ORTHOGONAL POLYNOMIAL SOLUTIONS FOR ELLIPTICAL CRACKS UNDER SHEAR LOADINGS

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SUMMARY

An infinite solid contains a flat elliptical crack, occupying the surface (in Cartesian coordinates)

$$\{(x, y, z): x = a\rho \cos \phi, y = b\rho \sin \phi, z = 0, 0 \leq \rho < 1, 0 \leq \phi < 2\pi\}.$$

The crack faces are subjected to prescribed shear stresses, with Cartesian components q_x and q_y; the corresponding crack-face displacements have components $\pm u_x$ and $\pm u_y$. We expand u_x, u_y, q_x and q_y as Fourier series in ϕ, and expand each Fourier component as a series of orthogonal polynomials in ρ. We obtain explicit relations (systems of linear algebraic equations) between the coefficients in these series, and derive simple formulae for the stress-intensity factors. Our systems of equations (i) yield the known analytical solutions for uniform shear and simple torsion, (ii) reduce to those obtained by Krenk for the penny-shaped crack ($a = b$), and (iii) are computationally attractive for arbitrary polynomial shear loadings.

1. Introduction

CONSIDER a flat elliptical crack in an otherwise unbounded homogeneous isotropic elastic solid. Let (x, y, z) be Cartesian coordinates, so that the crack occupies the region

$$\Omega = \{(x, y, z): 0 \leq \rho < 1, 0 \leq \phi < 2\pi, z = 0\}, \tag{1.1}$$

where

$$x = a\rho \cos \phi, \qquad y = b\rho \sin \phi \tag{1.2}$$

and $0 < b \leq a$; Ω is an elliptical region in the plane $z = 0$. A basic problem in fracture mechanics is to determine the displacement of the crack faces when they are subjected to arbitrary equal and opposite prescribed tractions. (This is sufficient to determine the displacement field throughout the solid: use the Somigliana formula.) An efficient scheme for solving this problem could also be used as part of an alternating technique for computing the stresses around an elliptical crack in a finite body. In the present paper, we describe such a scheme.

We begin with a brief description of previous work on the static loading of elliptical cracks. The earliest solution was given by Green and Sneddon (**1**), who considered a pressurized crack (normal component of the traction prescribed on Ω). This problem is symmetric about the plane $z = 0$, and can be reduced to the determination of a single harmonic function. For the special case of constant pressure, Green and Sneddon recognised that this potential problem had a known solution, proportional to $V^{(1)}$, where

$$V^{(\alpha)}(x, y, z) = \int_\lambda^\infty \frac{\{\omega(s)\}^\alpha\, ds}{\{Q(s)\}^{\frac{1}{2}}},$$

$$\omega(s) = 1 - \frac{x^2}{a^2+s} - \frac{y^2}{b^2+s} - \frac{z^2}{s},$$

$$Q(s) = s(s + a^2)(s + b^2),$$

λ is an ellipsoidal coordinate, defined as the positive root of $\omega(s) = 0$ and $\alpha > -\frac{1}{2}$ is a real number. The harmonic functions $V^{(\alpha)}$, and their partial derivatives with respect to x and y, have been used to treat polynomial loadings of the crack. Here, we shall consider their use for shear loadings of the crack (references to work on pressurized elliptical cracks are given in (**2**)).

When the crack faces are subjected to equal and opposite tangential tractions, the problem is antisymmetric about $z = 0$, and can be reduced to the determination of a pair of harmonic functions, $\Phi(x, y, z)$ and $\Psi(x, y, z)$, say (see section 3). For a constant shear, Kassir and Sih (**3**) took Φ and Ψ proportional to $V^{(1)}$, and then determined the two constants of proportionality by imposing the boundary conditions on Ω. This approach has been developed by Smith and Sorensen (**4**), Kassir and Sih (**5**, Chapter 3), Vijayakumar and Atluri (**6**) and Nishioka and Atluri (**7**). These last two papers contain a well-developed, systematic treatment, which we shall now describe. (We do this so as to put our own work into context; see section 6.) Thus, suppose that the prescribed tangential components of the traction on Ω, q_x and q_y, have the forms (**6**)

$$q_x(x, y) = \sum_{m=0}^{M} \sum_{n=0}^{m} A_{m-n}^n x^{2m-2n} y^{2n} \tag{1.3a}$$

and

$$q_y(x, y) = xy \sum_{m=0}^{M-1} \sum_{n=0}^{m} B_{m-n}^n x^{2m-2n} y^{2n}. \tag{1.3b}$$

Then take

$$\Phi(x, y, z) = \sum_{m=0}^{M} \sum_{n=0}^{m} C_{m-n}^n F_{2m,2n} \tag{1.4a}$$

and

$$\Psi(x, y, z) = \sum_{m=0}^{M-1} \sum_{n=0}^{m} D_{m-n}^n F_{2m+2,2n+1}, \tag{1.4b}$$

where

$$F_{mn}(x, y, z) = \frac{\partial^m V^{(m+1)}}{\partial x^{m-n}\, \partial y^n}$$

and C_m^n, D_m^n are unknown coefficients. Calculate the tangential tractions on Ω from Φ and Ψ, and expand them as polynomials in x and y (this is the complicated step: similar expansions of $F_{mn}(x, y, 0)$ are required). Equate the coefficients in these polynomials with the corresponding coefficients in (1.3), leading to systems of linear algebraic equations for the determination of C_m^n, D_m^n; see section 6.

The problem solved by Kassir and Sih (**3**) (uniform shear) was first solved by Eshelby (**8**). This problem has also been discussed by other authors. Chen (**9**) considered an elastic material with transverse isotropy; he also sketched a method for linear loadings. Shibuya (**10**) used dual integral equations and a conformal mapping between Ω and the unit circle. Kostrov and Das (**11**) have evaluated the stress field around the crack in some detail.

Eshelby's approach (**8**) was based on Somigliana's formula. This has also been used by Willis (**12**). His method is applicable to anisotropic media, and also leads to the following analogue of Galin's theorem.

THEOREM. *Suppose that the faces of an elliptical crack are loaded by equal and opposite tangential tractions, with Cartesian components* $q_x(x, y)$ *and* $q_y(x, y)$. *Assume that*

$$q_x(x, y) = Q_1(x, y) \quad \text{and} \quad q_y(x, y) = Q_2(x, y), \tag{1.5}$$

where Q_i $(i = 1, 2)$ *are polynomials of degree* n *in* x *and* y. *Then the tangential displacements of the crack faces have Cartesian components* $\pm u(x, y)$ *and* $\pm v(x, y)$, *where*

$$u(x, y) = (1-\rho^2)^{\frac{1}{2}} P_1(x, y), \qquad v(x, y) = (1-\rho^2)^{\frac{1}{2}} P_2(x, y), \tag{1.6}$$

$$\rho = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^{\frac{1}{2}}, \tag{1.7}$$

and P_i $(i = 1, 2)$ *are also polynomials of degree* n *in* x *and* y.

