WAVE MOTION 7 (1985) 177-193
NORTH-HOLLAND

ON THE T-MATRIX FOR WATER-WAVE SCATTERING PROBLEMS

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Received 12 March 1984, Revised 21 June 1984

A rigid cylinder of infinite length is floating in the free surface of deep water. The cylinder is held fixed and a given time-harmonic wave of small amplitude is incident upon it. The corresponding linear two-dimensional boundary-value problem for a velocity potential ϕ is treated using the null-field method, and an expression for the T-matrix is obtained. (The T-matrix connects the diffraction potential away from the cylinder to the given incident potential.) Fundamental properties of the T-matrix are derived from considerations of energy and reciprocity. For regular wavetrains incident from the right or from the left, there are well-known relations between the corresponding reflection and transmission coefficients; these relations are recovered by specialising the equations satisfied by the T-matrix. Two extensions to water of constant finite depth are described: one uses multipole potentials whilst the other uses Havelock wavemaker functions; this second approach also leads to a new method for treating the problem of waves in a semi-infinite channel with an end-wall of arbitrary shape.

1. Introduction

A rigid cylinder of infinite length is floating in the free surface of deep water. The cylinder is held fixed and a given time-harmonic wave of small amplitude is incident upon it. This wave is scattered and induces hydrodynamic forces on the cylinder. To model this situation, we make the usual assumptions of classical hydrodynamics [1]: we assume that the water is incompressible and inviscid, that the motion is irrotational, and that the wave crests are parallel to the generators of the cylinder. Consequently, we obtain the following linear, two-dimensional boundary-value problem for a velocity potential $\mathrm{Re}\{\phi(P)\,\mathrm{e}^{-\mathrm{i}\omega t}\}$.

Scattering problem $\mathbb{S}$

Find a function $\phi(P)$ such that ϕ satisfies Laplace's equation,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi=0 \quad \text{in } D, \tag{1.1}$$

the free-surface condition

$$K\phi+\frac{\partial \phi}{\partial y}=0 \quad \text{on } F \tag{1.2}$$

and the boundary condition

$$\frac{\partial \phi}{\partial n}=0 \quad \text{on } \partial D, \tag{1.3}$$

$\phi-\phi_{\mathrm{I}}\equiv\phi_{\mathrm{D}}$ satisfies a radiation condition, and $|\mathrm{grad}\,\phi|\to 0$ as $y\to\infty$. (The exact kinematic and dynamic boundary conditions on the moving free surface have been linearised and combined to give (1.2).)

0165-2125/85/$3.30

Here, D, F and ∂D denote the fluid domain, the mean free surface and the wetted surface of the cylinder, respectively. x and y are Cartesian coordinates, chosen so that the y-axis is vertical (y increasing with depth) with F occupying a portion of the line $y=0$; the remaining portion of this line is denoted by F_- and is assumed to contain the origin O. D_- denotes the interior of the cylinder, i.e. the region with boundary $\partial D \cup F_-$. Capital letters P, Q denote points of D, lower-case letters p, q denote points of ∂D, and P_-, Q_- denote points of D_-. $\partial/\partial n_q$ denotes normal differentiation at the point q, in the direction from ∂D into D, and r_P is the length OP. See Fig. 1.

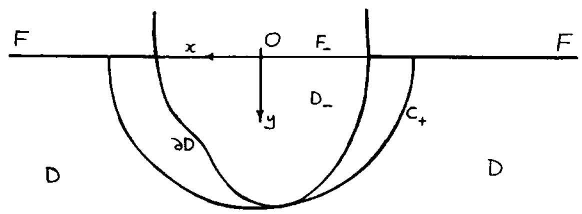

Fig. 1. The floating cylinder.

$K=\omega^2/g$, where ω is the radian frequency and g is the acceleration due to gravity, is a given positive number. ϕ, ϕ_{I} and ϕ_{D} are called the total, incident and diffraction potentials, respectively. ϕ_{I} is given, satisfies (1.1) everywhere in $y>0$ (except possibly at a finite number of isolated points in D) and (1.2) everywhere on $y=0$.

John [2] has proved that, under certain geometrical restrictions on ∂D, $\mathbb{S}$ is uniquely solvable for all values of K. We shall henceforth assume that these restrictions are met.

One method for solving $\mathbb{S}$ is the null-field method [3, 4]. This requires the solution of an infinite set of moment-like equations for $\phi(q)$ (these are called the *null-field equations*); an integral representation then gives $\phi(P)$ for all $P \in D$. Let C_+ (C_-) be the escribed (inscribed) semicircle to ∂D, centered on O. Then, for P outside C_+, we can formally reduce the null-field equations and the integral representation to

$$c_m = \sum_{n=1}^{\infty} T_{mn} d_n, \quad m=1,2,\dots$$

where d_n are the (known) coefficients in a certain expansion of $\phi_{\mathrm{I}}(P_-)$ for P_- inside C_- and c_m are the coefficients in a certain expansion of $\phi_{\mathrm{D}}(P)$ for P outside C_+ (see Section 3). T_{mn} is usually known as the *T-matrix*. Such a matrix was first introduced in the theory of quantum scattering [5, Section 7.2.2] and was later used by Waterman [6] to treat electromagnetic scattering problems; it is now used widely in classical scattering theory [7].

In this paper, we shall derive some fundamental properties of the T-matrix for $\mathbb{S}$, namely

$$T_{mn} = T_{nm} \tag{1.4}$$

and

$$\frac{2}{\pi}\,\mathrm{Im}(T_{mn}) + T_{1m}T_{1n}^* + T_{2m}T_{2n}^* = 0, \tag{1.5}$$

where the asterisk denotes the complex conjugate. These results are obtained (in Section 4) from reciprocity and energy considerations for $\mathbb{S}$. In fact, it is well known that if these considerations are valid for *any* given scattering process, then similar results will obtain; see, e.g. [8].

The results (1.4) and (1.5) can be used to provide independent checks on numerical calculations. It should also be possible to use (1.4) and (1.5) as constraints in a numerical method for finding approximations to T; this approach has been used successfully by Waterman [8] and Werby and Green [9].

In Section 5, we consider specific incident waves, namely regular waves incident from $x=+\infty$ or from $x=-\infty$; let $r_\pm$ and $t_\pm$ denote the corresponding complex reflection and transmission coefficients. We show that the known relationships between these coefficients [10] can be recovered by specialising (1.4) and (1.5).

In the last two sections, we consider the more complicated situation of finite depth. We describe two approaches: the first is a natural generalisation of the infinite-depth approach, but leads to a complicated analogue of (1.5); the second approach uses Havelock wavemaker functions—besides yielding a relation similar to (1.5), it also leads to a novel method for treating problems involving a semi-infinite channel with an end-wall of arbitrary shape. Possible extensions to three dimensions are described.

