is one merely truncates the infinite series of known (but complicated) terms and takes this as an approximation to the exact solution. However, the convergence of the infinite series has not been demonstrated, although Twersky (**9**) has shown that it can be summed in the far field for widely-spaced cylinders.

In 1964, Burke and Twersky (**5**) wrote: 'Today, the separations of variables derivations . . . are only of academic interest. It has been shown (**10**) that [the same equations] can be obtained for nonseparable as well as separable shapes without knowing the addition theorems for the special functions of the system . . .'. The method referred to uses the plane-wave (Sommerfeld) representation for $H_0^{(1)}(z)$ (**11**, §7.3.5), and leads to a complicated formal solution to the problem, but not at all points in D, nor on ∂D.

2.2. *Numerical solutions*

In the last twenty years the problem of scattering by two circular cylinders has continued to attract attention. Thus, Ohkusu (**12**) has rediscovered Twersky's method (**8**), whilst Záviška's method (**7**) has been rediscovered by several authors: see, for example, Olaofe (**13**), Young and Bertrand (**14**) and, in the context of water waves, Spring and Monkmeyer (**15**). It is clear that for circular cylinders Záviška's direct method is very efficient for numerical calculations.

For cylinders of arbitrary cross-section, integral-equation methods have been used. Thus, Isaacson (**16**) and Sorensen (**17**) have represented u as a distribution of simple wave sources over ∂D, and then solved the corresponding integral equation of the second kind for the unknown source strength. This integral equation is not uniquely solvable at a certain set of values of k (the irregular values: see section 4 below); indeed, Radlinski (**18**) has demonstrated this numerically, by comparing the far-field solution obtained from the solution of the integral equation with that obtained using Záviška's method for two circular cylinders.

One new method for treating multiple-scattering problems has appeared since Twersky's first survey. It is called the *null-field* (or *T-matrix*) method. This method was first devised by Waterman (**19**) for a single scatterer, and was later extended to several scatterers by Peterson and Ström (**20**). It can be considered to be a generalization of Záviška's method to arbitrary cylinders (in that it uses cylindrical wave-functions and their addition theorems, and leads to infinite systems of linear algebraic equations). The null-field method is widely used for obtaining numerical solutions to multiple-scattering problems; see, for example, (**21**) or (**22**). Actually, one motivation for the present work is the theoretical justification of the null-field method, using the author's earlier work on single-scatterer problems (**23, 24**).

3. Scattering by a single cylinder

Let D denote the unbounded region exterior to a simple closed Lyapunov curve ∂D. The title problem reduces to the following linear boundary-value problem.

Problem $\mathcal{P}_1$. Find a function $u(P)$ which satisfies the two-dimensional Helmholtz equation

$$(\nabla^2 + k^2)u(P) = 0 \quad \text{in } D, \tag{3.1}$$

the Neumann boundary condition

$$\frac{\partial u}{\partial n_p}(p) = f(p) \quad \text{on } \partial D \tag{3.2}$$

and the radiation condition

$$r_P^{\frac{1}{2}}\left(\frac{\partial u}{\partial r_P} - iku\right) \to 0 \quad \text{as} \quad r_P \to \infty. \tag{3.3}$$

The function $f(p)$ is prescribed on ∂D, and k is a positive real constant. We shall use the following notation: capital letters P, Q denote points of D; lower-case letters denote points of ∂D; and $\partial/\partial n_q$ denotes normal differentiation at the point q, in the direction from D towards ∂D. For $\mathcal{P}_1$, we choose the origin O at some point in D_-, the complement of $D \cup \partial D$. Finally $\mathbf{r}_P$ is the position vector of P with respect to O, and $r_P = |\mathbf{r}_P|$.

Typically, $\mathcal{P}_1$ is solved by integral-equation methods; for a summary, see (**25**) or (**26**). Let

$$G_0(P, Q) \equiv G_0(\mathbf{r}_P, \mathbf{r}_Q) = \tfrac{1}{2}i\pi H_0^{(1)}(k\,|\mathbf{r}_P - \mathbf{r}_Q|), \tag{3.4}$$

and then look for a solution of $\mathcal{P}_1$ in the form

$$u(P) = \int_{\partial D} \mu(q) G_0(P, q)\, ds_q; \tag{3.5}$$

applying the boundary condition (3.2) yields a Fredholm integral equation of the second kind for the unknown source strength $\mu(q)$, namely

$$\pi\mu(p)+\int_{\partial D}\mu(q)\frac{\partial}{\partial n_p}G_0(p,q)\,ds_q=f(p). \tag{3.6}$$

It is well known that this equation is uniquely solvable, except when k^2 coincides with an eigenvalue of the corresponding interior Dirichlet problem; we denote the (discrete) set of these *irregular* values of k^2 by IV (∂D).