Walpole (**13**) has also given a proof of this theorem for isotropic media.

The basic method used by Willis and Walpole is as follows. First obtain a pair of (integral) equations connecting q_x and q_y with u and v. Then, assume that u and v can be written as (1.6), and deduce that q_x and q_y can be written as (1.5). The equations connecting the coefficients in P_i to those in Q_i are complicated. However, Gladwell (**14**) has shown, for pressurized cracks, that they simplify if the Cartesian polynomials are replaced by Fourier series in ϕ and series of orthogonal polynomials in ρ (see (1.2)). Similar expansions have been used in (**2**) for pressurized elliptical cracks, and have also been used for arbitrary loadings of a penny-shaped crack $(a = b)$; references are given in (**2**).

In the present paper, we shall use methods similar to those used in (**2**) to treat polynomial shear loadings of a flat elliptical crack. These methods combine polynomial expansions of u and v with certain integral representations of Φ and Ψ, involving two-dimensional Fourier transforms (these are discussed briefly in section 2). We derive systems of simultaneous linear algebraic equations, connecting the coefficients in the expansions of u and v to the coefficients in similar expansions of the (given) loads, q_x and q_y. These systems appear to be new; they reduce to the relations found by Krenk (**15**) for polynomial shear loadings of a penny-shaped crack; they can be properly truncated (see section 6); and they are simpler than the corresponding systems obtained by Atluri *et al.* (**6**, **7**). In section 5, we derive simple formulae for the stress-intensity factors, $k_2(\phi)$ and $k_3(\phi)$. Finally, in section 7, we solve a few particular problems, and compare our solutions with those of other authors.

2. Two-dimensional Fourier transforms

Define the two-dimensional Fourier transform of $f(\mathbf{x})$ by

$$F(\boldsymbol{\xi}) = \mathscr{F}_2[f(\mathbf{x}); \boldsymbol{\xi}] = \frac{1}{2\pi}\int_{\mathbb{R}^2} f(\mathbf{x}) \exp\{i\boldsymbol{\xi}\,.\,\mathbf{x}\}\, d\mathbf{x}, \tag{2.1}$$

where $\mathbf{x} = (x, y) \in \mathbb{R}^2$, $\boldsymbol{\xi} = (\xi, \eta) \in \mathbb{R}^2$, and $\boldsymbol{\xi}\,.\,\mathbf{x} = \xi x + \eta y$. Make the substitutions $x = a\rho\cos\phi$, $y = b\rho\sin\phi$, $\xi = (\lambda/a)\cos\psi$ and $\eta = (\lambda/b)\sin\psi$ in (2.1) to give

$$\mathscr{F}_2[f(\mathbf{x}); \boldsymbol{\xi}] = \frac{ab}{2\pi}\int_0^\infty \int_0^{2\pi} f(\mathbf{x}) \exp\{i\lambda\rho\cos(\psi - \phi)\}\rho\, d\phi\, d\rho. \tag{2.2}$$

Suppose, now, that $f(\mathbf{x})$ has the Fourier expansion

$$f(\mathbf{x}) = \sum_{m=0}^\infty f_m(\rho)\cos m\phi + \sum_{m=1}^\infty \bar{f}_m(\rho)\sin m\phi. \tag{2.3}$$

Then, since

$$\exp\{\pm ir\cos\theta\} = \sum_{n=0}^\infty \varepsilon_n(\pm i)^n J_n(r)\cos n\theta, \tag{2.4}$$

where $\varepsilon_0 = 1$ and $\varepsilon_n = 2$ for $n > 0$, (2.2) gives

$$\mathscr{F}_2[f(\mathbf{x}); \boldsymbol{\xi}] = ab\sum_{m=0}^\infty i^m \cos m\psi\, \mathscr{H}_m[f_m(\rho); \lambda] + ab\sum_{m=1}^\infty i^m \sin m\psi\, \mathscr{H}_m[\bar{f}_m(\rho); \lambda], \tag{2.5}$$

where

$$\mathscr{H}_m[f(\rho); \lambda] = \int_0^\infty f(\rho) J_m(\lambda\rho)\rho\, d\rho \tag{2.6}$$

denotes a Hankel transform of f. Similar results obtain for the inverse Fourier transform, which is defined by

$$f(x) = \mathscr{F}_2^{-1}[F(\boldsymbol{\xi}); \mathbf{x}] = \frac{1}{2\pi}\int_{\mathbb{R}^2} F(\boldsymbol{\xi}) \exp\{-i\boldsymbol{\xi}\,.\,\mathbf{x}\}\, d\boldsymbol{\xi}. \tag{2.7}$$

3. An elliptical crack subjected to shear loadings

The problem under consideration is equivalent to determining the displacement $\mathbf{u} = (u_x, u_y, u_z)$ and the corresponding stresses τ_{ij} in the half-space $z > 0$, when the boundary conditions are

$$\tau_{xz}(x, y, 0) = q_x(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{3.1}$$

$$\tau_{yz}(x, y, 0) = q_y(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{3.2}$$

$$u_x(x, y, 0) = 0, \quad \mathbf{x} \in \mathbb{R}^2 \backslash \Omega, \tag{3.3}$$

$$u_y(x, y, 0) = 0, \quad \mathbf{x} \in \mathbb{R}^2 \backslash \Omega, \tag{3.4}$$

$$\tau_{zz}(x, y, 0) = 0, \quad \mathbf{x} \in \mathbb{R}^2. \tag{3.5}$$

This problem can be reduced to the determination of two harmonic functions, $\Phi(x, y, z)$ and $\Psi(x, y, z)$ (**5**, **6**);

$$2\mu u_x = z\frac{\partial X}{\partial x} - 2(1-\nu)\frac{\partial \Phi}{\partial z}, \qquad 2\mu u_y = z\frac{\partial X}{\partial y} - 2(1-\nu)\frac{\partial \Psi}{\partial z},$$

and

$$2\mu u_z = z\frac{\partial X}{\partial z} - (1-2\nu)X,$$

where μ is the shear modulus, ν is Poisson's ratio and

$$X = \frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial y}.$$

This representation satisfies (3.5). Also

$$\tau_{xz}(x, y, 0) = -(1-\nu)\frac{\partial^2 \Phi}{\partial z^2} + \nu\frac{\partial X}{\partial x}, \tag{3.6}$$

$$\tau_{yz}(x, y, 0) = -(1-\nu)\frac{\partial^2 \Psi}{\partial z^2} + \nu\frac{\partial X}{\partial y}, \tag{3.7}$$

$$u(\mathbf{x}) \equiv u_x(x, y, 0) = \frac{-(1-\nu)}{\mu}\frac{\partial \Phi}{\partial z}, \tag{3.8}$$

and

$$v(\mathbf{x}) \equiv u_y(x, y, 0) = \frac{-(1-\nu)}{\mu}\frac{\partial \Psi}{\partial z}. \tag{3.9}$$