2. Reciprocity and energy relations

Let ϕ_1 and ϕ_2 be two scattering potentials, corresponding to two incident potentials, $\phi_{\mathrm{I}1}$ and $\phi_{\mathrm{I}2}$, i.e.

$$\phi_i = \phi_{\mathrm{I}i} + \phi_{\mathrm{D}i}, \quad i = 1, 2$$

where ϕ_i solves $\mathbb{S}$. Choose two distinct points, P_1 and P_2, in D, and set

$$\phi_{\mathrm{I}i}(P) = G(P; P_i), \quad i = 1, 2 \tag{2.1}$$

where $G(P; Q)$ is the potential at P due to a simple wave source at Q in the absence of the body (G is given explicitly by (3.1), below). By Green's theorem, we have

$$2\pi\{\phi_2(P_1) - \phi_1(P_2)\} = [\phi_1, \phi_2] - [\phi_1, \phi_2]_C$$

where

$$[\phi_1, \phi_2]_C = \int_C \left(\phi_1 \frac{\partial \phi_2}{\partial n} - \phi_2 \frac{\partial \phi_1}{\partial n}\right) \mathrm{d}s, \tag{2.2}$$

$[\phi_1, \phi_2] \equiv [\phi_1, \phi_2]_{\partial D}$, C is any simple path in D connecting the positive x-axis to the negative x-axis, $\partial/\partial n$ denotes normal differentiation on C in the outward direction (i.e. away from the body), and it is assumed that P_1 and P_2 lie in the fluid between C and ∂D. By (1.3), $[\phi_1, \phi_2] = 0$. Also, since ϕ_1 and ϕ_2 both satisfy the radiation condition (i.e. not merely $\phi_{\mathrm{D}1}$ and $\phi_{\mathrm{D}2}$), $[\phi_1, \phi_2]_C \to 0$ as the path C recedes to infinity. Hence

$$\phi_2(P_1) = \phi_1(P_2). \tag{2.3}$$

Since G is symmetric, we have

$$\phi_{\mathrm{I}2}(P_1) = G(P_1; P_2) = G(P_2; P_1) = \phi_{\mathrm{I}1}(P_2),$$

whence (2.3) reduces to

$$\phi_{\mathrm{D}2}(P_1) = \phi_{\mathrm{D}1}(P_2). \tag{2.4}$$

This *reciprocity relation* states that the diffraction potential at P_1 due to a wave source at P_2 is the same as the diffraction potential at P_2 due to a wave source at P_1. Reciprocity relations are also well known in other branches of scattering theory; see, e.g., [11, Section 1.32].

Note that the derivation of (2.4) is independent of the water depth, which need not be constant (the symmetry of G can be proved, using Green's theorem, without knowing G explicitly). Note also that

$\phi_1(P)$ is not known; if it were, we could apply Green's theorem to ϕ_D and ϕ_1 and obtain

$$\phi_D(P_1) = \frac{-1}{2\pi} \int_{\partial D} \phi_1(q) \frac{\partial}{\partial n_q} \phi_I(q) \, ds_q,$$

which is a *formula* for ϕ_D (ϕ_1 is known as the *exact Green function*, or *Neumann function*, for $\mathbb{S}$; see, e.g. [12, p. 38]).

Let us now consider the average flux of energy through a control contour C, namely

$$E = \tfrac{1}{2}\rho\omega \, \mathrm{Im} \int_C \phi^* \frac{\partial \phi}{\partial n} \, ds = -\tfrac{1}{4}i\rho\omega[\phi^*, \phi]_C.$$

If ϕ_I is regular inside C, Green's theorem gives

$$E = -\tfrac{1}{4}i\rho\omega[\phi^*, \phi] = 0,$$

by (1.3), i.e. the average flux of energy through C is zero. Writing $\phi = \phi_I + \phi_D$ and noting that $[\phi_I^*, \phi_I] = 0$ (because ϕ_I is regular in D_-), we obtain

$$0 = \tfrac{1}{2}\rho\omega \, \mathrm{Im}[\phi_I^*, \phi_D] + E_D \tag{2.5a}$$

where

$$E_D = -\tfrac{1}{4}i\rho\omega[\phi_D^*, \phi_D] \tag{2.5b}$$

is the average flux of energy through C due to the diffracted waves only. We shall refer to (2.5) as the *energy relation*.

The energy relation is usually stated for the special case in which ϕ_I corresponds to a train of regular surface waves. Moreover, it is usually expressed in terms of Kochin's H-function; for details, see, e.g., [1, p. 249] or [10]. Similar relations have been obtained in other fields, where they are known as 'forward-scattering' theorems, or 'cross-section' theorems, or 'optical' theorems; see, e.g., [11, Sections 8.6, 8.24] or [13–16].

3. The null-field equations and the T-matrix

The potential of a simple wave source is [17]

$$G(P, Q) \equiv G(x, y; \xi, \eta) = \tfrac{1}{2}\log\frac{(x-\xi)^2 + (y-\eta)^2}{(x-\xi)^2 + (y+\eta)^2} - 2 \int_0^\infty e^{-k(y+\eta)} \cos k(x-\xi) \frac{dk}{k-K}, \tag{3.1}$$

where the path of integration passes below the pole of the integrand at $k = K$. We apply Green's theorem in D to ϕ_D and G, and in D_- to ϕ_I and G, and then add the resulting equations to give

$$2\pi\phi_D(P) = -\int_{\partial D} \phi(q) \frac{\partial}{\partial n_q} G(P, q) \, ds_q \tag{3.2}$$

and

$$2\pi\phi_I(P_-) = \int_{\partial D} \phi(q) \frac{\partial}{\partial n_q} G(P_-, q) \, ds_q. \tag{3.3}$$

(3.2) is an integral representation for ϕ_D, valid for all $P \in D$. (3.3) is valid for all $P_- \in D_-$.

G has a bilinear expansion [17],

$$G(P, Q) = \sum_{m=1}^{\infty} \alpha_m(P)\Phi_m(Q) \tag{3.4}$$

for $r_P < r_Q$, where the functions α_m and Φ_m are defined in Appendix A; each is harmonic and satisfies the free-surface condition (1.2); α_m are regular, whilst the *multipole potentials* Φ_m are singular at O and satisfy the radiation condition. Henceforth, we shall simplify our notation and use a summation convention: sum over repeated suffices from 1 to ∞. Thus, (3.4) becomes

$$G(P, Q) = \alpha_m(P)\Phi_m(Q).$$

If we restrict P to lie outside C_+ and P_- to lie inside C_-, we can substitute (3.4) into (3.2) and (3.3) giving

$$2\pi\phi_D(P) = c_m\Phi_m(P) \tag{3.5}$$

and

$$2\pi\phi_I(P_-) = d_m\alpha_m(P_-), \tag{3.6}$$

where

$$c_m = -\langle\phi, \alpha_m\rangle, \quad m = 1, 2, \ldots, \tag{3.7}$$

$$d_m = \langle\phi, \Phi_m\rangle, \quad m = 1, 2, \ldots \tag{3.8}$$

and

$$\langle f, g\rangle = \int_{\partial D} f(q)\frac{\partial g(q)}{\partial n_q}\,\mathrm{d}s_q$$

The constants d_m $(m = 1, 2, \ldots)$ are known; they are given in terms of ϕ_I by

$$d_m = [\phi_I, \Phi_m]. \tag{3.9}$$

(3.8) are called the *null-field equations*; they are to be solved for $\phi(q)$ and are known to be uniquely solvable at all frequencies [3, 4]. Once $\phi(q)$ has been determined, $\phi_D(P)$ is given by (3.2), or, when P is outside C_+, by (3.5) and (3.7).