Several methods have been devised for overcoming the difficulty at the irregular values of k^2; for references see (**24**, p. 398; **26**, §3.6). Here we shall concentrate on just one of these, namely, the replacement of G_0 by a different fundamental solution. This method has been investigated by Ursell (**25, 3**), Jones (**2**) and Kleinman and Roach (**27**). Let

$$G_1(P,Q)\equiv G_1(\mathbf{r}_P,\mathbf{r}_Q)=G_0(\mathbf{r}_P,\mathbf{r}_Q)+\sum_{m=0}^{M}\sum_{\sigma=1}^{2}a_m^\sigma\psi_m^\sigma(\mathbf{r}_P)\psi_m^\sigma(\mathbf{r}_Q), \tag{3.7}$$

where a_m^σ are constants,

$$\psi_m^\sigma(\mathbf{r}_Q)=H_m^{(1)}(kr_Q)E_m^\sigma(\theta_Q), \tag{3.8}$$

$$E_m^1(\theta)=2^{\frac{1}{2}}\cos m\theta,\qquad E_m^2(\theta)=2^{\frac{1}{2}}\sin m\theta,\qquad m\geq 1,$$

$E_0^1=1$, $E_0^2=0$ and (r,θ) are circular polar coordinates centred on O. We now modify (3.5) and look for a solution of $\mathscr{P}_1$ in the form

$$u(P)=\int_{\partial D}\mu(q)G_1(P,q)\,ds_q \tag{3.9}$$

whence $\mu(q)$ satisfies

$$\pi\mu(p)+\int_{\partial D}\mu(q)\frac{\partial}{\partial n_p}G_1(p,q)\,ds_q=f(p). \tag{3.10}$$

The solvability of this integral equation is governed by the solvability of the corresponding homogeneous equation, namely

$$\pi\mu(p)+\int_{\partial D}\mu(q)\frac{\partial}{\partial n_p}G_1(p,q)\,ds_q=0. \tag{3.11}$$

THEOREM 3.1. *Suppose that the homogeneous integral equation* (3.11) *has a non-trivial solution* $\mu(q)$. *Then the interior potential*

$$U(P)=\int_{\partial D}\mu(q)G_1(P,q)\,ds_q,\qquad P\in D_- \tag{3.12}$$

vanishes on ∂D.

Proof (**25**, pp. 120, 123). (The corresponding theorem with G_1 replaced by G_0 is classical.) Define $U(P)$ for $P\in D$ by (3.12); $\partial U/\partial n$ vanishes on ∂D by (3.11). The uniqueness theorem for $\mathscr{P}_1$ (**25**, p. 120) then asserts that $U\equiv 0$

in D. The result follows by noting that U is continuous across the source distribution on ∂D.

If we can show that $U \equiv 0$ in D_-, it will follow that (3.11) has only the trivial solution and hence that the inhomogeneous equation (3.10) is uniquely solvable. Ursell (**25**) has shown that this can be achieved by (i) setting $M = \infty$, and (ii) making a special choice for the constants a_m^σ appearing in (3.7); this modification to G_0 eliminates *all* irregular values. Later, it was shown by Jones (**2**) that any given irregular value can be removed by keeping M finite and imposing only a mild restriction on a_m^σ. More recently, Kleinman and Roach (**27**) have shown that this can be achieved with just one non-zero coefficient a_m^σ (provided k^2 is a simple eigenvalue). Jones proved the following theorem (**2**).

THEOREM 3.2. *Suppose that*

$$|2a_m^\sigma + \tfrac{1}{2}i\pi| > \tfrac{1}{2}\pi, \qquad \sigma = 1, 2;\ m = 0, 1, \ldots, M. \tag{3.13}$$

Then every solution of the homogeneous integral equation (3.11) *is a solution of*

$$\pi\mu(p) + \int_{\partial D} \mu(q) \frac{\partial}{\partial n_p} G_0(p, q)\, ds_q = 0, \tag{3.14}$$

which also satisfies

$$A_m^\sigma \equiv \int_{\partial D} \mu(q)\psi_m^\sigma(\mathbf{r}_q)\, ds_q = 0, \qquad \sigma = 1, 2;\ m = 0, 1, \ldots, M.$$