Following (**2**), suppose that, for $\mathbf{x} \in \mathbb{R}^2$,

$$u(\mathbf{x}) = \tfrac{1}{2}a\sum_{m=0}^{\infty} u_m(\rho)\cos m\phi + \tfrac{1}{2}a\sum_{m=1}^{\infty} \tilde{u}_m(\rho)\sin m\phi, \tag{3.10}$$

and then take the harmonic function Φ as

$$\Phi(x, y, z) = \frac{\mu}{1-\nu}\mathscr{F}_2^{-1}[|\boldsymbol{\xi}|^{-1}U(\boldsymbol{\xi})\exp(-|\boldsymbol{\xi}|z); \mathbf{x}], \tag{3.11}$$

where

$$U(\xi) = \tfrac{1}{2}a^2 b \sum_{m=0}^{\infty} i^m \mathscr{H}_m[u_m(\rho); \lambda] \cos m\psi + \tfrac{1}{2}a^2 b \sum_{m=1}^{\infty} i^m \mathscr{H}_m[\bar{u}_m(\rho); \lambda] \sin m\psi, \tag{3.12}$$

$|\xi| = (\xi^2 + \eta^2)^{\frac{1}{2}}$ and $\lambda = (a^2\xi^2 + b^2\eta^2)^{\frac{1}{2}}$. Then (2.5) shows that (3.8) and (3.10) are consistent; (3.3) will be satisfied if $u_m(\rho)$ and $\bar{u}_m(\rho)$ vanish for $\rho \geq 1$.

Similarly, define $v(x)$ and $\Psi(x, y, z)$ by making the transformations $u \to v$, $\Phi \to \Psi$, $U \to V$, $u_m \to \bar{v}_m$ and $\bar{u}_m \to v_m$ in (3.10 to 3.12). Equation (3.4) will be satisfied if v_m and $\bar{v}_m$ vanish for $\rho \geq 1$.

The boundary conditions on the crack, (3.1) and (3.2), will be satisfied if u_m, $\bar{u}_m$, v_m and $\bar{v}_m$ are also chosen to satisfy

$$q_x(\mathbf{x}) = \frac{-\mu}{1-\nu} \mathscr{F}_2^{-1}[|\xi|^{-1} \{(\xi^2 + (1-\nu)\eta^2)U + \nu\xi\eta V\}; \mathbf{x}] \tag{3.13}$$

and

$$q_y(\mathbf{x}) = \frac{-\mu}{1-\nu} \mathscr{F}_2^{-1}[|\xi|^{-1} \{\nu\xi\eta U + ((1-\nu)\xi^2 + \eta^2)V\}; \mathbf{x}]. \tag{3.14}$$

These integral equations are coupled. However, they are very similar in form to the equation which arises in the analysis for the pressurized crack (**2**, equation (3.9)) and so we shall adopt the same method of solution here. Note that it does not seem to be advantageous to take linear combinations of (3.13) and (3.14). In fact, even for the penny-shaped crack, nothing is gained by making the natural choice of polar coordinates, and then solving for u_r and u_θ, given τ_{rz} and $\tau_{\theta z}$; see section 4.

Consider equation (3.13). Noting that $|\xi|^{-1} = (b/\lambda)(1 - k^2 \cos^2 \psi)^{-\frac{1}{2}}$, where $k^2 = 1 - (b/a)^2$, and using results from section 2, we can reduce (3.13) to

$$q_x(\mathbf{x}) = \mu \sum_{n=0}^{\infty} t_n(\rho) \cos n\phi + \mu \sum_{n=1}^{\infty} \bar{t}_n(\rho) \sin n\phi, \tag{3.15}$$

where

$$\pi(1-\nu)k' t_n(\rho) = -\tfrac{1}{2}\varepsilon_n \sum_{m=0}^{\infty} I^C_{mn}(A, B)\mathscr{L}_n[u_m; \rho] - \tfrac{1}{2}\varepsilon_n \sum_{m=1}^{\infty} I^{SC}_{mn}\mathscr{L}_n[v_m; \rho], \tag{3.16}$$

$$\pi(1-\nu)k' \bar{t}_n(\rho) = -\sum_{m=1}^{\infty} I^S_{mn}(A, B)\mathscr{L}_n[\bar{u}_m; \rho] - \sum_{m=0}^{\infty} I^{CS}_{mn}\mathscr{L}_n[\bar{v}_m; \rho], \tag{3.17}$$

$$\mathscr{L}_n[u_m; \rho] = \mathscr{H}_n[\lambda \mathscr{H}_m\{u_m(\rho); \lambda\}; \rho], \tag{3.18}$$

$$I^C_{mn}(A, B) = \tfrac{1}{4} i^m (-i)^n \int_0^{2\pi} \Lambda(A + B \cos 2\psi) \cos m\psi \cos n\psi \, d\psi, \tag{3.19}$$

$$I_{mn}^{S}(A, B) = \tfrac{1}{4}i^m(-i)^n \int_0^{2\pi} \Lambda(A + B\cos 2\psi)\sin m\psi \sin n\psi \, d\psi, \quad (3.20)$$

$$I_{mn}^{SC} = \tfrac{1}{4}k'\nu i^m(-i)^n \int_0^{2\pi} \Lambda \sin 2\psi \sin m\psi \cos n\psi \, d\psi, \quad (3.21)$$

$$I_{mn}^{CS} = (-1)^{m+n} I_{nm}^{SC}, \quad (3.22)$$

$$A(k) = 2 - \nu - k^2, \qquad B(k) = \nu - k^2, \quad (3.23)$$

$\Lambda = (1 - k^2\cos^2\psi)^{-\frac{1}{2}}$ and $k' = b/a$. Similarly, (3.14) reduces to

