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ON SINGLE INTEGRAL EQUATIONS FOR THE TRANSMISSION PROBLEM OF ACOUSTICS*

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Abstract. The transmission problem, namely scattering of time-harmonic waves in a compressible fluid by a fluid inclusion with different material properties, is usually formulated as a pair of coupled boundary integral equations over the interface S between the inclusion and the exterior fluid. In this paper, however, we consider methods for solving the transmission problem using a *single* integral equation over S for a *single* unknown function. In fact, we derive four different integral equations, using a hybrid of the direct (Green's theorem) and indirect (layer *ansatz*) methods, and give conditions for the unique-solvability of each and for the subsequent construction of the solution to the transmission problem. Some of our single integral equations are Fredholm integral equations of the second kind with weakly-singular kernels. Thus, these equations, all of which appear to be new, are attractive computationally.

Key words. transmission problem, single integral equations

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1. Introduction. Consider the scattering of time-harmonic acoustic waves by a smooth bounded obstacle, immersed in a compressible fluid. If the obstacle is impenetrable (e.g., its surface, S, may be rigid), then the corresponding velocity potential solves an exterior boundary-value problem for the Helmholtz equation. There are two familiar methods for reducing this problem to a boundary integral equation, namely the "direct" method (using Green's theorem) and the "indirect" method (using a layer *ansatz*); see, e.g., [2], [9]. Both methods yield a single integral equation for a single unknown function.

Suppose now that the obstacle is a fluid inclusion, whose material properties differ from those of the surrounding fluid. Such an obstacle is penetrable, for waves can propagate through the interface S. The single boundary condition (for impenetrable obstacles) is replaced by a pair of transmission conditions, guaranteeing the continuity of acoustic pressure and normal velocity across S. This leads to a *transmission problem* for the corresponding velocity potential, and it is this problem that we shall study here.

The transmission problem is usually reduced to a pair of coupled boundary integral equations for a pair of unknowns: for the direct method, see, e.g., [3], [8]; for the indirect method, see, e.g., [2, § 3.8], [12]. It turns out that these pairs of equations are uniquely solvable, i.e., there are no irregular frequencies. Other pairs of boundary integral equations (that are not uniquely solvable) can be derived (see, e.g., [16] and § 4 below), as can equations involving volume integrals (see, e.g., [18]).

In this paper, we shall be concerned mainly with methods for solving the transmission problem using a single boundary integral equation for a single unknown. Such an equation was first obtained by Maystre and Vincent [15], in two dimensions. Much of their subsequent work is concerned with scattering by an infinite interface between two different unbounded media, especially periodic interfaces (gratings); for a summary, see Chapter 3 of [17].

Marx [13], [14] extended the ideas in [15] to three dimensions and also to electromagnetic and time-dependent scattering problems. For the electromagnetic case,

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Glisson [6] rederived Marx's equation and noted that irregular frequencies may occur; he has also demonstrated this numerically [7].

The equation obtained by Maystre and Vincent [15] and Marx [13], [14] (which we shall henceforth call the MVM equation) is an integral equation of the first kind, with a weakly-singular kernel. It was derived using a pair of auxiliary potentials. A simpler derivation has been given recently by Knockaert and De Zutter [11]. They also derived a different integral equation. We observe that their equation (which we shall henceforth call the KDeZ equation) is the (Hermitian) adjoint of the MVM equation.

A similar method to that used in [11] had previously been given by DeSanto [5]. He obtained two different single integral equations for the infinite-interface problem; we describe his equations at the ends of §§ 5 and 6 below. Yet another single integral equation has been obtained by Wirgin [22], using both the interior and the exterior exact Green's functions, satisfying homogeneous Neumann conditions on S. These exact Green's functions are not available for an arbitrary S and would have to be determined by solving additional integral equations.

In the present paper, we begin (in § 4) by giving a fairly detailed discussion on methods for solving the transmission problem using pairs of coupled integral equations. In particular, we obtain some new results on the direct method, in its manifestation as a Fredholm system of the second kind [8].