Proof. We give Ursell's simplified proof of this result (**3**). For $P \in D_-$, we have

$$U(P) = \int_{\partial D} \mu(q) G_0(P, q)\, ds_q + \sum_{m=0}^{M} \sum_{\sigma=1}^{2} a_m^\sigma A_m^\sigma \psi_m^\sigma(\mathbf{r}_P),$$

where μ is a solution of (3.11). Let C_- denote the inscribed circle to ∂D, centred on O, and let D_N denote the interior of C_-. If we restrict P to lie in D_N, we find that

$$U(\mathbf{r}_P) = \tfrac{1}{2}i\pi \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} A_m^\sigma \hat{\psi}_m^\sigma(\mathbf{r}_P) + \sum_{m=0}^{M} \sum_{\sigma=1}^{2} a_m^\sigma A_m^\sigma \psi_m^\sigma(\mathbf{r}_P), \tag{3.15}$$

where

$$\hat{\psi}_m^\sigma(\mathbf{r}_Q) = J_m(kr_Q) E_m^\sigma(\theta_Q) \tag{3.16}$$

and we have used the expansion

$$G_0(\mathbf{r}_P, \mathbf{r}_Q) = \tfrac{1}{2}i\pi \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \hat{\psi}_m^\sigma(\mathbf{r}_P) \psi_m^\sigma(\mathbf{r}_Q), \tag{3.17}$$

which is valid for $r_P < r_Q$. Next, we consider the integral

$$I \equiv \int_C \left(U \frac{\partial U^*}{\partial n} - U^* \frac{\partial U}{\partial n} \right) ds,$$

where the asterisk denotes the complex conjugate and C is any circle concentric with C_- and lying in D_N. Using Green's theorem and Theorem 3.1, we see that

$$I = \int_{\partial D} \left(U \frac{\partial U^*}{\partial n} - U^* \frac{\partial U}{\partial n} \right) ds = 0.$$

We can also evaluate I directly using the following.

THEOREM 3.3. *Suppose that $U(\mathbf{r}_P)$, $P \in D_N$, has an expansion*

$$U(\mathbf{r}_P) = \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} A_m^\sigma \hat{\psi}_m^\sigma(\mathbf{r}_P) + \sum_{m=0}^{M} \sum_{\sigma=1}^{2} B_m^\sigma \psi_m^\sigma(\mathbf{r}_P).$$

Then

$$\frac{i}{8} \int_0^{2\pi} \left(U \frac{\partial U^*}{\partial r_P} - U^* \frac{\partial U}{\partial r_P} \right) r_P \, d\theta_P = \sum_{m=0}^{M} \sum_{\sigma=1}^{2} \{ |B_m^\sigma|^2 + \mathrm{Re}\, [B_m^\sigma (A_m^\sigma)^*] \}.$$

Proof. Substitute for U, use the orthogonality of the trigonometric functions and then simplify using the Wronskian, $H_m(z) H_m'^*(z) - H_m'(z) H_m^*(z) = -4i/(\pi z)$, where $H_m(z) \equiv H_m^{(1)}(z)$.

Thus, returning to the proof of Theorem 3.2, we find that

$$0 = I = 2i \sum_{m=0}^{M} \sum_{\sigma=1}^{2} |A_m^\sigma|^2 \{ |2a_m^\sigma + \tfrac{1}{2} i\pi|^2 - \tfrac{1}{4}\pi^2 \}. \tag{3.18}$$

Since a_m^σ satisfy the inequality (3.13), it follows that (3.18) can only be satisfied if $A_m^\sigma = 0$ for $\sigma = 1, 2$ and $m = 0, 1, \ldots, M$. It also follows that μ satisfies (3.14) (substitute (3.7) into (3.11)).

Suppose that (3.13) is satisfied. Then, from (3.15) we see that, for $P \in D_N$,

$$U(\mathbf{r}_P) = \tfrac{1}{2} i\pi \sum_{m=M+1}^{\infty} \sum_{\sigma=1}^{2} A_m^\sigma \hat{\psi}_m^\sigma(\mathbf{r}_P) = O(r_P^{M+1}) \quad \text{as} \quad r_P \to 0.$$

So, either $U \equiv 0$ in D_- or U is an eigenfunction of the interior Dirichlet problem. We may exclude the second alternative by taking M sufficiently large (**2**, Theorem 2).

This completes our review of the theory for scattering by a single cylinder. In the next section we shall show how this theory can be extended to treat scattering by two cylinders.

4. Scattering by two cylinders

Let ∂D^i, for $i = 1, 2$, be two simple, closed, disjoint, Lyapunov curves, and let $\partial D = \partial D^1 \cup \partial D^2$. Then the analogue of $\mathcal{P}_1$ for two scatterers is the following problem.

Problem $\mathcal{P}_2$. Find a function $u(P)$ which satisfies the two-dimensional Helmholtz equation in D, the boundary condition (3.2), and the radiation condition (3.3).

It is known that $\mathcal{P}_2$ has precisely one solution; see for example (**26**, §§3.3, 3.4).