$$q_y(\mathbf{x}) = \mu \sum_{n=0}^{\infty} \bar{s}_n(\rho)\cos n\phi + \mu \sum_{n=1}^{\infty} s_n(\rho)\sin n\phi, \quad (3.24)$$

where

$$\pi(1-\nu)k's_n(\rho) = -\sum_{m=0}^{\infty} I_{mn}^{CS}\mathscr{L}_n[u_m;\rho] - \sum_{m=1}^{\infty} I_{mn}^{S}(C, D)\mathscr{L}_n[v_m;\rho], \quad (3.25)$$

$$\pi(1-\nu)k'\bar{s}_n(\rho) = -\tfrac{1}{2}\varepsilon_n \sum_{m=1}^{\infty} I_{mn}^{SC}\mathscr{L}_n[\bar{u}_m;\rho] - \tfrac{1}{2}\varepsilon_n \sum_{m=0}^{\infty} I_{mn}^{C}(C, D)\mathscr{L}_n[\bar{v}_m;\rho], \quad (3.26)$$

$$C(k) = 2 - \nu - k^2(1-\nu) \quad \text{and} \quad D(k) = -\nu - k^2(1-\nu). \quad (3.27)$$

Note that (3.16) and (3.25) relate t_n and s_n to u_m and v_m, whilst (3.17) and (3.26) relate $\bar{t}_n$ and $\bar{s}_n$ to $\bar{u}_m$ and $\bar{v}_m$.

It is easy to show that $I_{2m,2n+1}^{\alpha} = I_{2m+1,2n}^{\alpha} = 0$, where α denotes C, S, SC or CS. Thus, (3.16) and (3.25) separate as follows:

$$\begin{aligned}\pi(1-\nu)k't_{2n}(\rho) = &-\tfrac{1}{2}\varepsilon_{2n} \sum_{m=0}^{\infty} I_{2m,2n}^{C}(A, B)\mathscr{L}_{2n}[u_{2m};\rho] \\ &- \tfrac{1}{2}\varepsilon_{2n} \sum_{m=1}^{\infty} I_{2m,2n}^{SC}\mathscr{L}_{2n}[v_{2m};\rho],\end{aligned} \quad (3.28)$$

$$\begin{aligned}\pi(1-\nu)k't_{2n+1}(\rho) = &-\sum_{m=0}^{\infty} I_{2m+1,2n+1}^{C}(A, B)\mathscr{L}_{2n+1}[u_{2m+1};\rho] \\ &- \sum_{m=0}^{\infty} I_{2m+1,2n+1}^{SC}\mathscr{L}_{2n+1}[v_{2m+1};\rho],\end{aligned} \quad (3.29)$$

$$\begin{aligned}\pi(1-\nu)k's_{2n}(\rho) = &-\sum_{m=0}^{\infty} I_{2m,2n}^{CS}\mathscr{L}_{2n}[u_{2m};\rho] \\ &- \sum_{m=1}^{\infty} I_{2m,2n}^{S}(C, D)\mathscr{L}_{2n}[v_{2m};\rho],\end{aligned} \quad (3.30)$$

$$\pi(1-\nu)k's_{2n+1}(\rho) = -\sum_{m=0}^{\infty} I^{CS}_{2m+1,2n+1}\mathscr{L}_{2n+1}[u_{2m+1};\rho]$$

$$-\sum_{m=0}^{\infty} I^{S}_{2m+1,2n+1}(C,D)\mathscr{L}_{2n+1}[v_{2m+1};\rho]. \quad (3.31)$$

Equations (3.17) and (3.26) separate in a similar manner.

As in (**2**), the integrals I^{α}_{mn} can simplified; in fact, they can all be written in terms of the basic integrals (see the Appendix)

$$F_m(k) = \int_0^{\frac{1}{2}\pi} (1-k^2\sin^2 x)^{-\frac{1}{2}} \cos 2mx \, dx, \quad (3.32)$$

$m = 0, 1, \ldots,$ and these are easily computed.

4. Polynomial solutions

As in (**2**), we follow Krenk (**15**) and write

$$u_m(\rho) = H(1-\rho)\rho^m \sum_{j=0}^{\infty} U_j^m \frac{\Gamma(m+\frac{1}{2})\, j!}{\Gamma(m+j+\frac{3}{2})} C^{m+\frac{1}{2}}_{2j+1}((1-\rho^2)^{\frac{1}{2}}) \quad (4.1)$$

and

$$v_m(\rho) = H(1-\rho)\rho^m \sum_{j=0}^{\infty} V_j^m \frac{\Gamma(m+\frac{1}{2})\, j!}{\Gamma(m+j+\frac{3}{2})} C^{m+\frac{1}{2}}_{2j+1}((1-\rho^2)^{\frac{1}{2}}), \quad (4.2)$$

where $H(t)$ is the Heaviside unit function, U_j^m and V_j^m are unknown coefficients, and $C_m^{\lambda}(x)$ is a Gegenbauer polynomial of degree m with index λ (**16**, §10.9); these polynomials are orthogonal (see (**2**, equation (1.4))).

From (**2**, §4), we have

$$\mathscr{L}_n[u_m;\rho] = 2\sum_{j=0}^{\infty} U_j^m L^{2j}_{m,n}(\rho), \quad (4.3)$$

where

$$L^{2j}_{m,n}(\rho) = \int_0^{\infty} \lambda J_n(\lambda\rho) j_{2j+m+1}(\lambda)\, d\lambda \quad (4.4)$$

and $j_m(z)$ is a spherical Bessel function. The integral (4.4) is evaluated in (**2**, §4) for $0 \le \rho < 1$; for example, with $l = j+m-n$, $L^{2j}_{2m,2n}(\rho)$ vanishes when l is a negative integer, but is proportional to

$$\rho^{2n}(1-\rho^2)^{-\frac{1}{2}} C^{2n+\frac{1}{2}}_{2l+1}((1-\rho^2)^{\frac{1}{2}}) \quad (4.5)$$

when $l = 0, 1, 2, \ldots$. These results can then be used to express the right-hand sides of (3.28) to (3.31) as series of terms like (4.5). So, if we express the left-hand sides in a similar form, we shall obtain relations between the coefficients. Thus, we suppose that $t_n(\rho)$ and $s_n(\rho)$ have the expansions

$$t_n(\rho) = \rho^n \sum_{j=0}^{\infty} T_j^n \frac{\Gamma(n+\frac{1}{2})\Gamma(j+\frac{3}{2})}{(n+j)!\,(1-\rho^2)^{\frac{1}{2}}} C_{2j+1}^{n+\frac{1}{2}}((1-\rho^2)^{\frac{1}{2}}) \tag{4.6}$$