However, the bulk of the paper is concerned with the derivation and analysis of single integral equations. We extend the ideas implicit in [11] and give a systematic derivation of four different integral equations. The method used shows that single integral equations can be derived by using a hybrid of the direct and indirect methods.

We give conditions for the unique-solvability of the four equations and for the subsequent construction of the solution to the transmission problem. The four equations can be grouped into two pairs, each pair consisting of one equation and its (Hermitian) adjoint. We discuss two special cases of each equation in detail. Of these eight, four are Fredholm integral equations of the second kind, two are of the first kind (the MVM and KDeZ equations) and two are hypersingular equations. Given that the transmission problem itself has at most one solution (see § 3 below), we prove that its solution can be obtained by solving any one of the eight equations, except for certain irregular values of the exterior wavenumber k_e: four equations lose unique-solvability when k_e^2 is an eigenvalue of the interior Dirichlet problem, and four lose it when k_e^2 is an eigenvalue of the interior Neumann problem. These eight equations are summarized in Table 1; see § 8 below.

In § 7, we derive two different Fredholm integral equations of the second kind, both of which are uniquely solvable for all values of k_e^2. These equations are reminiscent of the combined single-layer and double-layer integral equation and the combined Green's formula integral equation, both of which can be used to solve the exterior Dirichlet problem for all values of k_e^2 (see, e.g., [2, §§ 3.6, 3.9] for references).

Computational considerations suggest that it is better to solve a single integral equation for a single unknown, rather than a pair of equations for a pair of unknowns. This was the original philosophy behind Maystre and Vincent's work [15]. See also Appendix B of [5]. In the present paper, we derive two known, and several new single integral equations, and give a theoretical foundation for solvability which was hitherto absent.

2. Statement of the problem. Let B_i denote a bounded domain in either $\mathbf{R}^2$ or $\mathbf{R}^3$, with a smooth closed boundary S and simply-connected unbounded exterior, B_e. We consider the following problem.

Transmission problem. Find functions $u_e(P)$ and $u_i(P)$, which satisfy

$$(2.1)\qquad (\nabla^2+k_e^2)u_e(P)=0, \qquad P\in B_e,$$

$$(2.2)\qquad (\nabla^2+k_i^2)u_i(P)=0, \qquad P\in B_i,$$

and two *transmission conditions* on the interface:

$$(2.3a)\qquad u(p)=u_i(p), \qquad p\in S$$

and

$$(2.3b)\qquad \frac{\partial u}{\partial n_p}=\rho\frac{\partial u_i}{\partial n_p}, \qquad p\in S,$$

where the total potential in B_e,

$$(2.4)\qquad u(P)=u_e(P)+u_{inc}(P), \qquad P\in B_e,$$

and the given incident potential, u_{inc}, is assumed to satisfy (2.1) everywhere, except possibly at isolated points in B_e. In addition, u_e must satisfy a radiation condition

$$(2.5)\qquad r_P^s\left(\frac{\partial u_e}{\partial r_P}-ik_eu_e\right)\to 0 \quad \text{as } r_P\to\infty,$$

where $s=\frac{1}{2}$ in $\mathbf{R}^2$ and $s=1$ in $\mathbf{R}^3$.

The exterior wavenumber k_e, interior wavenumber k_i and density ratio ρ are given complex constants.

We shall use the following notation: capital letters P, Q denote points of $B_e\cup B_i$; lower-case letters p, q denote points of S; and $\partial/\partial n_q$ denotes normal differentiation at the point q, in the direction from S towards B_e. We choose the origin 0 at some point in B_i; $\mathbf{r}_P$ is the position vector of P with respect to 0, and $r_P=|\mathbf{r}_P|$.

It is known that the transmission problem has at most one solution, provided some restrictions are placed on the choice of k_e, k_i and ρ. Sufficient conditions are given in the next theorem.

UNIQUENESS THEOREM. *Let k_e be such that*

$$(2.6a)\qquad k_e \text{ is real and positive, or } \operatorname{Im}(k_e)>0.$$

Let k_i and ρ be such that

$$(2.6b)\qquad \rho\neq 0 \text{ and } \operatorname{Im}(\bar\rho k_e)\geqq 0 \text{ and } \operatorname{Im}(\rho\bar k_e k_i^2)\geqq 0.$$

Then the transmission problem has at most one solution.