We can use integral-equation methods to treat $\mathcal{P}_2$ in just the same way as we treated $\mathcal{P}_1$. In the current notation we try to represent u as a distribution of simple sources over ∂D, (3.5), leading to the integral equation (3.6); this is the equation solved by Isaacson (**16**), Sorensen (**17**) and Radlinski (**18**). It is easy to modify the classical arguments (**25**, p. 120) to show that the corresponding irregular values are $\mathrm{IV}(\partial D^1) \cup \mathrm{IV}(\partial D^2)$.

In order to obtain uniquely-solvable integral equations we shall replace the simple source G_0 by a different fundamental solution. Let D_-^i denote the interior region bounded by ∂D^i, for $i = 1, 2$, and let $D_- = D_-^1 \cup D_-^2$. Choose two origins, O^i, with $O^i \in D_-^i$, and let $\mathbf{r}_P^i$ denote the position vector of a point P with respect to O^i. Let

$$G_1(P, Q) = G_0(P, Q) + \sum_{m=0}^{M} \sum_{\sigma=1}^{2} a_m^\sigma \psi_m^\sigma(\mathbf{r}_P^1)\psi_m^\sigma(\mathbf{r}_Q^1) + \sum_{m=0}^{N} \sum_{\sigma=1}^{2} b_m^\sigma \psi_m^\sigma(\mathbf{r}_P^2)\psi_m^\sigma(\mathbf{r}_Q^2), \tag{4.1}$$

where a_m^σ and b_m^σ are constants. This fundamental solution is singular at both O^1 and O^2 (recall that $\psi_m^\sigma(\mathbf{r}_P^i)$ is singular at O^i). Note that it is essential that our fundamental solution has this property, for if we chose one that only had singularities at O^1, say, then we could not eliminate those irregular values associated with ∂D^2.

Look for a solution of $\mathcal{P}_2$ in the form (3.9), whence the source density $\mu(q)$ satisfies the integral equation (3.10). Moreover, the same arguments as before show that Theorem 3.1 is true (in the current notation); here, we need the uniqueness theorem for $\mathcal{P}_2$.

Let us now investigate the solvability of the integral equation (3.10) and look for an analogue of Theorem 3.2. Suppose that $\mu(q)$ is any solution of the homogeneous integral equation (3.11). Consider the interior potential $U(P)$, defined by (3.12), for $P \in D_-^1$. We restrict P to lie in $D_N^1 \subset D_-^1$, where D_N^i is the circular region bounded by C_-^i, the inscribed circle to ∂D^i centred on O^i, for $i = 1, 2$. Using (3.17) and (4.1), we obtain

$$U(\mathbf{r}_P^1) = \tfrac{1}{2} i\pi \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} A_m^\sigma \hat{\psi}_m^\sigma(\mathbf{r}_P^1) + \sum_{m=0}^{M} \sum_{\sigma=1}^{2} a_m^\sigma A_m^\sigma \psi_m^\sigma(\mathbf{r}_P^1) + \\ + \sum_{m=0}^{N} \sum_{\sigma=1}^{2} b_m^\sigma B_m^\sigma \psi_m^\sigma(\mathbf{r}_P^2), \tag{4.2}$$

for $P \in D_N^1$, where

$$A_m^\sigma = \int_{\partial D} \mu(q)\psi_m^\sigma(\mathbf{r}_q^1)\, ds_q \tag{4.3}$$

and

$$B_m^\sigma = \int_{\partial D} \mu(q)\psi_m^\sigma(\mathbf{r}_q^2)\, ds_q. \tag{4.4}$$

In order to use Theorem 3.3 we need the expansion of U to be in terms of functions centred on O^1, that is we need the following addition theorem that gives a formula for $\psi_m^\sigma(\mathbf{r}_P^2)$ in terms of functions of $\mathbf{r}_P^1$.

THEOREM 4.1 (**11**).

$$\psi_m^\sigma(\mathbf{r}_P^2) = \sum_{n=0}^{\infty} \sum_{\nu=1}^{2} S_{mn}^{\sigma\nu}(\mathbf{a})\hat{\psi}_n^\nu(\mathbf{r}_P^1),$$

where $\mathbf{r}_P^2 = \mathbf{r}_P^1 + \mathbf{a}$, $r_P^1 < |\mathbf{a}|$, *and the matrix* $S_{mn}^{\sigma\nu}$ *is defined in the Appendix.*

