INTEGRAL-EQUATION METHODS FOR MULTIPLE-SCATTERING PROBLEMS II. WATER WAVES

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SUMMARY

In a previous paper (**1**), integral-equation methods were used to solve the two-dimensional problem of acoustic scattering by a pair of sound-hard cylinders; uniquely-solvable integral equations were obtained by using a fundamental solution with additional singularities inside each cylinder. In the present paper this approach is extended to treat a two-dimensional water-wave problem, namely, the interaction between two horizontal cylinders of infinite length that are floating in the free surface of deep water with their generators parallel. Differences between the two problems are highlighted. Thus, unlike in (**1**), it is shown how a finite number of singularities inside each cylinder can be treated, giving a complete generalization of Ursell's result for a single cylinder (**2**). However, this generalization is conditional on the availability of a uniqueness theorem for the original boundary-value problem; at present such a theorem has not been proved. The analysis uses an addition theorem for Ursell's multipole potentials, which is proved in an Appendix and has wider applications.

1. Introduction

Two rigid cylinders, with their generators horizontal and parallel, are partially immersed in the free surface of deep water, and a surface wave is incident upon them. For beam seas (waves with their crests parallel to the cylinder generators) we can formulate this problem as a linear two-dimensional boundary-value problem for a velocity potential ϕ. It is this plane problem (labelled $\mathcal{P}_2$ below) that we shall study here, using the integral-equation method described in (**1**); henceforth, this paper will be referred to as I.

The plan of the paper follows that of I. Thus we begin with a brief literature survey and then describe (in section 3) Ursell's (**2**) integral-equation method for the simpler problem of scattering by one partially-immersed cylinder. In section 4 we show how Ursell's approach can be extended to treat $\mathcal{P}_2$. Unlike the corresponding acoustic problem considered in I, here we are able to prove a complete generalization of Ursell's result: our main result (Theorem 4.2) concerns a modified fundamental solution with a finite number of additional singularities inside *each* scatterer (cf. I, Theorem 4.2). However, we also have to assume that $\mathcal{P}_2$ has at most one solution; at present, this uniqueness theorem has not been proved.

2. A brief survey

The literature on scattering of water waves by two or more rigid bodies is quite extensive, but does not seem to have been surveyed previously. Here we shall restrict ourselves to two-dimensional interactions between a pair of partially-immersed cylinders.

2.1. *The method of multipoles*

Apart from some work on scattering by two thin vertical barriers (see (**3**) for references), the first problem to be studied extensively was the radiation problem for two half-immersed circular cylinders. Thus, Ohkusu (**4**) and Wang and Wahab (**5**) extended Ursell's multipole method (**6**) for one cylinder to analyse the heaving motion of a catamaran, consisting of two identical, rigidly-connected, half-immersed circular cylinders (we call this the 'semicircle-catamaran problem'). For this symmetrical problem, the velocity potential at a point P in the fluid can be represented as

$$\phi(P) = \sum_{n=0}^{\infty} a_n\{\Phi_n^1(\mathbf{r}_P^1) + \Phi_n^1(\mathbf{r}_P^2)\} + \sum_{n=0}^{\infty} b_n\{\Phi_n^2(\mathbf{r}_P^1) - \Phi_n^2(\mathbf{r}_P^2)\}, \tag{2.1}$$

where $\Phi_n^\sigma(\mathbf{r}_P^i)$ are the multipole potentials defined in sections 3 and 4. Equation (2.1) satisfies all the conditions of the problem, except the boundary condition on the wetted surfaces of the cylinders; applying this condition allows the coefficients a_n and b_n to be determined. Ohkusu (**4**) and Wang and Wahab (**5**) computed the wave amplitude at infinity and the virtual-mass coefficient, and found good agreement with the corresponding values obtained from their experiments. Ohkusu (**7**) has also made similar calculations for the swaying and rolling motions of the same catamaran, whilst Spencer and Sayer (**8**) have analysed the motions of two freely-floating identical circular cylinders.

2.2. *Integral-equation methods*

Several authors have used integral-equation methods to treat multiple-scattering problems. Most of these authors represented ϕ as a distribution of simple wave sources over the wetted surfaces of the cylinders, and then solved the corresponding integral equation of the second kind for the unknown source density (see section 4). Thus, Nordenstrøm *et al.* (**9**), Kim (**10**), Lee *et al.* (**11**) and Katory *et al.* (**12**) have all solved the semicircle-catamaran problem for deep water, whilst Chung and Coleman (**13**) have solved it for water of constant finite depth. The agreement between these solutions, those obtained using the method of multipoles, and those determined by experiment is generally very good. Other geometries have also been investigated, for example, two different rectangles (**10, 12**), two triangles (**11**), and bulbous-form catamarans (**10, 14**). It is noteworthy that none of these authors reported any difficulties at the irregular frequencies.

Integral equations can also be obtained by applying Green's theorem to ϕ and the simple logarithmic source potential. These equations have simple kernels but the range of integration includes the free surface, the bottom, and two vertical control surfaces at some distance from the floating cylinders. The radiation condition (3.4), or an approximation to it, is imposed on these vertical surfaces, and the bottom is included so as to obtain a finite range of integration. For details of the method, see (**15, 16**). This method has been used by Ho and Harten (**17**) to analyse the motion of one or two rectangular cylinders oscillating near a vertical wall, and by Ijima *et al.* (**18**) to compute the transmission coefficient for the semicircle-catamaran problem.