and

$$s_n(\rho) = \rho^n \sum_{j=0}^{\infty} S_j^n \frac{\Gamma(n+\frac{1}{2})\Gamma(j+\frac{3}{2})}{(n+j)!\,(1-\rho^2)^{\frac{1}{2}}} C_{2j+1}^{n+\frac{1}{2}}((1-\rho^2)^{\frac{1}{2}}), \tag{4.7}$$

where the coefficients T_j^n and S_j^n are known. Using the orthogonality of the Gegenbauer polynomials, we obtain the following relations from (3.28) and (3.30):

$$-\pi(1-\nu)k'T_0^0 = I_{00}^C(A, B)U_0^0, \tag{4.8}$$

$$\begin{aligned} -\pi(1-\nu)k'T_j^{2n} = {} & \varepsilon_{2n} \sum_{m=0}^{n+j} I_{2m,2n}^C(A, B)U_{n+j-m}^{2m} + \\ & + \varepsilon_{2n} \sum_{m=1}^{n+j} I_{2m,2n}^{SC} V_{n+j-m}^{2m}, \qquad n+j \geq 1, \quad n \geq 0, \end{aligned} \tag{4.9}$$

$$\begin{aligned} -\pi(1-\nu)k'S_j^{2n} = {} & 2 \sum_{m=0}^{n+j} I_{2m,2n}^{CS} U_{n+j-m}^{2m} + \\ & + 2 \sum_{m=1}^{n+j} I_{2m,2n}^S(C, D)V_{n+j-m}^{2m}, \qquad n \geq 1, \end{aligned} \tag{4.10}$$

for $j = 0, 1, \ldots$. Similarly, (3.29) and (3.31) give

$$-\pi(1-\nu)k'T_j^{2n+1} = 2 \sum_{m=0}^{n+j} \{I_{2m+1,2n+1}^C(A, B)U_{n+j-m}^{2m+1} + I_{2m+1,2n+1}^{SC} V_{n+j-m}^{2m+1}\} \tag{4.11}$$

and

$$-\pi(1-\nu)k'S_j^{2n+1} = 2 \sum_{m=0}^{n+j} \{I_{2m+1,2n+1}^{CS} U_{n+j-m}^{2m+1} + I_{2m+1,2n+1}^S(C, D)V_{n+j-m}^{2m+1}\}, \tag{4.12}$$

for $n = 0, 1, \ldots$ and $j = 0, 1, \ldots$.

The corresponding results for (3.17) (respectively (3.26)) are obtained as follows. First expand $\bar{u}_m(\bar{v}_m)$ by (4.1)((4.2)) with $U_j^m(V_j^m)$ replaced by $\bar{U}_j^m(\bar{V}_j^m)$. Secondly, assume that $\bar{t}_n(\bar{s}_n)$ has the expansion (4.6)((4.7)) with $T_j^n(S_j^n)$ replaced by $\bar{T}_j^n(\bar{S}_j^n)$. Then the coefficients $\bar{S}_j^n$, $\bar{T}_j^n$, $\bar{U}_j^n$ and $\bar{V}_j^n$ satisfy relations obtained from (4.8) to (4.12) by making the following transformations: $A \leftrightarrow C$, $B \leftrightarrow D$, $S_j^n \to \bar{T}_j^n$, $T_j^n \to \bar{S}_j^n$, $U_j^n \to \bar{V}_j^n$ and $V_j^n \to \bar{U}_j^n$.

The limit as $k \to 0$ corresponds to a penny-shaped crack. If we evaluate the integrals I_{mn}^α in this limit, we obtain the following results from (4.8) to

(4.12):

$$-2(1-\nu)T_0^0=(2-\nu)U_0^0,$$
$$-4(1-\nu)T_j^0=2(2-\nu)U_j^0-\nu(U_{j-1}^2+V_{j-1}^2), \qquad j\geqslant 1,$$
$$-(1-\nu)(T_j^1+S_j^1)=U_j^1+V_j^1, \qquad j\geqslant 0,$$
$$-2(1-\nu)(T_j^2+S_j^2)=(2-\nu)(U_j^2+V_j^2)-2\nu U_{j+1}^0, \qquad j\geqslant 0,$$
$$-2(1-\nu)(T_j^n+S_j^n)=(2-\nu)(U_j^n+V_j^n)-\nu(U_{j+1}^{n-2}-V_{j+1}^{n-2}), \qquad n\geqslant 3,\ j\geqslant 0,$$
$$-2(1-\nu)(T_0^n-S_0^n)=(2-\nu)(U_0^n-V_0^n), \qquad n\geqslant 1,$$
$$-2(1-\nu)(T_j^n-S_j^n)=(2-\nu)(U_j^n-V_j^n)-\nu(U_{j-1}^{n+2}+V_{j-1}^{n+2}), \qquad n\geqslant 1,\ j\geqslant 1.$$

These equations can be solved explicitly for U_j^n and V_j^n. Moreover, although these results pertain to the expansions of the stresses and displacements in Cartesian coordinates, they can be simply rearranged to give the corresponding results in plane polar coordinates, and these agree with those given by Krenk (**15**) and Martin (**17**).

5. The stress-intensity factors

Consider two points P and P_1, where P is on the edge of the crack and has Cartesian coordinates $(a\cos\phi,\ b\sin\phi,\ 0)$, and P_1 is in the plane of the crack and has coordinates $(a\rho\cos\phi_1,\ b\rho\sin\phi_1,\ 0)$ with $\rho>1$. Let s denote the distance between P and P_1. Let P_1 approach P along the normal at P. We define two stress-intensity factors at P by

$$k_x(\phi)=\operatorname*{Lim}_{s\to 0+}(2s)^{\frac{1}{2}}\tau_{xz}(a\rho\cos\phi_1,\ b\rho\sin\phi_1,\ 0) \tag{5.1}$$

and

$$k_y(\phi)=\operatorname*{Lim}_{s\to 0+}(2s)^{\frac{1}{2}}\tau_{yz}(a\rho\cos\phi_1,\ b\rho\sin\phi_1,\ 0). \tag{5.2}$$