Let $\mathbf{b}$ be the position vector of O^2 with respect to O^1, whence $\mathbf{r}_P^2 = \mathbf{r}_P^1 - \mathbf{b}$. Since $O^2 \notin D_N^1$ (the cylinders do not intersect), we have $r_P^1 < |-\mathbf{b}|$ whence

$$U(\mathbf{r}_P^1) = \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \left\{\tfrac{1}{2}i\pi A_m^\sigma + \sum_{n=0}^{N} \sum_{\nu=1}^{2} b_n^\nu B_n^\nu S_{nm}^{\nu\sigma}(-\mathbf{b})\right\}\hat{\psi}_m^\sigma(\mathbf{r}_P^1) + \\ + \sum_{m=0}^{M} \sum_{\sigma=1}^{2} a_m^\sigma A_m^\sigma \psi_m^\sigma(\mathbf{r}_P^1), \qquad P \in D_N^1. \tag{4.5}$$

Using Green's theorem, Theorem 3.3 (for D_N^1) and the fact that $U(\mathbf{r}_P^1)$ vanishes on ∂D^1, we obtain

$$0 = \sum_{m=0}^{M} \sum_{\sigma=1}^{2} |A_m^\sigma|^2 \{|a_m^\sigma|^2 + \tfrac{1}{2}\pi \operatorname{Im}(a_m^\sigma)\} + \\ + \operatorname{Re} \sum_{m=0}^{M} \sum_{n=0}^{N} \sum_{\sigma=1}^{2} \sum_{\nu=1}^{2} a_m^\sigma A_m^\sigma [b_n^\nu B_n^\nu S_{nm}^{\nu\sigma}(-\mathbf{b})]^*. \tag{4.6}$$

Suppose now that $P \in D_N^2$. Then we obtain

$$U(\mathbf{r}_P^2) = \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \left\{\tfrac{1}{2}i\pi B_m^\sigma + \sum_{n=0}^{M} \sum_{\nu=1}^{2} a_n^\nu A_n^\nu S_{nm}^{\nu\sigma}(\mathbf{b})\right\}\hat{\psi}_m^\sigma(\mathbf{r}_P^2) + \\ + \sum_{m=0}^{N} \sum_{\sigma=1}^{2} b_m^\sigma B_m^\sigma \psi_m^\sigma(\mathbf{r}_P^2), \qquad P \in D_N^2, \tag{4.7}$$

and

$$0 = \sum_{m=0}^{N} \sum_{\sigma=1}^{2} |B_m^\sigma|^2 \{|b_m^\sigma|^2 + \tfrac{1}{2}\pi \operatorname{Im}(b_m^\sigma)\} + \\ + \operatorname{Re} \sum_{m=0}^{M} \sum_{n=0}^{N} \sum_{\sigma=1}^{2} \sum_{\nu=1}^{2} a_m^\sigma A_m^\sigma [b_n^\nu B_n^\nu]^* S_{mn}^{\sigma\nu}(\mathbf{b}). \tag{4.8}$$

By comparison with the proof of Theorem 3.2, we expect to be able to deduce from (4.6) and (4.8) that A_m^σ and B_n^σ must vanish for $\sigma=1,2$, $m=0,1,\ldots,M$ and $n=0,1,\ldots,N$. Add (4.6) and (4.8); since, from (A1), $S_{mn}^{\sigma\nu}(\mathbf{b})=S_{nm}^{\nu\sigma}(-\mathbf{b})$, we obtain

$$0=\sum_{m=0}^{M}\sum_{\sigma=1}^{2}|A_m^\sigma|^2\{|a_m^\sigma|^2+\tfrac{1}{2}\pi\,\mathrm{Im}\,(a_m^\sigma)\}+\sum_{m=0}^{N}\sum_{\sigma=1}^{2}|B_m^\sigma|^2\{|b_m^\sigma|^2+\tfrac{1}{2}\pi\,\mathrm{Im}\,(b_m^\sigma)\}+$$
$$+2\,\mathrm{Re}\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{\sigma=1}^{2}\sum_{\nu=1}^{2}a_m^\sigma A_m^\sigma[b_n^\nu B_n^\nu]^*\hat{S}_{mn}^{\sigma\nu}(\mathbf{b}),\tag{4.9}$$

where $\hat{S}_{mn}^{\sigma\nu}=\mathrm{Re}\,(S_{mn}^{\sigma\nu})$. Let us simplify (4.9). Choose circular polar coordinates (r^1,θ^1) at O^1 so that $\mathbf{b}=(b,0)$; it follows that $S_{mn}^{12}(\mathbf{b})=S_{mn}^{21}(\mathbf{b})=0$ (see the Appendix), that is, the dependence on σ uncouples. Write

$$\mathcal{A}_m^\sigma=a_m^\sigma A_m^\sigma \quad\text{and}\quad \mathcal{B}_m^\sigma=b_m^\sigma B_m^\sigma.$$