2.3. *Other methods*

Leonard *et al.* (**19**) have used a finite-element method to solve the semicircle-catamaran problem for water of constant finite depth, and the corresponding problem with freely-floating cylinders. It may be observed that their results for the catamaran are in good qualitative agreement with those of Chung and Coleman (**13**).

Two approximate methods have been used to treat two-dimensional multiple-scattering problems. Alker (**20**) has used the method of matched asymptotic expansions to study the scattering of short waves by two partially-immersed cylinders that do *not* make plane vertical intersections with the free surface. He has shown, for example, that for a symmetric pair of cylinders there is an infinite number of frequencies at which there is no reflected wave.

Secondly, Ohkusu (**7**) has used a 'wide-spacing' approximation, in which only wave-like interactions between the cylinders are taken into account. This leads to an approximate solution to the multiple-scattering problem in terms of the solutions to various single-cylinder problems. An alternative treatment has been given by Srokosz and Evans (**21**). For the semicircle-catamaran problem, Ohkusu (**7**) obtained good agreement with the exact solution (**4, 5**), even when the assumption that the spacing between the cylinders is large compared to the wavelength is not valid. Other applications have been made by Ohkusu (**7, 22, 23**), Ohkusu and Takaki (**24**), Srokosz and Evans (**21**) and Masubuchi and Shinomiya (**25**).

Finally, it is also possible to extend the null-field method (**26**) to treat multiple-scattering problems. Some preliminary work on this extension has been done (**27**), and the method is currently being used to solve some particular multiple-scattering problems. The results of this research will be presented elsewhere.

3. Scattering by a single cylinder

Consider a rigid horizontal cylinder which is partially immersed in the free surface of deep water. Take Cartesian coordinates (x, y, z), with the z-axis parallel to the generators of the cylinder, the x-axis horizontal and the y-axis vertical (y increasing with depth) such that the free surface occupies a portion of the plane $y = 0$. We consider motions that are two-dimensional, and independent of z. For irrotational motion a velocity potential exists; for time-harmonic motion (with radian frequency ω) we can write this potential as $\mathrm{Re}\{\phi(P)e^{-i\omega t}\}$, where ϕ solves the following linear boundary-value problem (**15, 16**).

Problem $\mathcal{P}_1$. Find a function $\phi(P)$ which satisfies the two-dimensional Laplace equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi(P)=0 \quad \text{in } D, \tag{3.1}$$

the Neumann boundary condition

$$\frac{\partial \phi(p)}{\partial n_p}=f(p) \quad \text{on } \partial D \tag{3.2}$$

and the free-surface condition

$$K\phi+\frac{\partial \phi}{\partial y}=0 \quad \text{on } F. \tag{3.3}$$

In addition, there is the radiation condition

$$\frac{\partial \phi}{\partial r_P}-iK\phi \to 0 \quad \text{as } r_P \to \infty, \tag{3.4}$$

and the condition that the fluid motion vanishes as $y \to \infty$.

Here, we denote the fluid domain (in the (x, y)-plane) by D, the mean free surface by F and the wetted surface of the cylinder by ∂D. The function $f(p)$ is prescribed on ∂D and $K = \omega^2/g$, where g is the acceleration due to gravity, is a positive constant. As in I, capital letters denote points of D and lower-case letters denote points of ∂D. We write $\partial/\partial n_q$ for normal differentiation at the point q, in the direction from ∂D into D. The origin O is assumed to lie in F_-, the portion of the line $y = 0$ that is inside the cylinder; D_- denotes the interior region, that is the region with boundary $\partial D \cup F_-$. Finally $\mathbf{r}_P$ is the position vector of P with respect to O, and $r_P = |\mathbf{r}_P|$.

Let ∂D^* denote the union of ∂D and its mirror image in F. We say that ∂D has properties J if ∂D^* is convex and twice-differentiable. John (**28**) has shown that if ∂D has properties J, then $\mathcal{P}_1$ has precisely one solution.

Typically, $\mathcal{P}_1$ is solved by integral-equation methods; for a summary, see

(**26, 15, 16**). Let

$$G_0(P, Q) \equiv G_0(x, y; \xi, \eta)$$
$$= \tfrac{1}{2} \log \frac{(x-\xi)^2 + (y-\eta)^2}{(x-\xi)^2 + (y+\eta)^2} - 2 \int_0^\infty e^{-k(y+\eta)} \cos k(x-\xi) \frac{dk}{k-K}, \qquad (3.5)$$

and then look for a solution of $\mathcal{P}_1$ in the form

$$\phi(P) = \int_{\partial D} \mu(q) G_0(P, q)\, ds_q; \qquad (3.6)$$

applying the boundary condition (3.2) shows that the source density μ satisfies the integral equation I(3.6). This equation is uniquely solvable, except when K coincides with an eigenvalue of the corresponding 'interior wave-Dirichlet problem', where the Dirichlet condition $\phi = 0$ is satisfied on ∂D and the free-surface condition (3.3) is satisfied on F_-; we denote the set of these *irregular* values of K by IV(∂D).