To solve the null-field equations, we begin by choosing a basis for representing functions defined on ∂D; let $\{\phi_n(q)\}$, $n = 1, 2, \ldots$, be such a basis. Thus, we may write

$$\phi(q) = a_n\phi_n(q) \tag{3.10}$$

where a_n, $n = 1, 2, \ldots$, are unknown coefficients. Substituting (3.10) into (3.8) gives

$$Q_{mn}a_n = d_m, \quad m = 1, 2, \ldots \tag{3.11}$$

where

$$Q_{mn} = \langle\phi_n, \Phi_m\rangle. \tag{3.12}$$

Similarly, substituting (3.10) into (3.7) gives

$$c_m = -\hat{Q}_{mn}a_n, \quad m = 1, 2, \ldots \tag{3.13}$$

where

$$\hat{Q}_{mn} = \langle\phi_n, \alpha_m\rangle. \tag{3.14}$$

The system (3.11) is uniquely solvable, i.e. Q^{-1}, the inverse of the infinite matrix Q, exists. Thus, eliminating a_n between (3.11) and (3.13), we obtain

$$c_m = T_{mn} d_n, \quad m = 1, 2, \ldots \tag{3.15}$$

where

$$T_{mn} = -\hat{Q}_{ml} Q_{ln}^{-1} \tag{3.16}$$

is known as the *T-matrix*. Given T, we can determine the diffraction potential, ϕ_D, outside C_+ for any given incident potential, ϕ_I, *without* computing the values of ϕ on ∂D.

It is easy to see that the unique-solvability of $\mathbb{S}$ implies that T exists and is unique. This in turn implies that T is independent of the choice of basis $\{\phi_n\}$. However, this choice may be important in numerical calculations, when T must necessarily be truncated.

Numerical solutions of the null-field equations have been presented in [3, 4, 18]. In [3], simple choices for $\{\phi_n\}$ are used. In [4], a different method is used; this method is convergent, but is only applicable when ϕ_I corresponds to regular surface waves on deep water. Multiple-scattering problems are considered in [18].

In the next section, we shall derive two basic properties of the T-matrix. These are consequences of the reciprocity and energy relations given in Section 2, and do not depend on the *derivation* of the T-matrix given above; we merely require that the coefficients c_m and d_m occurring in the representations (3.5) and (3.6), respectively, are related through a matrix T_{mn} by (3.15).

4. Fundamental properties of the *T*-matrix

4.1. Reciprocity relation

Let

$$2\pi\phi_{Di}(P) = c_m^{(i)} \Phi_m(P) \quad \text{and} \quad 2\pi\phi_{Ii}(P_-) = d_m^{(i)} \alpha_m(P_-)$$

for $i = 1, 2$, P outside C_+ and P_- inside C_-. From (2.1) and (3.4), we obtain

$$d_m^{(i)} = 2\pi\Phi_m(P_i), \quad m = 1, 2, \ldots,$$

for $i = 1, 2$ and $P_i \in D$. Hence

$$\begin{aligned} 2\pi\{\phi_{D2}(P_1) - \phi_{D1}(P_2)\} &= c_m^{(2)}\Phi_m(P_1) - c_m^{(1)}\Phi_m(P_2) = \Phi_m(P_1) T_{mn} d_n^{(2)} - \Phi_m(P_2) T_{mn} d_n^{(1)} \\ &= 2\pi\{\Phi_m(P_1) T_{mn} \Phi_n(P_2) - \Phi_m(P_2) T_{mn} \Phi_n(P_1)\}. \end{aligned}$$

Thus, using the reciprocity relation (2.4), we obtain

$$0 = \Phi_m(P_1)\Phi_n(P_2)\{T_{mn} - T_{nm}\};$$

this identity must hold for all P_1 and P_2 outside C_+, whence

$$T_{mn} = T_{nm}, \tag{4.1}$$

i.e. the T-matrix is symmetric.

4.2. Energy relation

From (3.5), we have

$$4\pi^2[\phi_D^*, \phi_D] = 4\pi^2[\phi_D^*, \phi_D]_{C_+} = c_k^* c_j[\Phi_k^*, \Phi_j]_{C_+} = c_k^* c_j[\Phi_k^*, \Phi_j].$$

Similarly, if ϕ_I is regular inside C_+,

$$4\pi^2[\phi_I^*, \phi_D] = d_k^* c_j[\alpha_k^*, \Phi_j],$$

where we have also used (3.6). Substituting into the energy relation (2.5), we obtain

$$0 = d_m^* d_n\{T_{jn}[\alpha_m^*, \Phi_j] - T_{jm}^*[\alpha_n, \Phi_j^*] + T_{km}^* T_{jn}[\Phi_k^*, \Phi_j]\}.$$

This identity must hold for any incident potential (that is regular inside C_+) whence

$$T_{jn}[\alpha_m^*, \Phi_j] - T_{jm}^*[\alpha_n, \Phi_j^*] + T_{km}^* T_{jn}[\Phi_k^*, \Phi_j] = 0. \tag{4.2}$$

From [17], we have

$$[\alpha_m^*, \Phi_j] = [\alpha_m, \Phi_j^*] = 2\pi\delta_{jm} \tag{4.3}$$

and

$$[\Phi_k^*, \Phi_j] = 2\pi^2 \mathrm{i}\delta_{jk}(\delta_{j1} + \delta_{j2}) \quad \text{(no summation)} \tag{4.4}$$

where δ_{ij} is the Kronecker delta. Thus, (4.2) becomes

$$T_{mn} - T_{nm}^* + \pi\mathrm{i}(T_{1m}^* T_{1n} + T_{2m}^* T_{2n}) = 0$$

which, on using (4.1), reduces to

$$\frac{2}{\pi}\,\mathrm{Im}(T_{mn}) + T_{1m}T_{1n}^* + T_{2m}T_{2n}^* = 0. \tag{4.5}$$

Observe that only Φ_1 and Φ_2 can carry energy to infinity (these are the only two wavelike multipole potentials; see Appendix A); this accounts for the occurrence of the subscripts 1 and 2 in (4.5).