Then (4.9) becomes

$$0=\sum_{\sigma=1}^{2}I^\sigma,\tag{4.10}$$

where

$$I^\sigma=K^\sigma+\tfrac{1}{2}\pi\sum_{m=0}^{M}\mathrm{Im}\,(a_m^\sigma)\,|A_m^\sigma|^2+\tfrac{1}{2}\pi\sum_{m=0}^{N}\mathrm{Im}\,(b_m^\sigma)\,|B_m^\sigma|^2$$

and

$$K^\sigma=\sum_{m=0}^{M}|\mathcal{A}_m^\sigma|^2+\sum_{m=0}^{N}|\mathcal{B}_m^\sigma|^2+2\sum_{m=0}^{M}\sum_{n=0}^{N}\mathrm{Re}\,\{\mathcal{A}_m^\sigma\mathcal{B}_n^{\sigma*}\}\hat{S}_{mn}^{\sigma\sigma}(\mathbf{b}).\tag{4.11}$$

Suppose that we can show that $K^\sigma\geq 0$. Then if we take $\mathrm{Im}\,(a_m^\sigma)>0$ ($\sigma=1,\ 2;\ m=0,1,\ldots,M$) and $\mathrm{Im}\,(b_m^\sigma)>0$ ($\sigma=1,\ 2;\ m=0,1,\ldots,N$), it will follow that $I^\sigma>0$, provided that $A_m^\sigma\neq 0$ and $B_m^\sigma\neq 0$. But, from (4.10), $I^1+I^2=0$, that is, we must have $A_m^\sigma=0$ ($\sigma=1,\ 2;\ m=0,1,\ldots,M$) and $B_m^\sigma=0$ ($\sigma=1,\ 2;\ m=0,1,\ldots,N$).

So consider the quadratic form K^σ. Without loss of generality we can assume that $\mathcal{A}_m^\sigma$ and $\mathcal{B}_m^\sigma$ are *real*. For if they were complex we could split K^σ into the sum of two identical quadratic forms, one involving only the real parts of $\mathcal{A}_m^\sigma$ and $\mathcal{B}_m^\sigma$, and one involving only the imaginary parts. Suppressing the dependence on σ, we have

$$\begin{aligned}K&=\sum_{m=0}^{M}(\mathcal{A}_m)^2+\sum_{m=0}^{N}(\mathcal{B}_m)^2+2\sum_{m=0}^{M}\sum_{n=0}^{N}\mathcal{A}_m\mathcal{B}_n\hat{S}_{mn}(\mathbf{b})\\&=\sum_{m=0}^{M}\Big(\mathcal{A}_m+\sum_{n=0}^{N}\mathcal{B}_n\hat{S}_{mn}\Big)^2+\sum_{m=0}^{N}(\mathcal{B}_m)^2-\sum_{m=0}^{M}\Big(\sum_{n=0}^{N}\mathcal{B}_n\hat{S}_{mn}\Big)^2\\&=K_1+K_2,\end{aligned}\tag{4.12}$$

say, where K_1 denotes the first summation; clearly $K_1 \geq 0$. We can write K_2 as

$$K_2 = \sum_{m=0}^{N} (\mathscr{B}_m)^2 - \sum_{n=0}^{N} \sum_{k=0}^{N} \mathscr{B}_n \mathscr{B}_k C_{nk}(M), \tag{4.13}$$

where

$$C_{nk}^{\sigma}(M) = \sum_{m=0}^{M} \hat{S}_{mn}^{\sigma\sigma}(\mathbf{b}) \hat{S}_{mk}^{\sigma\sigma}(\mathbf{b}) = C_{kn}^{\sigma}(M). \tag{4.14}$$

Is $K_2 \geq 0$? We have been unable to answer this question for arbitrary M and N. However, we do have the following result, which is a partial generalization of Jones's theorem (Theorem 3.2).

THEOREM 4.2. *Suppose that, in* (4.1), *only one of M and N is finite: take $M = \infty$. Suppose also that*

$$\operatorname{Im}(a_m^{\sigma}) > 0, \qquad \sigma = 1, 2;\ m = 0, 1, 2, \ldots$$

and

$$\operatorname{Im}(b_m^{\sigma}) > 0, \qquad \sigma = 1, 2;\ m = 0, 1, \ldots, N.$$

Then every solution of the homogeneous integral equation (3.11) *is a solution of the homogeneous integral equation* (3.14) *which also satisfies*

$$A_m^{\sigma} = 0, \qquad \sigma = 1, 2;\ m = 0, 1, 2, \ldots$$

and

$$B_m^{\sigma} = 0, \qquad \sigma = 1, 2;\ m = 0, 1, \ldots, N.$$

Proof. In the light of the previous discussion, it suffices to show that $K_2 \geq 0$. In fact, $K_2 = 0$, because

$$C_{mn}^{\sigma}(\infty) = \delta_{mn}, \tag{4.15}$$

where δ_{ij} is the Kronecker delta; this result is proved in the Appendix, using two applications of the addition theorem for $\hat{\psi}_m^{\sigma}$.