Several methods have been devised for overcoming the difficulty at the irregular values of K—some of these have analogues in acoustics, others do not (for example, it is possible to obtain a uniquely-solvable integral equation by distributing additional sources over F_- (**29, 30**)); see (**2, 15, 26**) for references. As in I we shall concentrate on just one of these, namely, the replacement of G_0 by a different fundamental solution. This method has been investigated by Ursell (**2**). Let

$$G_1(P, Q) \equiv G_1(\mathbf{r}_P, \mathbf{r}_Q) = G_0(\mathbf{r}_P, \mathbf{r}_Q) + \sum_{m=0}^{M} \sum_{\sigma=1}^{2} a_m^\sigma \Phi_m^\sigma(\mathbf{r}_P) \Phi_m^\sigma(\mathbf{r}_Q), \qquad (3.7)$$

where a_m^σ are constants,

$$\Phi_0^1(\mathbf{r}_P) = \int_0^\infty e^{-ky} \cos kx \frac{dk}{k-K}, \qquad \Phi_0^2 = \frac{-1}{K} \frac{\partial}{\partial x} \Phi_0^1,$$

$$\Phi_m^1(\mathbf{r}_P) = \frac{\cos 2m\theta}{r^{2m}} + \frac{K}{2m-1} \frac{\cos(2m-1)\theta}{r^{2m-1}},$$

$$\Phi_m^2(\mathbf{r}_P) = \frac{\sin(2m+1)\theta}{r^{2m+1}} + \frac{K}{2m} \frac{\sin 2m\theta}{r^{2m}},$$

$m \geq 1$, and $\mathbf{r}_P = (x, y)$ has circular polar coordinates given by $x = r \sin \theta$, $y = r \cos \theta$. The Φ_m^σ are called *multipole potentials* (**6**); they are harmonic, except at O where they are singular, and they satisfy the radiation and free-surface conditions. We modify (3.6) and look for a solution of $\mathcal{P}_1$ in the form

$$\phi(P) = \int_{\partial D} \mu(q) G_1(P, q)\, ds_q, \qquad (3.8)$$

whence $\mu(q)$ satisfies the integral equation I(3.10), namely

$$\pi\mu(p)+\int_{\partial D}\mu(q)\frac{\partial}{\partial n_p}G_1(p,q)\,ds_q=f(p). \tag{3.9}$$

Ursell (**2**) has analysed the solvability of this equation. He proved the next three theorems, which are the analogues of I, Theorems 3.1 to 3.3.

THEOREM 3.1. *Suppose that the homogeneous integral equation* I(3.11) *has a non-trivial solution* $\mu(q)$. *Then the interior potential* U, *defined by* I(3.12) *for* $P\in D_-$, *vanishes on* ∂D.

The proof of Theorem 3.1 uses John's uniqueness theorem for $\mathcal{P}_1$ (**28**).

THEOREM 3.2. *Suppose that*

$$\operatorname{Im}(a_m^\sigma)>0,\qquad \sigma=1,2;\ m=0,1,\ldots,M.$$

Then every solution of the homogeneous integral equation I(3.11) *is a solution of the homogeneous integral equation* I(3.14) *which also satisfies*

$$A_m^\sigma\equiv\int_{\partial D}\mu(q)\Phi_m^\sigma(\mathbf{r}_q)\,ds_q=0,\qquad \sigma=1,2;\ m=0,1,\ldots,M.$$

(*Actually, the condition on* a_0^σ *can be replaced by a weaker condition, namely,* $|\pi a_0^\sigma+i|>1$ *for* $\sigma=1,2$.)

Ursell's proof of this theorem is similar to his proof of I, Theorem 3.2. One ingredient is the expansion (cf. I(3.17))

$$G_0(\mathbf{r}_P,\mathbf{r}_Q)=\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\alpha_m^\sigma(\mathbf{r}_P)\Phi_m^\sigma(\mathbf{r}_Q), \tag{3.10}$$

which is valid for $r_P<r_Q$, where

$$\alpha_0^1(\mathbf{r}_P)=-2e^{-Ky}\cos Kx,\qquad \alpha_0^2=-2e^{-Ky}\sin Kx,$$

$$\alpha_m^1(\mathbf{r}_P)=\frac{-2(2m-1)!}{K^{2m}}\sum_{q=2m}^{\infty}\frac{(-Kr)^q}{q!}\cos q\theta,$$

and

$$\alpha_m^2(\mathbf{r}_P)=\frac{2(2m)!}{K^{2m+1}}\sum_{q=2m+1}^{\infty}\frac{(-Kr)^q}{q!}\sin q\theta;$$

α_m^σ are regular harmonic functions which satisfy the free-surface condition (3.3). A second ingredient is the following.

THEOREM 3.3. *Suppose that* $U(\mathbf{r}_P)$, $P\in D_N$, *has an expansion*

$$U(\mathbf{r}_P)=\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}A_m^\sigma\alpha_m^\sigma(\mathbf{r}_P)+\sum_{m=0}^{M}\sum_{\sigma=1}^{2}B_m^\sigma\Phi_m^\sigma(\mathbf{r}_P).$$

Then

$$\frac{1}{4\pi i}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\left(U\frac{\partial U^*}{\partial r_P}-U^*\frac{\partial U}{\partial r_P}\right)r_P\,d\theta_P=\sum_{\sigma=1}^{2}\left\{\tfrac{1}{2}\pi\,|B_0^\sigma|^2+\sum_{m=0}^{M}\operatorname{Im}\left[B_m^\sigma(A_m^\sigma)^*\right]\right\}.$$

Here, D_N is the semicircular region bounded by F_- and the lower half of C_-, where C_- is the inscribed circle to ∂D^*, centred on O.