The analysis in (**2**, §5) is immediately applicable, and shows that

$$k_x(\phi)=\frac{\mu}{2(1-\nu)}\left(\frac{a}{b}\right)^{\frac{1}{2}}\mathscr{A}^{-\frac{1}{4}}X(\phi) \tag{5.3}$$

and

$$k_y(\phi)=\frac{\mu}{2(1-\nu)}\left(\frac{a}{b}\right)^{\frac{1}{2}}\mathscr{A}^{-\frac{1}{4}}Y(\phi), \tag{5.4}$$

where

$$X(\phi)=a^2\sum_{j=0}^{\infty}(-1)^j\{(A+B\cos 2\phi)U_j(\phi)+k'\nu\sin 2\phi V_j(\phi)\}, \tag{5.5}$$

$$Y(\phi)=a^2\sum_{j=0}^{\infty}(-1)^j\{k'\nu\sin 2\phi U_j(\phi)+(C+D\cos 2\phi)V_j(\phi)\}, \tag{5.6}$$

$$U_j(\phi) = U_j^0 + \sum_{m=1}^{\infty} (U_j^m \cos m\phi + \bar{U}_j^m \sin m\phi), \tag{5.7}$$

$$V_j(\phi) = \bar{V}_j^0 + \sum_{m=1}^{\infty} (\bar{V}_j^m \cos m\phi + V_j^m \sin m\phi) \tag{5.8}$$

and

$$\mathscr{A}(\phi) = a^2 \sin^2 \phi + b^2 \cos^2 \phi. \tag{5.9}$$

In fracture mechanics, it is conventional to define two different stress-intensity factors at P by

$$k_2(\phi) = \lim_{s \to 0+} (2s)^{\frac{1}{2}} \tau_{nz}(a\rho \cos \phi_1, b\rho \sin \phi_1, 0) \tag{5.10}$$

and

$$k_3(\phi) = \lim_{s \to 0+} (2s)^{\frac{1}{2}} \tau_{tz}(a\rho \cos \phi_1, b\rho \sin \phi_1, 0), \tag{5.11}$$

where (n, t, z) are Cartesian coordinates at P, with n pointing in the direction of the normal at P towards P_1. We have

$$\tau_{nz} = n_x \tau_{xz} + n_y \tau_{yz} \quad \text{and} \quad \tau_{tz} = t_x \tau_{xz} + t_y \tau_{yz},$$

where

$$(n_x, n_y) = (t_y, -t_x) = \mathscr{A}^{-\frac{1}{2}}(b \cos \phi, a \sin \phi).$$

Since

$$a^2\{b \cos \phi(A + B \cos 2\phi) + a \sin \phi(k'\nu \sin 2\phi)\} = 2\mathscr{A}b \cos \phi,$$

$$a^2\{b \cos \phi(k'\nu \sin 2\phi) + a \sin \phi(C + D \cos 2\phi)\} = 2\mathscr{A}a \sin \phi,$$

$$a^2\{-a \sin \phi(A + B \cos 2\phi) + b \cos \phi(k'\nu \sin 2\phi)\} = -2(1-\nu)\mathscr{A}a \sin \phi,$$

and

$$a^2\{-a \sin \phi(k'\nu \sin 2\phi) + b \cos \phi(C + D \cos 2\phi)\} = 2(1-\nu)\mathscr{A}b \cos \phi,$$

where we have used (3.23) and (3.27), we obtain

$$k_2(\phi) = \frac{\mu}{1-\nu}\left(\frac{a}{b}\right)^{\frac{1}{2}} \mathscr{A}^{-\frac{1}{4}} \sum_{j=0}^{\infty} (-1)^j \{b \cos \phi U_j(\phi) + a \sin \phi V_j(\phi)\} \tag{5.12}$$

and

$$k_3(\phi) = \mu\left(\frac{a}{b}\right)^{\frac{1}{2}} \mathscr{A}^{-\frac{1}{4}} \sum_{j=0}^{\infty} (-1)^j \{-a \sin \phi U_j(\phi) + b \cos \phi V_j(\phi)\}. \tag{5.13}$$

For the penny-shaped crack, these formulae reduce to those given by Krenk (**15**).

6. Truncation

Consider (4.9) and (4.10), which are coupled infinite systems of equations for the unknown coefficients U_j^{2m} and V_j^{2m} in terms of the known coefficients

S_j^{2m} and T_j^{2m}. A discussion on the proper truncation of such systems is given in (2, §6). There, it was shown that the infinite systems for the pressurized crack can be split into a sequence of finite uncoupled systems. Similar simplifications occur here. Thus, for example, suppose we consider prescribed loadings of the form

$$q_x(\mathbf{x}) = \mu \sum_{n=0}^{N} t_{2n}(\rho) \cos 2n\phi \tag{6.1a}$$

and

$$q_y(\mathbf{x}) = \mu \sum_{n=1}^{N} s_{2n}(\rho) \sin 2n\phi, \tag{6.1b}$$

where

$$t_{2n}(\rho) = \rho^{2n} \sum_{j=0}^{N-n} T_j^{2n} \frac{\Gamma(2n+\frac{1}{2})\Gamma(j+\frac{3}{2})}{(2n+j)!\,(1-\rho)^{\frac{1}{2}}} C_{2j+1}^{2n+\frac{1}{2}}((1-\rho^2)^{\frac{1}{2}}),$$

and $s_{2n}(\rho)$ has a similar expansion with T_j^{2n} replaced by S_j^{2n} (the loading (1.3) can be written in this form), then we obtain the following systems for the determination of U_j^{2n} and V_j^{2n}:

$$-\pi(1-\nu)k'T_0^0 = I_{00}^C(A, B)U_0^0, \tag{6.2a}$$

$$-\pi(1-\nu)k'T_{l-n}^{2n} = \varepsilon_{2n} \sum_{m=0}^{l} I_{2m,2n}^C(A, B)U_{l-m}^{2m} + \varepsilon_{2n} \sum_{m=1}^{l} I_{2m,2n}^{SC} V_{l-m}^{2m}, \tag{6.2b}$$

for $0 \leqslant n \leqslant N,\ n \leqslant l \leqslant N$ but $l \neq 0$, and

$$-\pi(1-\nu)k'S_{l-n}^{2n} = 2\sum_{m=0}^{l} I_{2m,2n}^{CS} U_{l-m}^{2m} + 2\sum_{m=1}^{l} I_{2m,2n}^S(C, D)V_{l-m}^{2m}, \tag{6.2c}$$

for $1 \leqslant n \leqslant N$ and $n \leqslant l \leqslant N$. All other coefficients U_j^{2m} and V_j^{2m} can be set to zero. For each l in $0 \leqslant l \leqslant N$, (6.2) gives $2l+1$ equations for $2l+1$ unknowns. For example, $l=0$ gives just (6.2a), whilst $l=1$ (for $N \geqslant 1$) gives

$$-\frac{\pi(1-\nu)k'}{2}\begin{pmatrix} 2T_1^0 \\ T_0^2 \\ S_0^2 \end{pmatrix} = \begin{pmatrix} I_{00}^C(A, B) & I_{20}^C(A, B) & I_{20}^{SC} \\ I_{02}^C(A, B) & I_{22}^C(A, B) & I_{22}^{SC} \\ I_{02}^{CS} & I_{22}^{CS} & I_{22}^S(C, D) \end{pmatrix}\begin{pmatrix} U_0^1 \\ U_0^2 \\ V_0^2 \end{pmatrix}. \tag{6.3}$$

Note that the 3×3 matrix appearing in (6.3) is symmetric; this is typical.