Theorem 4.2 is useful from an analytical point of view. In particular, if we let $N \to \infty$, we obtain an integral equation which is uniquely solvable for all values of k^2 and, hence, we can prove an existence theorem for $\mathscr{P}_2$. However, Theorem 4.2 is less useful for the analysis of a possible numerical method, in which M and N must *both* be finite. To illustrate the difficulties with the present approach we conclude this section with two examples, corresponding to $M = N = 0$ and $M = N = 1$.

EXAMPLE 1. Let $M = N = 0$. From (4.13) we have $K_2^{\sigma} = (\mathscr{B}_0^{\sigma})^2\{1 - C_{00}^{\sigma}(0)\}$, whilst from (4.14) and the Appendix we have $C_{00}^{\sigma}(0) = \{\hat{S}_{00}^{\sigma\sigma}(\mathbf{b})\}^2 = \delta_{1\sigma} J_0^2(kb)$.

Thus,

$$K_2^2 = (\mathcal{B}_0^2)^2 \geq 0$$

and

$$K_2^1 = (\mathcal{B}_0^1)^2\{1 - J_0^2(kb)\} \geq 0,$$

since $J_0(x) \leq 1$ for all $x \geq 0$. It follows that $\mathcal{A}_0^\sigma = \mathcal{B}_0^\sigma = 0$, for $\sigma = 1, 2$.

EXAMPLE 2. Let $M = N = 1$. We have

$$K_2^\sigma = (\mathcal{B}_0^\sigma)^2\{1 - C_{00}^\sigma(1)\} + (\mathcal{B}_1^\sigma)^2\{1 - C_{11}^\sigma(1)\} - 2\mathcal{B}_0^\sigma\mathcal{B}_1^\sigma C_{01}^\sigma(1).$$

Suppose that $\Delta^\sigma = \{1 - C_{00}^\sigma(1)\} > 0$. Then,

$$\Delta^\sigma K_2^\sigma = \{\Delta^\sigma \mathcal{B}_0^\sigma - C_{01}^\sigma(1)\mathcal{B}_1^\sigma\}^2 + (\mathcal{B}_1^\sigma)^2\Lambda^\sigma,$$

where $\Lambda^\sigma = \Delta^\sigma\{1 - C_{11}^\sigma(1)\} - \{C_{01}^\sigma(1)\}^2$. It follows that $K_2^\sigma \geq 0$, as desired, if $\Lambda^\sigma \geq 0$. Now, from (4.14) and results in the Appendix, we have

$$\Delta^1 = 1 - J_0^2 - 2J_1^2, \qquad \Delta^2 = 1,$$
$$\Lambda^1 = \Delta^1\{1 - 2J_1^2 - (J_0 - J_2)^2\} - 2J_1^2J_2^2$$

and

$$\Lambda^2 = 1 - (J_0 + J_2)^2,$$

where $J_n = J_n(kb)$. Since

$$1 = J_0^2 + 2\sum_{n=1}^{\infty} J_n^2 \tag{4.16}$$

and $kb > 0$, we have $\Delta^1 > 0$. Similarly, since $J_0 + J_2 = 2J_1/(kb)$ and $J_1(x) \leq \frac{1}{2}x$ for all $x \geq 0$, we have $\Lambda^2 \geq 0$. It remains to show that $\Lambda^1(kb) \geq 0$ for all $kb > 0$. We have been unable to do this; it is easy to show that $\Lambda^1(k) \sim \alpha k^8$ as $k \to 0$, $\Lambda^1(k) \sim 1 - \beta/k$ as $k \to \infty$ (where α and β are positive constants), and $\Lambda^1(k)$ is not a monotonic function of k. We expect similar difficulties to arise when M and N take on larger (finite) values.

5. Conclusions

In this paper we have considered integral-equation methods for solving a multiple-scattering problem in acoustics, namely, the two-dimensional problem of scattering by a pair of sound-hard cylinders. We obtained a class of integral equations which are uniquely solvable at all frequencies, that is irregular values of k do not occur. To do this, we replaced the simple wave source G_0 by a new fundamental solution that has additional singularities inside both scatterers; this is a generalization of a method used by Jones (**2**). However, our result (Theorem 4.2) is not the best possible, for the fundamental solution used has an *infinite* number of singularities inside one scatterer (cf. (**25**)) and a finite number inside the other. For computational reasons we should prefer to have only a finite number (L, say) of additional singularities, and a result of the following form. *The integral equation* (3.10) *is uniquely solvable at any given value of k, provided that $L > L_0$, where L_0 depends on k.* However, the examples at the end of section 4 suggest that such a result may be difficult to obtain.