From these results, it can be shown that the integral equation (3.9) is uniquely solvable at any given value of K, provided that M is sufficiently large; see (**2**, p. 148).

4. Scattering by two cylinders

Consider two rigid cylinders that are partially immersed in the free surface of deep water, with their generators horizontal and parallel. In the xy-plane we denote the fluid domain by D, the free surface by F and the wetted surfaces of the cylinders by ∂D^i, for $i=1,2$. Let $\partial D=\partial D^1\cup\partial D^2$. The analogue of $\mathcal{P}_1$ for two cylinders is the following problem.

Problem $\mathcal{P}_2$. Find a function $\phi(P)$ which satisfies Laplace's equation (3.1) in D, the boundary condition (3.2) on ∂D, the free-surface condition, the radiation condition, and the condition that the fluid motion vanishes as $y\to\infty$.

In order to make some progress with the analysis of $\mathcal{P}_2$, we make the following hypothesis.

Uniqueness assumption. Problem $\mathcal{P}_2$ has at most one solution, that is, the only solution of the homogeneous problem ($f=0$) is the trivial solution, $\phi\equiv 0$.

To the author's knowledge, this result has not been proved. John's proof (**28**) for one cylinder does not seem to extend to two cylinders. A proof may also impose some restrictions on the geometry: here, we shall suppose that ∂D^1 and ∂D^2 each have properties J (this also eliminates corner-singularities from the integral equation below; see, for example (**28**, §7)).

Although we do not have a uniqueness theorem for $\mathcal{P}_2$, uniqueness can be proved for some other configurations. Thus, John's proof succeeds for two (or more) floating *three-dimensional* bodies (each having a wetted surface which is bounded and has properties J);† the essential difference between this problem and $\mathcal{P}_2$ is the connectivity of the free-surface (note that John's proof also fails for a single floating torus). We also have two results for *totally submerged* bodies: Schnute (**31**) has proved uniqueness for a pair of widely-spaced circular cylinders, whilst a theorem due to Maz'ja guarantees

† I am indebted to Ralph Kleinman for this observation.

uniqueness for any two (or more) bodies which individually satisfy a certain geometrical condition for the same choice of origin; see (**32**) for details.

We shall use integral-equation methods to treat $\mathscr{P}_2$, and begin by trying to represent ϕ as a distribution of sources over ∂D, (3.6), leading to the integral equation I(3.6). It is easy to show that the corresponding irregular values are $\mathrm{IV}(\partial D^1) \cup \mathrm{IV}(\partial D^2)$, if the uniqueness assumption is correct.

We now replace G_0 by a different fundamental solution. Let F_-^i denote the portion of the line $y=0$ inside the ith cylinder, and let D_-^i denote the interior region bounded by F_-^i and ∂D^i, for $i=1,2$. Let $D_- = D_-^1 \cup D_-^2$ and $F_- = F_-^1 \cup F_-^2$. Choose two origins O^i, with $O^i \in F_-^i$, and let $\mathbf{r}_P^i$ denote the position vector of P with respect to O^i. Let

$$G_1(P,Q) = G_0(P,Q) + \sum_{m=0}^{M} \sum_{\sigma=1}^{2} a_m^\sigma \Phi_m^\sigma(\mathbf{r}_P^1)\Phi_m^\sigma(\mathbf{r}_Q^1) + \sum_{m=0}^{N} \sum_{\sigma=1}^{2} b_m^\sigma \Phi_m^\sigma(\mathbf{r}_P^2)\Phi_m^\sigma(\mathbf{r}_Q^2), \tag{4.1}$$

where a_m^σ and b_m^σ are constants, and then look for a solution of $\mathscr{P}_2$ in the form (3.8), whence μ satisfies the integral equation (3.9). Then it is straightforward to show that Theorem 3.1 is true (in the current notation), whenever the uniqueness assumption is correct.

Suppose now that μ is any solution of the homogeneous equation I(3.11), and consider the interior potential $U(P)$, defined by I(3.12), for $P \in D_N^1$, where D_N^i is the semicircular region bounded by F_-^i and the lower half of C_-^i, and C_-^i is the inscribed semicircle to ∂D^i centred on O^i, for $i=1,2$. Using (3.10) and (4.1) we obtain

$$U(\mathbf{r}_P^1) = \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} A_m^\sigma \alpha_m^\sigma(\mathbf{r}_P^1) + \sum_{m=0}^{M} \sum_{\sigma=1}^{2} a_m^\sigma A_m^\sigma \Phi_m^\sigma(\mathbf{r}_P^1) + \sum_{m=0}^{N} \sum_{\sigma=1}^{2} b_m^\sigma B_m^\sigma \Phi_m^\sigma(\mathbf{r}_P^2), \tag{4.2}$$

for $P \in D_N^1$, where

$$A_m^\sigma = \int_{\partial D} \mu(q)\Phi_m^\sigma(\mathbf{r}_q^1)\, ds_q \tag{4.3}$$

and

$$B_m^\sigma = \int_{\partial D} \mu(q)\Phi_m^\sigma(\mathbf{r}_q^2)\, ds_q. \tag{4.4}$$

In order to use Theorem 3.3 we need an addition theorem for Φ_m^σ; we have the following.