5. The Kreisel–Meyer relations

In this section, we shall consider particular incident waves, namely regular surface waves. We show that the well-known Kreisel–Meyer relations [10] can be obtained by specialising the fundamental relations satisfied by the T-matrix.

Consider a train of regular surface waves, propagating from $x = +\infty$ towards the cylinder; the corresponding potential is

$$\phi_I = A_+\,\mathrm{e}^{-Ky-\mathrm{i}Kx}$$

where A_+ is a complex constant. This incident wave will be partially reflected and partially transmitted. We define r_+ and t_+ by

$$\phi(P) \sim \begin{cases} A_+\,\mathrm{e}^{-Ky}(\mathrm{e}^{-\mathrm{i}Kx} + r_+\,\mathrm{e}^{\mathrm{i}Kx}) & \text{as } x \to +\infty, \\ A_+ t_+\,\mathrm{e}^{-Ky-\mathrm{i}Kx} & \text{as } x \to -\infty; \end{cases} \tag{5.1}$$

r_+ and t_+ are the (complex) reflection and transmission coefficients. Since $\phi = \phi_{\mathrm{I}} + \phi_{\mathrm{D}}$, we have

$$\phi_{\mathrm{D}}(P) \sim \begin{cases} A_+ r_+ \,\mathrm{e}^{-Ky+\mathrm{i}Kx} & \text{as } x \to +\infty, \\ A_+(t_+ - 1)\,\mathrm{e}^{-Ky-\mathrm{i}Kx} & \text{as } x \to -\infty. \end{cases} \tag{5.2}$$

ϕ_{D} is also given by (3.5); using the asymptotic behaviour of Φ_m given in Appendix A, we find that

$$\phi_{\mathrm{D}}(P) \sim \begin{cases} \tfrac{1}{2}(c_1 + \mathrm{i}c_2)\,\mathrm{e}^{-Ky+\mathrm{i}Kx} & \text{as } x \to +\infty, \\ \tfrac{1}{2}(-c_1 + \mathrm{i}c_2)\,\mathrm{e}^{-Ky-\mathrm{i}Kx} & \text{as } x \to -\infty. \end{cases} \tag{5.3}$$

Comparing (5.2) and (5.3), we see that

$$c_1 = A_+(r_+ - t_+ + 1) \tag{5.4}$$

and

$$c_2 = -\mathrm{i}A_+(r_+ + t_+ - 1). \tag{5.5}$$

Now, from (3.6) and (A.1), we have

$$d_1 = \mathrm{i}\pi A_+, \qquad d_2 = -\pi A_+$$

and $d_n = 0$ for $n > 2$. Thus, from (3.15), we have

$$c_1 = \pi A_+(\mathrm{i}T_{11} - T_{12}) \quad \text{and} \quad c_2 = \pi A_+(\mathrm{i}T_{21} - T_{22}),$$

whence (5.4) and (5.5) give

$$r_+ - t_+ + 1 = \pi(\mathrm{i}T_{11} - T_{12}) \tag{5.6}$$

and

$$r_+ + t_+ - 1 = -\pi(T_{21} + \mathrm{i}T_{22}). \tag{5.7}$$

Similarly, for a regular wave from $x = -\infty$, we have

$$\phi_{\mathrm{I}} = A_-\,\mathrm{e}^{-Ky+\mathrm{i}Kx} \quad \text{and} \quad \phi(P) \sim \begin{cases} A_- t_-\,\mathrm{e}^{-Ky+\mathrm{i}Kx} & \text{as } x \to +\infty, \\ A_-\,\mathrm{e}^{-Ky}(\mathrm{e}^{\mathrm{i}Kx} + r_-\mathrm{e}^{-\mathrm{i}Kx}) & \text{as } x \to -\infty. \end{cases} \tag{5.8}$$

In this case, we have

$$d_1 = -\mathrm{i}\pi A_-, \qquad d_2 = -\pi A_-$$

and $d_n = 0$ for $n > 2$. It follows that

$$r_- - t_- + 1 = \pi(\mathrm{i}T_{11} + T_{12}) \tag{5.9}$$

and

$$r_- + t_- - 1 = \pi(T_{21} - \mathrm{i}T_{22}). \tag{5.10}$$

If we eliminate r_+ between (5.6) and (5.7), and r_- between (5.9) and (5.10), we obtain

$$t_+ - t_- = \pi(T_{12} - T_{21})$$

whence the symmetry of the T-matrix, (4.1), shows that

$$t_+ = t_- = t, \text{ say.} \tag{5.11}$$

From (5.6), (5.7), (5.9) and (5.10), we have

$$\pi i T_{11} = 1 - t + \tfrac{1}{2}(r_- + r_+), \tag{5.12}$$

$$\pi i T_{22} = 1 - t - \tfrac{1}{2}(r_- + r_+), \tag{5.13}$$

and

$$\pi T_{12} = \pi T_{21} = \tfrac{1}{2}(r_- - r_+). \tag{5.14}$$

If we substitute these into (4.5), we obtain

$$|r_- + r_+ - 2t|^2 + |r_- - r_+|^2 - 4 = 0, \tag{5.15}$$

$$|r_- + r_+ + 2t|^2 + |r_- - r_+|^2 - 4 = 0, \tag{5.16}$$

and

$$|r_-|^2 - |r_+|^2 + 2i\,\mathrm{Im}(tr_+^* + t^* r_-) = 0, \tag{5.17}$$

where (5.15), (5.16) and (5.17) correspond to $m = n = 1$, $m = n = 2$ and $m = 1$, $n = 2$ (or $m = 2$, $n = 1$), respectively. Taking the real and imaginary parts of (5.17), gives

$$|r_+| = |r_-| \tag{5.18}$$

and

$$\mathrm{Im}(tr_+^* + tr_-^*) = 0. \tag{5.19}$$

Adding (5.15) and (5.16) gives

$$|r_+|^2 + |r_-|^2 + 2|t|^2 = 2,$$

whence (5.18) implies that

$$|r_+|^2 + |t|^2 = 1 \tag{5.20$_+$}$$

and

$$|r_-|^2 + |t|^2 = 1. \tag{5.20$_-$}$$

Subtracting (5.15) and (5.16) gives

$$\mathrm{Re}(tr_+^* + t^* r_-) = 0;$$

combining this with (5.19) shows that

$$tr_+^* + t^* r_- = 0. \tag{5.21}$$

Using (5.18), (5.21) can be rewritten as

$$2\arg(t) - \arg(r_+) - \arg(r_-) = \pi \textit{ modulo } 2\pi. \tag{5.22}$$

The results (5.11), (5.18), (5.20) and (5.22) are all well known: we shall call them the *Kreisel-Meyer relations*; they are derived systematically by Newman [10] for water of constant finite depth (he also derives the corresponding results in three dimensions). Kreisel [19] obtained (5.18) and (5.20), and their generalisations to the situation where the asymptotic depths of water are different at $x = \pm\infty$. R. Meyer, in an appendix to a paper by Biesel and Le Méhauté [20], proved (5.11), (5.18), (5.20) and (5.22). He also showed that, by a suitable shift of origin, it can be arranged that $\arg(r_+) = \arg(r_-)$, whence $r_+ = r_-$; however, this shift cannot be determined *a priori* and, moreover, it depends on K.