If we consider prescribed loadings of the form

$$q_x(\mathbf{x}) = \mu \sum_{n=0}^{N} t_{2n+1}(\rho) \cos(2n+1)\phi \tag{6.4a}$$

and

$$q_y(\mathbf{x}) = \mu \sum_{n=0}^{N} s_{2n+1}(\rho) \sin(2n+1)\phi, \tag{6.4b}$$

we find that

$$-\pi(1-\nu)k'T_{l-n}^{2n+1} = 2\sum_{m=0}^{l}\{I_{2m+1,2n+1}^{C}(A,B)U_{l-m}^{2m+1} + I_{2m+1,2n+1}^{SC}V_{l-m}^{2m+1}\} \tag{6.5}$$

and

$$-\pi(1-\nu)k'S_{l-n}^{2n+1} = 2\sum_{m=0}^{l}\{I_{2m+1,2n+1}^{CS}U_{l-m}^{2m+1} + I_{2m+1,2n+1}^{S}(C,D)V_{l-m}^{2m+1}\},$$

which hold for $0 \leq n \leq N$ and $n \leq l \leq N$. For each l in $0 \leq l \leq N$, (6.5) gives $2l+2$ equations in $2l+2$ unknowns. For example, $l=0$ gives

$$\frac{-\pi(1-\nu)k'}{2}\begin{pmatrix} T_0^1 \\ S_0^1 \end{pmatrix} = \begin{pmatrix} I_{11}^C(A,B) & I_{11}^{SC} \\ I_{11}^{CS} & I_{11}^S(C,D) \end{pmatrix}\begin{pmatrix} U_0^1 \\ V_0^1 \end{pmatrix}. \tag{6.6}$$

Loadings with other symmetries lead to similar systems; these can be obtained by making the transformations given after (4.12).

In section 1, we briefly described the method used by Atluri *et al.* (**6**, **7**). They obtained systems of equations whose structure is similar to (6.2) and (6.5). However, there are some differences, which may render the present scheme more efficient for computational purposes.

(i) For each l in $0 \leq l \leq N$, the systems (6.2) and (6.5) are uncoupled from all the others, whereas the systems in (**6**, **7**) are weakly coupled, that is, in order to solve one of their systems for $l = L$, say, they first need the solutions for the larger systems with $L < l \leq N$ (see (**6**, p. 92)).

(ii) The matrices occurring in (6.2) and (6.5) are symmetric; see, for example, (6.3) and (6.6). Moreover, they are simpler than the corresponding matrices in (**6**, **7**).

(iii) As we have expanded q_x and q_y in terms of orthogonal functions, we can write down explicit formulae for the load coefficients (S_j^n, $\bar{S}_j^n$, T_j^n and $\bar{T}_j^n$), involving weighted integrals of q_x or q_y over Ω.

Finally, we mention again the solution sketched by Smith and Sorensen (**4**). They considered a cubic loading of the crack faces, given by

$$q_x(\mathbf{x}) = \sum_{m=0}^{3}\sum_{n=0}^{m} A_{m-n}^{m}x^{m-n}y^{n}, \qquad q_y(\mathbf{x}) = \sum_{m=0}^{3}\sum_{n=0}^{m} B_{m-n}^{m}x^{m-n}y^{n}.$$

For such a loading, there are 20 independent load coefficients; in our notation, they are T_0^0, T_1^0, T_0^1, T_1^1, T_0^2, T_0^3, S_0^1, S_1^1, S_0^2, S_0^3 and 10 more obtained by making the transformations $T \to \bar{S}$ and $S \to \bar{T}$. To determine the corresponding coefficients in the expansion of $u(\mathbf{x})$ and $v(\mathbf{x})$, we must solve 8 uncoupled systems of equations, two each of sizes 1×1, 2×2, 3×3 and 4×4. This compares favourably with Smith and Sorensen, who proposed solving a single system of size 20×20!

7. Three examples

To illustrate our method, we shall consider some particular loadings of the crack. We begin with the two simplest examples, namely

(i) $$q_x(\mathbf{x}) = q_1, \qquad q_y(\mathbf{x}) = 0, \tag{7.1}$$

(ii) $$q_x(\mathbf{x}) = 0, \qquad q_y(\mathbf{x}) = q_2, \tag{7.2}$$

where q_1 and q_2 are constants. If we set $q_1 = q_0 \cos\beta$ and $q_2 = q_0 \sin\beta$, and then add the solutions to (i) and (ii), we obtain the solution for uniform shear at an angle β to the major axis of the ellipse.

(i) Since $C_1^\lambda(x) = 2\lambda x$, we obtain

$$T_0^0 = 2q_1/\pi\mu,$$

whence (6.2a) gives

$$U_0^0 = (1-\nu)k^2k'q_1/\mu\Omega_1,$$

where

$$\Omega_1(k) = -\tfrac{1}{2}k^2 I_{00}^C(A, B) = (\nu - k^2)E - \nu k'^2 K.$$

Hence, $u_y(\mathbf{x}) = 0$,

$$u_x(\mathbf{x}) = U_0^0 a(1-\rho^2)^{\frac{1}{2}} = \frac{(1-\nu)k^2 b q_1}{\mu\Omega_1}(1-\rho^2)^{\frac{1}{2}}, \tag{7.3}$$

$$k_2(\phi) = (b/a)^{\frac{1}{2}}\mathscr{A}^{-\frac{1}{4}}(k^2/\Omega_1)q_1 b\cos\phi, \tag{7.4}$$

and

$$k_3(\phi) = -(1-\nu)(b/a)^{\frac{1}{2}}\mathscr{A}^{-\frac{1}{4}}(k^2/\Omega_1)q_1 a\sin\phi, \tag{7.5}$$

where $\mathscr{A}(\phi)$ is defined by (5.9). Equation (7.3) was first obtained by Eshelby (**8**) (with an erratum given in (**18**); see also (**11**)). Equations (7.4) and (7.5) were obtained by Kassir and Sih (**3**) (but the factor k'^2 should be replaced by $k'^{\frac{1}{2}}$ in (**5**, equations 3.54a,b)) and by Shibuya (**10**).