THEOREM 4.1.

$$\Phi_m^\sigma(\mathbf{r}_P^2) = \sum_{n=0}^{\infty} \sum_{\nu=1}^{2} S_{mn}^{\sigma\nu}(\mathbf{a})\alpha_n^\nu(\mathbf{r}_P^1),$$

where $\mathbf{r}_P^2 = \mathbf{r}_P^1 + \mathbf{a}$, $r_P^1 < |\mathbf{a}|$, *and the matrix* $S_{mn}^{\sigma\nu}$ *is defined in the Appendix.*

This theorem appears to be new. It can be proved either by introducing complex variables (**27**) or by using integral representations; both proofs are sketched in the Appendix.

Let $\mathbf{b}$ be the position vector of O^2 with respect to O^1. Then, using Theorem 4.1, (4.2) becomes

$$U(\mathbf{r}_P^1)=\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\left\{A_m^\sigma+\sum_{n=0}^{N}\sum_{\nu=1}^{2}b_n^\nu B_n^\nu S_{nm}^{\nu\sigma}(-\mathbf{b})\right\}\alpha_m^\sigma(\mathbf{r}_P^1)+$$
$$+\sum_{m=0}^{M}\sum_{\sigma=1}^{2}a_m^\sigma A_m^\sigma \Phi_m^\sigma(\mathbf{r}_P^1),\qquad P\in D_N^1. \tag{4.5}$$

Using Green's theorem, the free-surface condition (3.3), Theorem 3.3 (for D_N^1) and the fact that U vanishes on ∂D^1, we obtain

$$0=\tfrac{1}{2}\pi\sum_{\sigma=1}^{2}|a_0^\sigma A_0^\sigma|^2+\sum_{m=0}^{M}\sum_{\sigma=1}^{2}|A_m^\sigma|^2\,\mathrm{Im}\,(a_m^\sigma)+$$
$$+\mathrm{Im}\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{\sigma=1}^{2}\sum_{\nu=1}^{2}a_m^\sigma A_m^\sigma[b_n^\nu B_n^\nu S_{nm}^{\nu\sigma}(-\mathbf{b})]^*. \tag{4.6}$$

Similarly, we obtain

$$U(\mathbf{r}_P^2)=\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\left\{B_m^\sigma+\sum_{n=0}^{M}\sum_{\nu=1}^{2}a_n^\nu A_n^\nu S_{nm}^{\nu\sigma}(\mathbf{b})\right\}\alpha_m^\sigma(\mathbf{r}_P^2)+$$
$$+\sum_{m=0}^{N}\sum_{\sigma=1}^{2}b_m^\sigma B_m^\sigma \Phi_m^\sigma(\mathbf{r}_P^2),\qquad P\in D_N^2 \tag{4.7}$$

and

$$0=\tfrac{1}{2}\pi\sum_{\sigma=1}^{2}|b_0^\sigma B_0^\sigma|^2+\sum_{m=0}^{N}\sum_{\sigma=1}^{2}|B_m^\sigma|^2\,\mathrm{Im}\,(b_m^\sigma)$$
$$-\mathrm{Im}\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{\sigma=1}^{2}\sum_{\nu=1}^{2}a_m^\sigma A_m^\sigma[b_n^\nu B_n^\nu]^* S_{mn}^{\sigma\nu}(\mathbf{b}). \tag{4.8}$$

Adding (4.6) and (4.8), and using (A7), we find that

$$0=\sum_{m=0}^{M}\sum_{\sigma=1}^{2}|A_m^\sigma|^2\,\mathrm{Im}\,(a_m^\sigma)+\sum_{m=0}^{N}\sum_{\sigma=1}^{2}|B_m^\sigma|^2\,\mathrm{Im}\,(b_m^\sigma)+\tfrac{1}{2}\pi L,$$

where

$$L=\sum_{\sigma=1}^{2}\{|\mathscr{A}_0^\sigma|^2+|\mathscr{B}_0^\sigma|^2\}-\frac{4}{\pi}\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{\sigma=1}^{2}\sum_{\nu=1}^{2}\mathrm{Re}\,(\mathscr{A}_m^\sigma\mathscr{B}_n^{\nu *})\,\mathrm{Im}\,(S_{mn}^{\sigma\nu}(\mathbf{b})),$$
$$\mathscr{A}_m^\sigma=a_m^\sigma A_m^\sigma\quad\text{and}\quad\mathscr{B}_m^\sigma=b_m^\sigma B_m^\sigma.$$

Now from the Appendix it can be seen that $\mathrm{Im}\,(S_{mn}^{\sigma\nu})=0$ unless $m=n=0$; moreover, from (A8) and (A9), we have

$$\mathrm{Im}\,\{S_{00}^{11}(\mathbf{b})\}=\mathrm{Im}\,\{S_{00}^{22}(\mathbf{b})\}=-\tfrac{1}{2}\pi\cos Kb$$

and

$$\operatorname{Im}\{S_{00}^{12}(\mathbf{b})\} = -\operatorname{Im}\{S_{00}^{21}(\mathbf{b})\} = \tfrac{1}{2}\pi \sin Kb,$$

where O^2 is located at $(b, 0)$ with respect to Cartesian coordinates (x_1, y_1) at O^1. Hence

$$\begin{aligned} L = \sum_{\sigma=1}^{2} \{|\mathscr{A}_0^\sigma|^2 + |\mathscr{B}_0^\sigma|^2\} + 2\cos Kb \operatorname{Re}(\mathscr{A}_0^1\mathscr{B}_0^{1*} + \mathscr{A}_0^2\mathscr{B}_0^{2*}) \\ -2\sin Kb \operatorname{Re}(\mathscr{A}_0^1\mathscr{B}_0^{2*} - \mathscr{A}_0^2\mathscr{B}_0^{1*}). \end{aligned}$$