6. Extension to finite depth, using multipole potentials

We can obtain similar results when the water is of constant finite depth, h, say. We begin by modifying $\mathbb{S}$, so that the condition on ϕ as $y \to \infty$ is replaced by

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on the bottom, } y = h. \tag{6.1}$$

The potential of a simple wave source is now given by [17]

$$\begin{aligned} G(x, y; \xi, \eta) = {} & \tfrac{1}{2} \log \frac{(x-\xi)^2 + (y-\eta)^2}{(x-\xi)^2 + (y+\eta)^2} \\ & - 2 \oint_0^\infty \frac{\cosh k(h-y) \cosh k(h-\eta) \cos k(x-\xi) \, \mathrm{d}k}{\cosh kh (k \sinh kh - K \cosh kh)} \\ & - 2 \int_0^\infty \mathrm{e}^{-kh} \frac{\sinh ky \sinh k\eta}{k \cosh kh} \cos k(x-\xi) \, \mathrm{d}k \end{aligned} \tag{6.2}$$

where the path of integration passes below the pole of the integrand at $k = k_0$, and k_0 is the unique positive real root of

$$K = k_0 \tanh k_0 h. \tag{6.3}$$

G has the bilinear expansion [17]

$$G(P, Q) = \alpha_m(P) \tilde{\Phi}_m(Q) \tag{6.4}$$

for $r_P < r_Q$, where the functions α_m are the same as for the case of infinite depth, and $\tilde{\Phi}_m$ are the finite-depth multipole potentials; $\tilde{\Phi}_m$ are defined in Appendix A—they are harmonic, satisfy the free-surface and radiation conditions, satisfy the bottom condition (6.1), and are singular at O. Using (6.4), we obtain all the formulae derived in Section 3, with Φ_m replaced by $\tilde{\Phi}_m$ (except that we do not have to make this replacement in (3.9)). In particular, we have

$$2\pi \phi_{\mathrm{D}}(P) = c_m \tilde{\Phi}_m(P), \quad P \text{ outside } C_+ \tag{6.5}$$

and

$$2\pi \phi_{\mathrm{I}}(P_-) = d_m \alpha_m(P_-), \quad P \text{ inside } C_- \tag{6.6}$$

with

$$c_m = T_{mn} d_n, \quad m = 1, 2, \ldots . \tag{6.7}$$

As before, the reciprocity relation implies that T is symmetric. The energy relation yields

$$T_{jn}[\alpha_m^*, \tilde{\Phi}_j] - T_{jm}^*[\alpha_n, \tilde{\Phi}_j^*] + T_{km}^* T_{jn}[\tilde{\Phi}_k^*, \tilde{\Phi}_j] = 0. \tag{6.8}$$

Ursell [17] has evaluated the bilinear products occuring here (see Appendix A); when these are substituted into (6.8), we eventually obtain

$$\frac{2}{\pi} \operatorname{Im}(T_{mn}) + A(P_m P_n^* + Q_m Q_n^*) = 0 \tag{6.9}$$

where

$$A(k_0h)=\frac{2\cosh^2 k_0h}{2k_0h+\sinh 2k_0h}, \tag{6.10}$$

$$P_m=\frac{k_0}{K}T_{1m}+\operatorname{sech}^2 k_0h\sum_{j=1}^{\infty}\frac{k_0^{2j+1}}{(2j)!}T_{m,2j+1} \tag{6.11}$$

and

$$Q_m=T_{2m}+\operatorname{sech}^2 k_0h\sum_{j=1}^{\infty}\frac{k_0^{2j}}{(2j-1)!}T_{m,2j+2}. \tag{6.12}$$

(6.9) is a finite-depth form of (4.5); moreover, as $h\to\infty$, (6.9) reduces to (4.5).

Consider a regular wave from $x=+\infty$, with potential

$$\phi_I=A_+Y_0(y)\,\mathrm{e}^{-\mathrm{i}k_0x}$$

where

$$Y_0(y)=\cosh k_0(h-y)\operatorname{sech} k_0h, \tag{6.13}$$

and define r_+ and t_+ by

$$\phi(P)\sim\begin{cases}A_+Y_0(y)(\mathrm{e}^{-\mathrm{i}k_0x}+r_+\,\mathrm{e}^{\mathrm{i}k_0x}) & \text{as } x\to+\infty,\\ A_+Y_0(y)t_+\,\mathrm{e}^{-\mathrm{i}k_0x} & \text{as } x\to-\infty.\end{cases}$$

it is shown in Appendix B that the coefficients d_m in (6.6) are given by

$$d_1=\mathrm{i}\pi(k_0/K)A_+,\qquad (2m)!d_{2m+1}=\mathrm{i}\pi A_+k_0^{2m+1}\operatorname{sech}^2 k_0h,$$

$$d_2=-\pi A_+,\qquad (2m-1)!d_{2m+2}=-\pi A_+k_0^{2m}\operatorname{sech}^2 k_0h,$$

for $m\geqslant 1$. Complicated expressions for r_+ and t_+ can then be found by using (6.5), (6.7) and the asymptotic behaviour of the multipole potentials, $\tilde{\Phi}_m$ (see Appendix A); cf. the derivation of (5.6) and (5.7). Similarly, for a regular wave from $x=-\infty$,

$$\phi_I=A_-Y_0(y)\,\mathrm{e}^{\mathrm{i}k_0x}$$

and r_- and t_- are defined by

$$\phi(P)\sim\begin{cases}A_-Y_0(y)t_-\,\mathrm{e}^{\mathrm{i}k_0x} & \text{as } x\to+\infty,\\ A_-Y_0(y)(\mathrm{e}^{\mathrm{i}k_0x}+r_-\,\mathrm{e}^{-\mathrm{i}k_0x}) & \text{as } x\to-\infty.\end{cases}$$