Proof. We sketch a proof; similar arguments were used in [3] and [12]. Let v_e and v_i solve the homogeneous transmission problem in which $u_{inc}\equiv 0$. Two applications of Green's theorem and use of (2.3) give

$$\int_{S_R} v_e\frac{\partial \bar v_e}{\partial n}\,ds=\int_{B_{e,R}}\{|\text{grad } v_e|^2-\bar k_e^2|v_e|^2\}\,dV+\int_{B_i}\bar\rho\{|\text{grad } v_i|^2-\bar k_i^2|v_i|^2\}\,dV,$$

where $S_R=\{P\in B_e\colon r_P=R\}$, $B_{e,R}=\{P\in B_e\colon r_P<R\}$ and the overbar denotes the complex conjugate. Multiply by k_e and take the imaginary part to give

$$(2.7)\qquad \begin{aligned}\operatorname{Im}\left\{k_e\int_{S_R} v_e\frac{\partial \bar v_e}{\partial n}\,ds\right\}&=\operatorname{Im}(k_e)\int_{B_{e,R}}\{|\text{grad } v_e|^2+|k_ev_e|^2\}\,dV\\&\quad+\operatorname{Im}(\bar\rho k_e)\int_{B_i}|\text{grad } v_i|^2\,dV+\operatorname{Im}(\rho\bar k_e k_i^2)\int_{B_i}|v_i|^2\,dV.\end{aligned}$$

If $\operatorname{Im}(k_e)>0$, the left-hand side of (2.7) tends to zero as $R\to\infty$ (since v_e decays exponentially), whence (2.6) imply that

$$\int_{B_e} |\operatorname{grad} v_e|^2 \, dV = 0,$$

and the result follows by standard arguments [12]. If k_e is real and positive, the right-hand side of (2.7) is nonnegative, whereas the radiation condition (2.5) gives

$$k_e \int_{S_R} |v_e|^2 \, ds + \operatorname{Im} \int_{S_R} v_e \frac{\partial \bar{v}_e}{\partial n} \, ds = o(1) \quad \text{as } R \to \infty.$$

The result follows by appealing to Rellich's lemma [2, Lemma 3.11].

If we have uniqueness (i.e., if the homogeneous transmission problem, with parameters k_e, k_i and ρ, has only the trivial solution), we shall say that $U(k_e; k_i; \rho)$ holds. Later, we shall also require uniqueness with k_e and k_i interchanged, and ρ replaced by σ, say; thus, we would then require that $U(k_i; k_e; \sigma)$ holds. We shall also use the following notation: if $U(k_e; k_i; \rho)$ holds and k_i satisfies (2.6a), we say that $U'(k_e; k_i; \rho)$ holds.

The conditions of the Uniqueness Theorem are met in all of the following examples.

Example 1. All parameters are real [2, Thm. 3.40]

$$0 < k_e < \infty, \quad 0 \leqq k_i < \infty, \quad 0 < \rho < \infty.$$

Example 2. The scattering of surface water waves by a seaweed farm [4] leads to

$$0 < k_e < \infty, \quad \operatorname{Im}(k_i^2) > 0, \quad \rho = 1.$$

Example 3. The cooking of meat in a microwave oven [19] leads to

$$0 < k_e < \infty, \quad \operatorname{Im}(k_i) > 0, \quad \rho = k_e/k_i.$$

Note that uniqueness does not obtain for *all* choices of k_e, k_i and ρ; for a simple counterexample, see [12].