As in I, we can assume that $\mathscr{A}_0^\sigma$ and $\mathscr{B}_0^\sigma$ are real, whence

$$\begin{aligned} L &= (\mathscr{A}_0^1)^2 + (\mathscr{A}_0^2)^2 + (\mathscr{B}_0^1)^2 + (\mathscr{B}_0^2)^2 + 2\cos Kb(\mathscr{A}_0^1\mathscr{B}_0^1 + \mathscr{A}_0^2\mathscr{B}_0^2) \\ &\quad -2\sin Kb(\mathscr{A}_0^1\mathscr{B}_0^2 - \mathscr{A}_0^2\mathscr{B}_0^1) \\ &= (\mathscr{B}_0^1 + \mathscr{A}_0^1\cos Kb + \mathscr{A}_0^2\sin Kb)^2 + (\mathscr{B}_0^2 - \mathscr{A}_0^1\sin Kb + \mathscr{A}_0^2\cos Kb)^2 \geq 0. \end{aligned}$$

We have thus proved the following generalization of Ursell's theorem (Theorem 3.2).

THEOREM 4.2. *Suppose that*

$$\operatorname{Im}(a_m^\sigma) > 0, \qquad \sigma = 1, 2;\ m = 0, 1, \ldots, M$$

and

$$\operatorname{Im}(b_m^\sigma) > 0, \qquad \sigma = 1, 2;\ m = 0, 1, \ldots, N.$$

Suppose also that the uniqueness assumption is correct. Then every solution of the homogeneous integral equation I(3.11) *is a solution of the homogeneous integral equation* I(3.14) *which also satisfies*

$$A_m^\sigma = 0, \qquad \sigma = 1, 2;\ m = 0, 1, \ldots, M$$

and

$$B_m^\sigma = 0, \qquad \sigma = 1, 2;\ m = 0, 1, \ldots, N.$$

An immediate consequence of this theorem is the following existence theorem.

THEOREM 4.3. *Suppose that the uniqueness assumption is correct, that is, suppose that $\mathscr{P}_2$ has at most one solution. Then $\mathscr{P}_2$ has precisely one solution.*

5. Conclusions

In this paper we have considered integral-equation methods for solving a multiple-scattering problem in the linear theory of surface water waves, namely, the two-dimensional problem of scattering by a pair of partially-immersed, floating, horizontal cylinders. At present there is no uniqueness theorem for this problem. However, if such a theorem does hold, then we have obtained a class of integral equations that are uniquely solvable at any given frequency.

In order to prove our results we used an addition theorem for the multipole potentials Φ_m^σ (Theorem 4.1). This can be used in two other ways. In the first place it can be used in the extension of the null-field method from one cylinder (**26**) to two (or more) cylinders; a formal derivation of the so-called T-matrix for two cylinders has been given by Bencheikh (**27**). Secondly, it can be used to adapt Záviška's method and Twersky's method (these are methods for solving the problem of acoustic scattering by two circular cylinders; see I, §2) to the semicircle-catamaran problem.

It is interesting to compare the results of the present paper with those obtained in I. Here we have obtained a complete generalization of Ursell's theorem (Theorem 3.2), that is, we can treat fundamental solutions G_1 with a finite number of singularities inside both scatterers. For the corresponding problem in acoustics we were only able to treat fundamental solutions with an infinite number of singularities inside one of the scatterers and a finite number inside the other; see I, Theorem 4.2. On the other hand, we did not need to make a uniqueness assumption in I, because such an assumption is already known to be correct for the exterior problems of acoustics.

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APPENDIX

Theorem. *Let a point P in the fluid have position vector $\mathbf{r}_P^i$ with respect to O^i, for $i=1,2$, where O^1 and O^2 are two origins in the mean free surface* $(y=0)$. *Then*

$$\Phi_m^\sigma(\mathbf{r}_P^2)=\sum_{n=0}^{\infty}\sum_{\nu=1}^{2}S_{mn}^{\sigma\nu}(\mathbf{a})\alpha_n^\nu(\mathbf{r}_P^1),$$

where $\mathbf{r}_P^2=\mathbf{r}_P^1+\mathbf{a}$, $r_P^1<|\mathbf{a}|$ and the matrix $S_{mn}^{\sigma\nu}$ is defined as follows:

$$S_{0n}^{11}=-\tfrac{1}{2}\Phi_n^1,\qquad S_{0n}^{12}=\tfrac{1}{2}\Phi_n^2,\qquad n\geqslant 0;$$

$$S_{00}^{21}=-\tfrac{1}{2}\Phi_0^2,\qquad S_{0n}^{21}=\frac{-n}{K}\Phi_n^2,$$

$$S_{00}^{22}=-\frac{1}{2}\left\{\Phi_0^1+\frac{1}{K^2}\Phi_1^1\right\},\qquad S_{0n}^{22}=-\frac{2n+1}{2K}\Phi_{n+1}^1,\qquad n\geqslant 1;$$