The corresponding coefficients d_m are given by (see Appendix B)

$$d_1=-\mathrm{i}\pi(k_0/K)A_-,\qquad (2m)!d_{2m+1}=-\mathrm{i}\pi A_-k_0^{2m+1}\operatorname{sech}^2 k_0h,$$

$$d_2=-\pi A_-,\qquad (2m-1)!d_{2m+2}=-\pi A_-k_0^{2m}\operatorname{sech}^2 k_0h,$$

for $m\geqslant 1$, and expressions for r_- and t_- can be found. The symmetry of T then implies that $t_+=t_-=t$, say. We then obtain

$$1-t+\tfrac{1}{2}(r_-+r_+)=A\pi\mathrm{i}\left\{\frac{k_0}{K}P_1+\operatorname{sech}^2 k_0h\sum_{n=1}^{\infty}\frac{k_0^{2n+1}}{(2n)!}P_{2n+1}\right\}, \tag{6.14}$$

$$1-t-\tfrac{1}{2}(r_-+r_+)=A\pi\mathrm{i}\left\{Q_2+\operatorname{sech}^2 k_0h\sum_{n=1}^{\infty}\frac{k_0^{2n}}{(2n-1)!}Q_{2n+2}\right\}, \tag{6.15}$$

and two equivalent expressions for $r_- - r_+$:

$$\tfrac{1}{2}(r_- - r_+) = A\pi\left\{ P_2 + \operatorname{sech}^2 k_0 h \sum_{n=1}^{\infty} \frac{k_0^{2n}}{(2n-1)!} P_{2n+2} \right\} \tag{6.16}$$

$$= A\pi\left\{ \frac{k_0}{K} Q_1 + \operatorname{sech}^2 k_0 h \sum_{n=1}^{\infty} \frac{k_0^{2n+1}}{(2n)!} Q_{2n+1} \right\}. \tag{6.17}$$

We can use these, together with (6.9), to verify that r_+, r_- and t satisfy the Kreisel–Meyer relations. For example, consider (5.15): we have

$$|r_- + r_+ - 2t|^2 + |r_- - r_+|^2 - 4 = |\lambda|^2 + |r_- - r_+|^2 - 2(\lambda + \lambda^*) \tag{6.18}$$

where

$$\lambda = 2 - 2t + r_- + r_+.$$

From (6.11) and (6.14), we have

$$2(\lambda + \lambda^*) = -8A\pi\left\{ \frac{k_0^2}{K^2} \operatorname{Im}(T_{11}) + \frac{2k_0}{K} \operatorname{sech}^2 k_0 h \sum_{j=1}^{\infty} \frac{k_0^{2j+1}}{(2j)!} \operatorname{Im}(T_{1,2j+1}) \right.$$
$$\left. + \operatorname{sech}^4 k_0 h \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{k_0^{2j+2k+2}}{(2j)!(2k)!} \operatorname{Im}(T_{2k+1,2j+1}) \right\}.$$

Using (6.9), we obtain

$$2(\lambda + \lambda^*) = (2A\pi)^2 \left\{ \frac{k_0^2}{k^2} |P_1|^2 + \frac{k_0}{K} \operatorname{sech}^2 k_0 h \sum_{j=1}^{\infty} \frac{k_0^{2j+1}}{(2j)!} (P_1 P_{2j+1}^* + P_{2j+1} P_1^*) \right.$$
$$+ \operatorname{sech}^4 k_0 h \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{k_0^{2j+2k+2}}{(2j)!(2k)!} P_{2k+1} P_{2j+1}^*$$
$$\left. + \text{similar terms with } P_n \text{ replaced by } Q_n \right\}.$$

When this is substituted into (6.18), together with (6.14) and (6.17), it is seen that (5.15) is satisfied.

7. Extension to finite depth, using Havelock wavemaker functions

Most of the formulae in Section 6 for water of finite depth are very complicated, whereas the corresponding formulae for infinite depth are simple, e.g., compare (4.5) with (6.9). The reason for this is that *all* of the multipole potentials $\tilde{\Phi}_m$ are wave-like at infinity, whereas only Φ_1 and Φ_2 generate waves (see Appendix A). Thus, we wish to replace $\{\tilde{\Phi}_m\}$ with a different set of functions, most of which are wavefree (i.e. evanescent). The relevant functions are the *Havelock wavemaker functions*; the functions corresponding to $\{\alpha_m\}$ are suggested by the following alternative bilinear expansion for G (defined by (6.2)), which is given by, e.g. John [2],

$$G(P, Q) = \beta_m(P)\Psi_m(Q) \tag{7.1}$$

for $|x|<|\xi|$, where

$$\Psi_1(P)=A\pi\,\mathrm{sgn}(x)\,Y_0(y)\,\mathrm{e}^{\mathrm{i}k_0|x|},\qquad \Psi_2(P)=A\pi\mathrm{i}\,Y_0(y)\,\mathrm{e}^{\mathrm{i}k_0|x|},$$

$$\Psi_{2m+1}(P)=A_m\pi\,\mathrm{sgn}(x)\,Y_m(y)\,\mathrm{e}^{-k_m|x|},\qquad \Psi_{2m+2}(P)=A_m\pi\,Y_m(y)\,\mathrm{e}^{-k_m|x|},$$

$$\beta_1(P)=-2\,Y_0(y)\sin k_0x,\qquad \beta_2(P)=-2\,Y_0(y)\cos k_0x,$$

$$\beta_{2m+1}(P)=-2\,Y_m(y)\sinh k_mx,\qquad \beta_{2m+2}(P)=-2\,Y_m(y)\cosh k_mx.$$

Here, A and Y_0 are defined by (6.10) and (6.13), respectively, k_m, $m=1,2,\ldots$, are the positive real solutions of

$$K+k_m\tan k_mh=0,\qquad A_m=\frac{2\cos^2 k_mh}{2k_mh+\sin 2k_mh}\quad\text{and}\quad Y_m(y)=\frac{\cos k_m(h-y)}{\cos k_mh}.$$

Ψ_m are (linear combinations of) the well-known Havelock wavemaker functions [21, 22]: consider a semi-infinite channel $x>0$ with a wavemaker at $x=0$; suppose that $\partial\phi/\partial x=f(y)$ on $x=0$, $0<y<h$; then

$$\phi(x,y)=\sum_{m=0}^{\infty}a_m\Psi_{2m+2}(x,y)\quad\text{where}\quad a_m=-\frac{2}{\pi}\int_0^h f(y)\,Y_m(y)\,\mathrm{d}y,$$

since

$$\int_0^h Y_m(y)\,Y_n(y)\,\mathrm{d}y=\frac{\delta_{nm}}{2k_mA_m}\tag{7.2}$$

for m, $n\geqslant 0$, with $A_0\equiv A$. Ψ_m and β_m are harmonic functions that also satisfy the free-surface and bottom conditions; β_m are regular but Ψ_m are not smooth at $x=0$; Ψ_m satisfy the radiation conditions as $x\to\pm\infty$.