(ii) We have

$$\bar{S}_0^0 = 2q_2/\pi\mu,$$

whence

$$\bar{V}_0^0 = (1-\nu)k^2k'q_2/\mu\Omega_2,$$

where

$$\Omega_2(k) = -\tfrac{1}{2}k^2 I_{00}^C(C, D) = \nu k'^2 K - (k^2 + \nu k'^2)E.$$

Hence, $u_x(\mathbf{x}) = 0$,

$$u_y(\mathbf{x}) = \frac{(1-\nu)k^2 b q_2}{\mu\Omega_2}(1-\rho^2)^{\frac{1}{2}}, \tag{7.6}$$

$$k_2(\phi) = (b/a)^{\frac{1}{2}}\mathscr{A}^{-\frac{1}{4}}(k^2/\Omega_2)q_2 a\sin\phi \tag{7.7}$$

and

$$k_3(\phi) = (1-\nu)(b/a)^{\frac{1}{2}}\mathscr{A}^{-\frac{1}{4}}(k^2/\Omega_2)q_2 b \cos\phi. \tag{7.8}$$

Again, (7.7) and (7.8) have been given by Kassir and Sih (**3**) and Shibuya (**10**).

We now consider a linear loading of the crack faces. Specifically, take

$$q_x(\mathbf{x}) = \tau_1(y/b) \quad \text{and} \quad q_y(\mathbf{x}) = \tau_2(x/a); \tag{7.9}$$

for suitable choices of the constants τ_1 and τ_2, this corresponds to a simple torsional loading. The only non-zero load coefficients are

$$\tilde{T}_0^1 = 4\tau_1/3\pi\mu \quad \text{and} \quad \tilde{S}_0^1 = 4\tau_2/3\pi\mu.$$

The corresponding coefficients $\tilde{U}_0^1$ and $\tilde{V}_0^1$ satisfy (cf. (6.6))

$$\frac{-\pi(1-\nu)k'}{2}\begin{pmatrix}\tilde{S}_0^1\\ \tilde{T}_0^1\end{pmatrix} = \begin{pmatrix} I_{11}^C(C,D) & I_{11}^{SC} \\ I_{11}^{CS} & I_{11}^S(A,B)\end{pmatrix}\begin{pmatrix}\tilde{V}_0^1\\ \tilde{U}_0^1\end{pmatrix},$$

whence

$$3\mu\Omega_3\tilde{U}_0^1 = -2(1-\nu)k'\{\tau_1 I_{11}^C(C,D) - \tau_2 I_{11}^{SC}\}$$

and

$$3\mu\Omega_3\tilde{V}_0^1 = -2(1-\nu)k'\{\tau_2 I_{11}^S(A,B) - \tau_1 I_{11}^{SC}\},$$

where

$$\Omega_3(k) = I_{11}^C(C,D)I_{11}^S(A,B) - (I_{11}^{SC})^2$$

and we have noted that $I_{11}^{SC} = I_{11}^{CS}$. From the Appendix, we have

$$4I_{11}^C(C,D) = 2C(F_0 - F_1) + D(F_2 - 2F_1 + F_0),$$
$$4I_{11}^S(A,B) = 2A(F_0 + F_1) - B(F_2 + 2F_1 + F_0)$$

and

$$4I_{11}^{SC} = k'\nu(F_0 - F_2).$$

Then, we have

$$u_x(\mathbf{x}) = a(y/b)\tilde{U}_0^1(1-\rho^2)^{\frac{1}{2}}, \qquad u_y(\mathbf{x}) = a(x/a)\tilde{V}_0^1(1-\rho^2)^{\frac{1}{2}},$$

$$k_2(\phi) = \frac{-\mu}{1-\nu}(a/b)^{\frac{1}{2}}\mathscr{A}^{-\frac{1}{4}}\{b\tilde{U}_0^1 + a\tilde{V}_0^1\}\sin\phi\cos\phi \tag{7.10}$$

and

$$k_3(\phi) = \mu(a/b)^{\frac{1}{2}}\mathscr{A}^{-\frac{1}{4}}\{a\tilde{U}_0^1\sin^2\phi - b\tilde{V}_0^1\cos^2\phi\}. \tag{7.11}$$

After a lengthy calculation, it may be verified that (7.10) and (7.11) agree with the formulae given by Smith and Sorensen (**4**) and by Kassir and Sih (**5**, equations (A3.7i,j)).

We have been unable to find any published solutions for loadings other than (7.1), (7.2) and (7.9). However, our method is systematic, and so we can, in principle, treat higher-order polynomial loadings. In particular, the discussion at the end of section 6 shows that it is not too difficult to obtain the analytical solution to Smith and Sorensen's problem (cubic loading).

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APPENDIX

The integrals I^{α}_{mn} can all be expressed in terms of $F_m(k)$, as defined by (3.32). We have the following relations:

$$
\begin{aligned}
&I^{C}_{2m,2n} \pm I^{S}_{2m,2n} = AF_{m\mp n} - \tfrac{1}{2}B\{F_{m\mp n+1} + F_{m\mp n-1}\},\\
&I^{C}_{2m+1,2n+1} + I^{S}_{2m+1,2n+1} = I^{C}_{2m,2n} + I^{S}_{2m,2n},\\
&I^{C}_{2m-1,2n+1} - I^{S}_{2m-1,2n+1} = -(I^{C}_{2m,2n} - I^{S}_{2m,2n}),\\
&I^{SC}_{2m,2n} \pm I^{CS}_{2m,2n} = \tfrac{1}{2}k'\nu\{F_{m\pm n+1} - F_{m\pm n-1}\},\\
&I^{SC}_{2m-1,2n+1} + I^{CS}_{2m-1,2n+1} = -(I^{SC}_{2m,2n} + I^{CS}_{2m,2n}),\\
&I^{SC}_{2m+1,2n+1} - I^{CS}_{2m+1,2n+1} = -(I^{SC}_{2m,2n} - I^{CS}_{2m,2n}),
\end{aligned}
$$

where $I^{C}_{mn} = I^{C}_{mn}(A, B)$ and $I^{S}_{mn} = I^{S}_{mn}(A, B)$.

We have

$$F_0 = K(k), \qquad k^2F_1 = 2E - (1 + k'^2)K$$

and

$$(2m + 1)k^2F_{m+1} = 4m(k^2 - 2)F_m - (2m - 1)F_{m-1}$$

for $m \geqslant 1$, where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind, respectively.