$$S_{m0}^{11}=-\tfrac{1}{2}\Phi_m^1,\qquad S_{mn}^{11}=-\frac{1}{2}\left\{\binom{2m+2n-1}{2m-1}2n\Phi_{m+n}^1-\binom{2m+2n-3}{2m-1}\frac{K^2}{2n-1}\Phi_{m+n-1}^1\right\},$$

$$S_{m0}^{12}=\frac{m}{K}\Phi_m^2,\qquad S_{mn}^{12}=\frac{1}{2}\left\{\binom{2m+2n}{2m-1}(2n+1)\Phi_{m+n}^2-\binom{2m+2n-2}{2m-1}\frac{K^2}{2n}\Phi_{m+n-1}^2\right\},$$

$$S_{m0}^{21}=-\tfrac{1}{2}\Phi_m^2,\qquad S_{mn}^{21}=-\frac{1}{2}\left\{\binom{2m+2n}{2m}2n\Phi_{m+n}^2-\binom{2m+2n-2}{2m}\frac{K^2}{2n-1}\Phi_{m+n-1}^2\right\},$$

$$S_{m0}^{22}=-\frac{2m+1}{2K}\Phi_{m+1}^1,$$

$$S_{mn}^{22}=-\frac{1}{2}\left\{\binom{2m+2n+1}{2m}(2n+1)\Phi_{m+n+1}^1-\binom{2m+2n-1}{2m}\frac{K^2}{2n}\Phi_{m+n}^1\right\},$$

for m, $n \geq 1$. *Here, all functions have argument* $\mathbf{a}$ *and* $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$.

Proof. The result for $m = 0$, $\sigma = 1$, follows from the bilinear expansion (3.10), namely

$$G_0(P, Q) = \sum_{n=0}^{\infty} \sum_{\nu=1}^{2} \alpha_n^\nu(\mathbf{r}_P^1)\Phi_n^\nu(\mathbf{r}_Q^1)$$

(which holds for $r_P^1 < r_Q^1$), and the relation

$$\Phi^1(\mathbf{r}_P^2) = -\tfrac{1}{2}G_0(P, O^2).$$

The result for $m = 0$, $\sigma = 2$, is obtained by differentiation. For $m \geq 1$, we give two methods: the first is a complex-variable method and the second uses integral representations.

Method 1. (This method is described fully in (**27**).) Let

$$z_j = r_P^j \exp(i\theta_P^j) = y_j + ix_j, \quad j = 1, 2 \qquad \text{and} \qquad z_a = ae^{i\delta}, \quad \delta = \pm\tfrac{1}{2}\pi.$$

We have

$$(2m-1)\Phi_m^1(r, \theta) = \left(K - \frac{\partial}{\partial y}\right)\phi_m^1(r, \theta), \tag{A1}$$

where $\phi_m^1(r, \theta) = \cos((2m-1)\theta)/r^{2m-1}$, that is

$$\phi_m^1(\mathbf{r}_P^2) = \operatorname{Re}(z_2^{-(2m-1)}). \tag{A2}$$

Now, $z_2 = z_1 + z_a$, whence

$$z_2^{-n} = \sum_{k=0}^{\infty} b_k^{(n)} z_1^k z_a^{-(n+k)} \quad \text{for} \quad r_P^1 < a, \tag{A3}$$

where $b_k^{(n)} = (-1)^k\binom{n+k-1}{n-1}$. Combining (A1) to (A3), we find that

$$\Phi_m^1(\mathbf{r}_P^2) = I_c^{(m)}(\mathbf{r}_P^1; \mathbf{a}) + I_s^{(m)}(\mathbf{r}_P^1; \mathbf{a}), \tag{A4}$$

where

$$(2m-1)I_c^{(m)}(\mathbf{r}_P^1; \mathbf{a}) = \left(K - \frac{\partial}{\partial y_1}\right) \sum_{n=0}^{\infty} b_n^{(2m-1)} \operatorname{Re}(z_1^n) \operatorname{Re}(z_a^{-(2m+n-1)})$$

and

$$(2m-1)I_s^{(m)}(\mathbf{r}_P^1; \mathbf{a}) = -\left(K - \frac{\partial}{\partial y_1}\right) \sum_{n=1}^{\infty} b_n^{(2m-1)} \operatorname{Im}(z_1^n) \operatorname{Im}(z_a^{-(2m+n-1)}).$$

Consider $I_c^{(m)}$. Since $\delta = \pm\frac{1}{2}\pi$, it follows that $\operatorname{Re}(z_a^n) = 0$ if n is odd, whilst $\operatorname{Re}(z_a^{-2m}) = \Phi_m^1(a, \delta)$. Also,

$$\left(K - \frac{\partial}{\partial y}\right) \operatorname{Re}\{(re^{i\theta})^{2m+1}\} = -(2m+1)\hat{\Phi}_m^1(r, \theta),$$

where

$$\hat{\Phi}_m^1(r, \theta) = r^{2m} \cos 2m\theta - \frac{K}{2m+1} r^{2m+1} \cos(2m+1)\theta \tag{A5}$$

is a regular harmonic function that satisfies the free-surface condition. Hence

$$I_c^{(m)}(r,\theta;\mathbf{a})=\sum_{n=0}^{\infty}B_n^{(m)}\Phi_{m+n}^1(a,\delta)\hat{\Phi}_n^1(r,\theta), \tag{A6}$$

where

$$B_n^{(m)}=\frac{-(2n+1)}{2m-1}b_{2n+1}^{(2m-1)}=\binom{2n+2m-1}{2m-1}.$$

A similar expansion can be found for $I_s^{(m)}$ by introducing the regular functions

$$\hat{\Phi}_m^2(r,\theta)=r^{2m+1}\sin(2m+1)\theta-\frac{K}{2m+2}r^{2m+2}\sin(2m+2)\theta,\qquad m\geqslant 0.$$