Suppose that

$$2\pi\phi_{\mathrm{D}}(P)=\tilde{c}_m\Psi_m(P)\quad\text{for } |x|>X\tag{7.3}$$

and

$$2\pi\phi_{\mathrm{I}}(P)=\tilde{d}_m\beta_m(P)\quad\text{for } |x|<X,\tag{7.4}$$

where X is a positive constant. Let

$$\tilde{c}_m=T_{mn}\tilde{d}_n,\quad m=1,2,\ldots\tag{7.5}$$

for some T_{mn}. Reciprocity then implies that T is symmetric, as before, whilst it can be shown that the energy relation (2.5) yields

$$\frac{2}{\pi}\,\mathrm{Im}(T_{mn})+A(T_{1m}T_{1n}^*+T_{2m}T_{2n}^*)=0,\tag{7.6}$$

which should be compared with (4.5).

For $|x|>X$, (3.2) and (7.1) give (7.3) with

$$\tilde{c}_m=-\langle\phi,\beta_m\rangle.\tag{7.7}$$

However, for a *floating* body, we cannot use the same procedure to reduce (3.3) to (7.4) (since there will always be points $q\equiv(\xi,\eta)\in\partial D$ with $|\xi|<|x|$). Instead, suppose we use (6.4) to give (6.6), namely

$$2\pi\phi_{\mathrm{I}}(P_-)=d_m\alpha_m(P_-),\tag{7.8}$$

with

$$d_m = \langle \phi, \tilde{\Phi}_m \rangle, \quad m = 1, 2, \ldots; \tag{7.9}$$

these are the null-field equations for finite depth. Using (3.10) and eliminating a_n, $n = 1, 2, \ldots$, we obtain

$$\tilde{c}_m = \tilde{T}_{mn} d_n \tag{7.10}$$

where

$$\tilde{T}_{mn} = -\hat{R}_{mk} \tilde{Q}_{kn}^{-1}, \qquad \tilde{Q}_{mn} = \langle \phi_n, \tilde{\Phi}_m \rangle \quad \text{and} \quad \hat{R}_{mn} = \langle \phi_n, \beta_m \rangle.$$

Thus, we have a viable procedure for computing (an approximation to) $\tilde{T}_{mn}$.

Now, if we can relate (7.4) and (7.8), i.e. if we can find the matrix S where

$$\tilde{d}_m = S_{mn} d_n, \quad m = 1, 2, \ldots,$$

then, comparing (7.5) and (7.10), we obtain

$$\tilde{T}_{mn} = T_{mk} S_{kn}, \quad m, n = 1, 2, \ldots. \tag{7.11}$$

Properties of $\tilde{T}$ can then be derived from those for T.

The matrix S is known explicitly: suppose

$$\tilde{d}_m \beta_m(x, y) = d_m \alpha_m(x, y); \tag{7.12}$$

set $x = 0$ to give

$$-2 \sum_{m=0}^{\infty} \tilde{d}_{2m+2} Y_m(y) = \sum_{n=1}^{\infty} d_{2n} \alpha_{2n}(0, y);$$

the orthogonality of $\{Y_m(y)\}$, (7.2), then yields

$$\tilde{d}_{2m+2} = -k_m A_m \sum_{n=1}^{\infty} d_{2n} \int_0^h \alpha_{2n}(0, y) Y_m(y) \, dy, \quad m = 0, 1, 2, \ldots;$$

$\tilde{d}_{2m+1}$ are obtained by first differentiating (7.12) with respect to x. Actually, it is also possible to calculate S^{-1} explictly; for a typical example, see Appendix B.

Finally, we observe that if ∂D intersects the bottom (physically, this means that the fluid domain is split into two separate domains, each one of which corresponds to a 'generalised wavemaker problem', i.e. a semi-infinite channel with an end-wall or arbitrary shape), then

$$T_{mn} = -\hat{R}_{mk} R_{kn}^{-1} \quad \text{where } R_{mn} = \langle \phi_n, \Psi_m \rangle.$$

The corresponding null-field equations are

$$\tilde{d}_m = \langle \phi, \Psi_m \rangle, \quad m = 1, 2, \ldots, \tag{7.13}$$

and these could be used to solve the generalised wavemaker problem. Furthermore, if ϕ_1 corresponds to a regular surface wave, the method of projection [4] can be used to yield a convergent numerical scheme for solving (7.13). This approach should also be useful for treating various three-dimensional problems, e.g., scattering by a right circular cone which is resting on the bottom and which pierces the free surface. These new methods are currently under investigation.

Appendix A. Multipole potentials

From [17], we have (3.4) with

$$\begin{aligned}
&\Phi_2(P) = \int_0^\infty \mathrm{e}^{-ky} \cos kx \frac{\mathrm{d}k}{k-K}, \qquad \Phi_1(P) = -\frac{1}{K}\frac{\partial}{\partial x}\Phi_2(P),\\
&\Phi_{2m+2}(P) = \frac{\cos 2m\theta}{r^{2m}} + \frac{K}{2m-1}\frac{\cos(2m-1)\theta}{r^{2m-1}}, \qquad \Phi_{2m+1}(P) = \frac{\sin(2m+1)\theta}{r^{2m+1}} + \frac{K}{2m}\frac{\sin 2m\theta}{r^{2m}},\\
&\alpha_2(P) = -2\,\mathrm{e}^{-Ky}\cos Kx, \qquad \alpha_1(P) = -2\,\mathrm{e}^{-Ky}\sin Kx,\\
&\alpha_{2m+2}(P) = \frac{-2(2m-1)!}{K^{2m}}\sum_{q=2m}^{\infty}\frac{(-Kr)^q}{q!}\cos q\theta, \qquad \alpha_{2m+1}(P) = \frac{2(2m)!}{K^{2m+1}}\sum_{q=2m+1}^{\infty}\frac{(-Kr)^q}{q!}\sin q\theta,
\end{aligned} \tag{A.1}$$

$m = 1, 2, \ldots$, and the point $P \equiv (x, y)$ has circular polar coordinates given by $x = r\sin\theta$, $y = r\cos\theta$ (with $r = r_p$). Note that Φ_{2m} and α_{2m} (Φ_{2m-1} and α_{2m-1}) are even (odd) functions of x, $m = 1, 2, \ldots$.

For water of constant finite depth, h, say, the multipole potentials are given by [17]

$$\tilde{\Phi}_2(P) = \int_0^\infty \frac{\cosh k(h-y)\cos kx\,\mathrm{d}k}{k\sinh kh - K\cosh kh}, \qquad \tilde{\Phi}_1(P) = -\frac{1}{K}\frac{\partial}{\partial x}\tilde{\Phi}_2(P),$$

$$\tilde{\Phi}_{2m+2}(P) = \Phi_{2m+2}(P) - \frac{1}{(2m-1)!}\int_0^\infty \frac{\mathrm{e}^{-kh}(K+k)(K\sinh ky - k\cosh ky)k^{2m-2}\cos kx\,\mathrm{d}k}{k\sinh kh - K\cosh kh},$$

$$\tilde{\Phi}_{2m+1}(P) = \Phi_{2m+1}(P) - \frac{1}{(2m)!}\int_0^\infty \frac{\mathrm{e}^{-kh}(K+k)(K\sinh ky - k\cosh ky)k^{2m-1}\sin kx\,\mathrm{d}k}{k\sinh kh - K\cosh kh}.$$

Asymptotic behaviour of the multipole potentials as $|x| \to \infty$

We have

$$\Phi_1 \sim \pm\pi\,\mathrm{e}^{-Ky\pm\mathrm{i}Kx} \quad \text{and} \quad \Phi_2 \sim \pi\mathrm{i}\,\mathrm{e}^{-Ky\pm\mathrm{i}Kx} \quad \text{as } x \to \pm\infty.$$

For $m > 2$, Φ_m is a wavefree potential, i.e. it decays algebraically as $|x| \to \infty$.