Theorem 4.2 can be used to prove the existence of a solution to problem $\mathcal{P}_2$; this result has been obtained previously by other methods (**26**). It can also be used to analyse the null-field equations for $\mathcal{P}_2$ (**20, 21**); this work will be described elsewhere.

The method described in this paper can also be used to solve other scattering problems. In a sequel we shall use it to treat certain water-wave problems, namely, the scattering of a surface wave by a pair of partially-immersed horizontal cylinders. It can also be used to treat three-dimensional problems. If the number of scatterers is increased the present method may become unwieldy, for it would require the addition of singularities inside each of them; for such a configuration other integral-equation methods may be more suitable (**26**).

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APPENDIX

We have (**11**, 7.15(34))

$$\psi_m^\sigma(\mathbf{r}_P^2) = \sum_{n=0}^{\infty} \sum_{\nu=1}^{2} S_{mn}^{\sigma\nu}(\mathbf{a}^2)\hat{\psi}_n^\nu(\mathbf{r}_P^1),$$

where $\mathbf{r}_P^2 = \mathbf{r}_P^1 + \mathbf{a}^2$, $r_P^1 < |\mathbf{a}^2|$ and the matrix $S_{mn}^{\sigma\nu}$ is as follows (all functions have argument $\mathbf{a}^2$, which is the position vector of O^1 with respect to O^2).

$$\begin{aligned}
&S_{0n}^{11} = (-1)^n\psi_n^1, \qquad S_{0n}^{12} = (-1)^n\psi_n^2, \qquad n \geq 0;\\
&S_{m0}^{11} = \psi_m^1, \qquad S_{mm}^{11} = \psi_0^1 + 2^{-\frac{1}{2}}(-1)^m\psi_{2m}^1,\\
&S_{m0}^{21} = \psi_m^2, \qquad S_{mm}^{22} = \psi_0^1 - 2^{-\frac{1}{2}}(-1)^m\psi_{2m}^1,\\
&S_{mn}^{11} = 2^{-\frac{1}{2}}\{\psi_{m-n}^1 + (-1)^n\psi_{m+n}^1\}, \qquad m \neq n,\\
&S_{mn}^{12} = 2^{-\frac{1}{2}}\{-\psi_{m-n}^2 + (-1)^n\psi_{m+n}^2\},\\
&S_{mn}^{21} = 2^{-\frac{1}{2}}\{\psi_{m-n}^2 + (-1)^n\psi_{m+n}^2\},\\
&S_{mn}^{22} = 2^{-\frac{1}{2}}\{\psi_{m-n}^1 - (-1)^n\psi_{m+n}^1\}, \qquad m \neq n,
\end{aligned}$$

and m, $n \geq 1$. Moreover, it is easy to show that

$$S_{mn}^{\sigma\nu}(\mathbf{a}^2) = S_{nm}^{\nu\sigma}(-\mathbf{a}^2). \tag{A1}$$

There is a similar addition theorem for $\hat{\psi}_m^\sigma$; we have (**11**, 7.7.2(6))

$$\hat{\psi}_m^\sigma(\mathbf{r}_P^2) = \sum_{n=0}^{\infty} \sum_{\nu=1}^{2} \hat{S}_{mn}^{\sigma\nu}(\mathbf{a}^2)\hat{\psi}_n^\nu(\mathbf{r}_P^1), \tag{A2}$$

where $\hat{S}_{mn}^{\sigma\nu} = \mathrm{Re}\,(S_{mn}^{\sigma\nu})$, for k real, and there is no restriction on r_P^1. Equation (4.16) may be derived from (A2).

Suppose we set $\mathbf{r}_P^1 = \mathbf{r}_P^2 - \mathbf{a}^2$, and then use (A2) to replace $\hat{\psi}_n^\nu(\mathbf{r}_P^1)$ in (A2); we find that

$$\hat{\psi}_m^\sigma(\mathbf{r}_P^2) = \sum_{k=0}^{\infty} \sum_{\mu=1}^{2} \hat{\psi}_k^\mu(\mathbf{r}_P^2) \sum_{n=0}^{\infty} \sum_{\nu=1}^{2} \hat{S}_{mn}^{\sigma\nu}(\mathbf{a}^2)\hat{S}_{nk}^{\nu\mu}(-\mathbf{a}^2).$$

Then (4.15) follows by using $\hat{S}_{nk}^{\nu\mu}(-\mathbf{a}^2) = \hat{S}_{kn}^{\mu\nu}(\mathbf{a}^2)$ and $\hat{S}_{mn}^{12}(\mathbf{b}) = \hat{S}_{mn}^{21}(\mathbf{b}) = 0$.