Finally, we shall always assume below that

$$1 + \rho \neq 0.$$

3. Potential theory. We begin by introducing two free-space wave sources, G_α, defined by

$$G_\alpha(P, Q) = \begin{cases} -\tfrac{1}{2} i H_0^{(1)}(k_\alpha R) & \text{in } \mathbf{R}^2, \\ -\exp(i k_\alpha R)/(2\pi R) & \text{in } \mathbf{R}^3 \end{cases} \tag{3.1}$$

where $R = |\mathbf{r}_P - \mathbf{r}_Q|$ and $\alpha = e$ or i. $G_e(G_i)$ satisfies (2.1) ((2.2)) everywhere, except at $P = Q$. Also, G_e satisfies the radiation condition (2.5) if $\operatorname{Im}(k_e) \geqq 0$.

Next, we define single-layer and double-layer potentials by

$$(S_\alpha \mu)(P) = \int_S \mu(q) G_\alpha(P, q) \, ds_q, \qquad P \in B_e \cup B_i \tag{3.2}$$

and

$$(D_\alpha \nu)(P) = \int_S \nu(q) \frac{\partial}{\partial n_q} G_\alpha(P, q) \, ds_q, \qquad P \in B_e \cup B_i, \tag{3.3}$$

respectively. $(S_\alpha \mu)(P)$ is continuous in P as P crosses S, whereas both D_α and the normal derivative of S_α exhibit jumps given by

$$D_\alpha \nu = (\mp I + \bar{K}_\alpha^*) \nu \tag{3.4}$$

and

$$\frac{\partial}{\partial n_p} S_\alpha \mu = (\pm I + K_\alpha)\mu, \tag{3.5}$$

respectively, where, in each case, the upper (lower) sign corresponds to $P \to p \in S$ from $B_e(B_i)$. Here, K_α and $\bar{K}_\alpha^*$ are boundary integral operators defined by

$$K_\alpha \mu = \int_S \mu(q) \frac{\partial}{\partial n_p} G_\alpha(p, q)\, ds_q, \qquad p \in S \tag{3.6}$$

and

$$\bar{K}_\alpha^* \nu = \int_S \nu(q) \frac{\partial}{\partial n_q} G_\alpha(p, q)\, ds_q, \qquad p \in S, \tag{3.7}$$

where $\bar{K}_\alpha^*$ is the Hermitian adjoint of K_α: the asterisk denotes the adjoint with respect to the inner product in $L_2(S)$, defined by

$$(u, v) = \int_S u(q)\bar{v}(q)\, ds_q. \tag{3.8}$$

In all of the above formulae, it is sufficient that the densities μ and ν be continuous on S. However, we shall also require the normal derivative of the double-layer potential $D_\alpha \nu$, defined by

$$N_\alpha \nu = \frac{\partial}{\partial n_p}(D_\alpha \nu). \tag{3.9}$$

Continuity of ν is sufficient to ensure that the right-hand side of (3.9) is continuous across S. However, the existence of $N_\alpha \nu$ requires that ν be smoother: a sufficient condition is that ν have Hölder-continuous first tangential derivatives on S [2, p. 62].

The adjoints of S_α and N_α are given by

$$S_\alpha^* = \bar{S}_\alpha \quad \text{and} \quad N_\alpha^* = \bar{N}_\alpha. \tag{3.10}$$

We shall also make use of the formulae

$$N_\alpha S_\alpha = -I + K_\alpha^2 \tag{3.11}$$

and

$$S_\alpha N_\alpha = -I + (\bar{K}_\alpha^*)^2. \tag{3.12}$$

The results (3.10), (3.11) and (3.12) are proved in [20].

We shall make extensive use of Green's theorem. Thus, if we apply Green's theorem in B_e to u_e and G_e, we obtain the Helmholtz formula

$$\begin{array}{ll} \text{(3.13a)} & 2u_e(P) \\ \text{(3.13b)} & 0 \end{array} \Bigg\} = \int_S \left\{ G_e(P, q) \frac{\partial u_e}{\partial n_q} - u_e(q) \frac{\partial}{\partial n_q} G_e(P, q) \right\} ds_q \qquad \begin{array}{l} P \in B_e, \\ P \in B_i. \end{array}$$