For finite depth, we have

$$\tilde{\Phi}_1 \sim \pm\pi(k_0/K)AY_0\,\mathrm{e}^{\pm\mathrm{i}k_0x}, \qquad (2m)!\tilde{\Phi}_{2m+1} \sim \pm\pi k_0^{2m+1}\operatorname{sech}^2(k_0h)AY_0\,\mathrm{e}^{\pm\mathrm{i}k_0x},$$

$$\tilde{\Phi}_2 \sim \pi\mathrm{i}AY_0\,\mathrm{e}^{\pm\mathrm{i}k_0x}, \qquad (2m-1)!\tilde{\Phi}_{2m+2} \sim \pi\mathrm{i}k_0^{2m}\operatorname{sech}^2(k_0h)AY_0\,\mathrm{e}^{\pm\mathrm{i}k_0x}$$

as $x \to \pm\infty$, where k_0, A and Y_0 are defined by (6.3), (6.10) and (6.13), respectively, and $m = 1, 2, \ldots$. (Note that, as $h \to \infty$, $k_0 \to K$, $A \to 1$ and, for fixed y ($0 \le y < h$), $Y_0 \to \mathrm{e}^{-Ky}$.)

Bilinear products

From [17], we have

$$[\alpha_m^*, \tilde{\Phi}_j] = [\alpha_m, \tilde{\Phi}_j^*] = 2\pi\delta_{jm} \quad \text{and} \quad [\tilde{\Phi}_k^*, \tilde{\Phi}_j] = 2\pi^2\mathrm{i}A\Omega_{kj},$$

where $\Omega_{kj} = \Omega_{jk}$, $\Omega_{2m,2n+1} = 0$,

$$\Omega_{11} = k_0^2/K^2, \qquad K(2m)!\Omega_{1,2m+1} = k_0^{2m+2}\operatorname{sech}^2 k_0h,$$

$$\Omega_{22} = 1, \qquad (2m-1)!\Omega_{2,2m+2} = k_0^{2m}\operatorname{sech}^2 k_0h$$

$$(2m)!(2n)!\Omega_{2m+1,2n+1} = k_0^{2m+2n+2}\operatorname{sech}^4 k_0h$$

and

$$(2m-1)!(2n-1)!\Omega_{2m+2,2n+2}=k_0^{2m+2n}\operatorname{sech}^4 k_0h.$$

Appendix B

Consider a regular surface wave, propagating from $x=+\infty$ on water of depth h. We seek the coefficients d_m^+ in the expansion

$$2\pi\phi_1(x,y)\equiv 2\pi Y_0(y)\,\mathrm{e}^{-\mathrm{i}k_0x}=d_m^+\alpha_m(x,y). \tag{B.1}$$

Set $x=0$, whence

$$2\pi Y_0(y)=\sum_{n=0}^{\infty} d_{2n+2}^+\alpha_{2n+2}(0,y). \tag{B.2}$$

We have

$$Y_0(y)=\cosh k_0y-(K/k_0)\sinh k_0y=1-Ky+\sum_{m=1}^{\infty}p_m(Ky)(k_0/K)^{2m} \tag{B.3}$$

where

$$p_m(Ky)=\frac{(Ky)^{2m}}{(2m)!}\left(1-\frac{Ky}{2m+1}\right).$$

From Appendix A, we have

$$\alpha_2(0,y)=-2\left(1-Ky+\sum_{m=1}^{\infty}p_m(Ky)\right) \tag{B.4}$$

and

$$\alpha_{2n+2}(0,y)=-\frac{2(2n-1)!}{K^{2n}}\sum_{m=n}^{\infty}p_m(Ky)$$

for $n=1,2,\ldots$. Hence

$$\sum_{n=1}^{\infty} d_{2n+2}^+\alpha_{2n+2}(0,y)=-2\sum_{m=1}^{\infty}p_m(Ky)\sum_{n=1}^{m}\frac{(2n-1)!}{K^{2n}}d_{2n+2}^+. \tag{B.5}$$

Substituting (B.3)-(B.5) into (B.2), and comparing coefficients gives

$$2\pi=-2d_2^+ \quad\text{and}\quad 2\pi\left(\frac{k_0}{K}\right)^{2m}=-2\left\{d_2^++\sum_{n=1}^{m}\frac{(2n-1)!}{K^{2n}}d_{2n+2}^+\right\}$$

for $m=1,2,\ldots$. Hence $d_2^+=-\pi$ and

$$1-\left(\frac{k_0}{K}\right)^{2m}=\frac{1}{\pi}\sum_{n=1}^{m}\frac{(2n-1)!}{K^{2n}}d_{2n+2}^+ \quad\text{for } m=1,2,\ldots.$$

Since

$$1-x^m=(1-x)\sum_{n=1}^{m}x^{n-1},$$

we obtain

$$d_{2n+2}^{+}=\frac{\pi(K^2-k_0^2)k_0^{2n-2}}{(2n-1)!}=-\frac{\pi k_0^{2n}\operatorname{sech}^2 k_0h}{(2n-1)!}.$$

The odd coefficients can be obtained by differentiating (B.1) with respect to x noting that

$$\frac{\partial}{\partial x}\alpha_1=K\alpha_2 \quad\text{and}\quad \frac{\partial}{\partial x}\alpha_{2n+1}=2n\alpha_{2n+2}$$

for $n=1,2,\ldots$ and then setting $x=0$; the results are

$$d_1^{+}=\pi\mathrm{i}(k_0/K) \quad\text{and}\quad (2n)!d_{2n+1}^{+}=\pi\mathrm{i}k_0^{2n+1}\operatorname{sech}^2 k_0h.$$

For a regular wave from $x=-\infty$, we have

$$2\pi\phi_I(x,y)\equiv 2\pi Y_0(y)\,\mathrm{e}^{\mathrm{i}k_0x}=d_m^{-}\alpha_m(x,y);$$

it is easy to see that

$$d_{2n+1}^{-}=-d_{2n+1}^{+} \quad\text{and}\quad d_{2n+2}^{-}=d_{2n+2}^{+} \quad \text{for } n=0,1,\ldots.$$

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