Thus we have an expansion for $\Phi_m^1(\mathbf{r}_P^2)$ in terms of $\hat{\Phi}_n^\sigma(\mathbf{r}_P^1)$. We can rewrite this expansion in terms of $\alpha_n^\sigma(\mathbf{r}_P^1)$: for example, from the definitions in section 3, we have

$$\hat{\Phi}_0^1=-\tfrac{1}{2}\{\alpha_0^1-K^2\alpha_1^1\}$$

and

$$\hat{\Phi}_m^1=-\tfrac{1}{2}\left\{2m\alpha_m^1-\frac{K^2}{2m+1}\alpha_{m+1}^1\right\}\qquad\text{for } m\geqslant 1.$$

Substituting these into (A6), and rearranging, gives

$$I_c^{(m)}(\mathbf{r};\mathbf{a})=-\tfrac{1}{2}B_0^{(m)}\Phi_m^1(\mathbf{a})\alpha_0^1(\mathbf{r})-\frac{1}{2}\sum_{n=1}^{\infty}\left\{2nB_n^{(m)}\Phi_{m+n}^1(\mathbf{a})-\frac{K^2}{2n-1}B_{n-1}^{(m)}\Phi_{m+n-1}^1(\mathbf{a})\right\}\alpha_n^1(\mathbf{r}).$$

The theorem (for $\sigma=1$) follows by combining this expansion with the corresponding expansion for $I_s^{(m)}$.

Method 2. From (**2**, equation A.6), we have

$$\begin{aligned}\Phi_m^1(\mathbf{r}_P^2)&=\frac{1}{(2m-1)!}\int_0^\infty k^{2m-2}(k+K)e^{-ky_2}\cos kx_2\,dk\\&=\frac{1}{(2m-1)!}\int_0^\infty k^{2m-2}(k+K)e^{-k(y_1+a\cos\delta)}\times\\&\qquad\times\{\cos(kx_1)\cos(ka\sin\delta)-\sin(kx_1)\sin(ka\sin\delta)\}\,dk.\end{aligned}$$

If we use

$$e^{-ky}\cos kx=\sum_{n=0}^{\infty}\frac{(-kr)^n}{n!}\cos n\theta\quad\text{and}\quad e^{-ky}\sin kx=-\sum_{n=1}^{\infty}\frac{(-kr)^n}{n!}\sin n\theta,$$

we obtain (A4), where

$$(2m-1)!\,I_c^{(m)}(r,\theta;\mathbf{a})=\sum_{n=0}^{\infty}\frac{(-r)^n}{n!}I_{2m+n-2}^+(\mathbf{a})\cos n\theta,$$

$$(2m-1)!\,I_s^{(m)}(r,\theta;\mathbf{a})=\sum_{n=1}^{\infty}\frac{(-r)^n}{n!}I_{2m+n-2}^-(\mathbf{a})\sin n\theta,$$

$$I_q^+(\mathbf{a})=\int_0^\infty k^q(k+K)e^{-ka\cos\delta}\cos(ka\sin\delta)\,dk$$

and

$$I_q^-(\mathbf{a}) = \int_0^\infty k^q(k+K)e^{-ka\cos\delta}\sin(ka\sin\delta)\,dk.$$

Consider $I_c^{(m)}$. Eliminate all even powers of r (including r^0) in favour of odd powers and $\mathring{\Phi}_m^1$, using (A5). Thus

$$(2m-1)!\,I_c^{(m)} = \mathring{\Phi}_0^1 I_{2m-2}^+ + \sum_{n=1}^\infty \frac{1}{(2n)!}\mathring{\Phi}_n^1 I_{2m+2n-2}^+ + \\ + \sum_{n=0}^\infty \frac{r^{2n+1}}{(2n+1)!}\{KI_{2m+2n-2}^+ - I_{2m+2n-1}^+\}\cos(2n+1)\theta.$$

But Φ_{m+n}^1 satisfies the free-surface condition and this implies that the term in braces approaches zero as $\delta \to \pm\frac{1}{2}\pi$. Moreover

$$I_{2m-2}^+(\mathbf{a}) = (2m-1)!\,\Phi_m^1(\mathbf{a}),$$

and so we recover (A6). The rest of the proof is as before.

By inspection, it can be verified that

$$S_{mn}^{\sigma\nu}(\mathbf{a}) = S_{nm}^{\nu\sigma}(-\mathbf{a}). \tag{A7}$$

Also, all quantities are real except Φ_0^1 and Φ_0^2:

$$\operatorname{Im}\{\Phi_0^1(r,\theta)\} = \pi e^{-Ky}\cos Kx \tag{A8}$$

and

$$\operatorname{Im}\{\Phi_0^2(r,\theta)\} = \pi e^{-Ky}\sin Kx, \tag{A9}$$

where $x = r\sin\theta$ and $y = r\cos\theta$.