Similarly, applying Green's theorem in B_i to u_{inc} and G_e, and adding the result to (3.13), we obtain

$$\begin{array}{ll} \text{(3.14a)} & 2u_e(P) \\ \text{(3.14b)} & -2u_{inc}(P) \end{array} \Bigg\} = \int_S \left\{ G_e(P, q) \frac{\partial u}{\partial n_q} - u(q) \frac{\partial}{\partial n_q} G_e(P, q) \right\} ds_q \qquad \begin{array}{l} P \in B_e, \\ P \in B_i, \end{array}$$

where we have used (2.4). Also, applying Green's theorem in B_i to u_i and G_i, we obtain

$$\begin{matrix} (3.15a) \\ (3.15b) \end{matrix} \quad \left.\begin{matrix} 0 \\ -2u_i(P) \end{matrix}\right\} = \int_S \left\{ G_i(P,q)\frac{\partial u_i}{\partial n_q} - u_i(q)\frac{\partial}{\partial n_q}G_i(P,q)\right\} ds_q \qquad \begin{matrix} P \in B_e, \\ P \in B_i. \end{matrix}$$

Letting $P \to p \in S$ in (3.14) and (3.15), we obtain

$$(I + \bar{K}_e^*)u - S_e\frac{\partial u}{\partial n} = 2u_{inc}(p), \qquad p \in S. \tag{3.16}$$

and

$$(I - \bar{K}_i^*)u_i + S_i\frac{\partial u_i}{\partial n} = 0, \qquad p \in S. \tag{3.17}$$

We shall also need the normal derivatives of (3.14) and (3.15), evaluated on S; these are

$$(I - K_e)\frac{\partial u}{\partial n} + N_e u = 2\frac{\partial u_{inc}}{\partial n} \tag{3.18}$$

and

$$(I + K_i)\frac{\partial u_i}{\partial n} - N_i u_i = 0. \tag{3.19}$$

4. Pairs of coupled integral equations. The standard method for solving the transmission problem is to solve a pair of coupled boundary integral equations. We shall describe a few variants.

4.1. Indirect method. Set

$$u_e(P) = (S_e\mu)(P) + \rho(D_e\nu)(P), \qquad P \in B_e \tag{4.1a}$$

and

$$u_i(P) = c(S_i\mu)(P) + (D_i\nu)(P), \qquad P \in B_i, \tag{4.1b}$$

where c is a constant at our disposal. Imposing the transmission conditions (2.3) gives

$$\begin{aligned} &(1+\rho)\nu - (\rho\bar{K}_e^* - \bar{K}_i^*)\nu - (S_e - cS_i)\mu = u_{inc}, \\ &(1+\rho c)\mu + \rho(N_e - N_i)\nu + (K_e - \rho c K_i)\mu = -\frac{\partial u_{inc}}{\partial n}. \end{aligned} \tag{4.2}$$

This is a pair of coupled integral equations to be solved for $\mu(q)$ and $\nu(q)$. Note that the combination $N_e - N_i$ occurs; this is an integral operator with a weakly-singular kernel [8], [12]. We have the following.

THEOREM 4.1. *Assume that* (i) $U'(k_e; k_i; \rho)$ *holds. Choose c so that* (ii) $1+\rho c \neq 0$ *and* (iii) $U(k_i; k_e; \rho c)$ *holds. Then the system* (4.2) *is uniquely solvable.*

This theorem is proved by Kress and Roach [12]. The constant c can always be chosen so that conditions (ii) and (iii) are satisfied: from the Uniqueness Theorem, we see that it is sufficient to set

$$\arg(c) = \arg(k_i) - \arg(\rho) \quad \text{if } \operatorname{Re}(k_e) \geqq 0$$

and

$$\arg(c) = \arg(k_i) - \arg(\rho) - \pi \quad \text{if } \operatorname{Re}(k_e) < 0,$$

adjusting $|c|$ if necessary, in accordance with condition (ii). We note that conditions (i) to (iii) are all satisfied in Example 1 ($k_i \neq 0$; take $c = 1$), Example 2 (take $c = k_i$) and Example 3 (take $c = k_i^2$).

If (4.2) is uniquely solvable, then the transmission problem has precisely one solution, for any u_{inc}, and this solution can be represented by (4.1), where μ and ν solve (4.2). In particular, irregular frequencies do not occur with this method. Numerical solutions of (4.2), with $c = 1$, have been presented by Rokhlin [21].

4.2. Direct method. If we use (2.3) in (3.14a) and (3.15b), we obtain the representations

$$2u_e(P) = \left(S_e \frac{\partial u}{\partial n}\right)(P) - (D_e u)(P), \qquad P \in B_e \tag{4.3a}$$

and

$$-2u_i(P) = \frac{1}{\rho}\left(S_i \frac{\partial u}{\partial n}\right)(P) - (D_i u)(P), \qquad P \in B_i. \tag{4.3b}$$

Similarly, using (2.3) in (3.16)-(3.19), we obtain

$$(I + \bar{K}_e^*)u - S_e \frac{\partial u}{\partial n} = 2u_{inc}, \tag{4.4}$$

$$\rho(I - \bar{K}_i^*)u + S_i \frac{\partial u}{\partial n} = 0, \tag{4.5}$$

$$(I - K_e)\frac{\partial u}{\partial n} + N_e u = 2\frac{\partial u_{inc}}{\partial n}, \tag{4.6}$$

$$(I + K_i)\frac{\partial u}{\partial n} - \rho N_i u = 0. \tag{4.7}$$

These are four boundary integral equations in the two unknowns $u(q)$ and $\partial u/\partial n_q$. Which equations, or linear combination of equations, should we solve?

The simplest choice is to use only (4.4) and (4.5). However, it is known that this pair is not always uniquely solvable: irregular frequencies occur; see, e.g., Morita [16].

A second choice can be motivated by considering (4.3) as a layer *ansatz*; on S, we have

$$2(u_e + u_{inc} - u_i) = \left(S_e + \frac{1}{\rho} S_i\right)\frac{\partial u}{\partial n} - (\bar{K}_e^* + \bar{K}_i^*)u + 2u_{inc} \tag{4.8a}$$

and

$$2\left(\frac{\partial u_e}{\partial n} + \frac{\partial u_{inc}}{\partial n} - \rho\frac{\partial u_i}{\partial n}\right) = (K_e + K_i)\frac{\partial u}{\partial n} - (N_e + \rho N_i)u + 2\frac{\partial u_{inc}}{\partial n} \tag{4.8b}$$

and so the transmission conditions (2.3) give

$$\begin{aligned} &\rho(\bar{K}_e^* + \bar{K}_i^*)u - (\rho S_e + S_i)\frac{\partial u}{\partial n} = 2\rho u_{inc}, \\ &(N_e + \rho N_i)u - (K_e + K_i)\frac{\partial u}{\partial n} = 2\frac{\partial u_{inc}}{\partial n}. \end{aligned} \tag{4.9}$$

This system is ρ (4.4) − (4.5) and (4.6) − (4.7). Costabel and Stephan [3] have shown that (4.9) is always uniquely solvable, and the corresponding $u_e(P)$ and $u_i(P)$, given by (4.3), solve the transmission problem; they require that both $U(k_e;\, k_i;\, \rho)$ and $U(k_i;\, k_e;\, \rho^{-1})$ hold. Note that (4.9) is a system of the first kind, involving the hypersingular operator $N_e+\rho N_i$.

A third choice can be motivated by a desire to obtain a system of the second kind with weakly-singular kernels. Thus, we consider (4.4) + (4.5) and ρ(4.6) + (4.7):

$$\begin{aligned}(1+\rho)u+(\bar{K}_e^*-\rho\bar{K}_i^*)u-(S_e-S_i)\frac{\partial u}{\partial n}&=2u_{inc},\\ (1+\rho)\frac{\partial u}{\partial n}+\rho(N_e-N_i)u-(\rho K_e-K_i)\frac{\partial u}{\partial n}&=2\rho\frac{\partial u_{inc}}{\partial n}.\end{aligned}\tag{4.10}$$

This system has been studied by Kittappa and Kleinman [8]; they give references to earlier work, and prove that (4.10) is solvable by iteration for sufficiently small $|k_e-k_i|$ and $|1-\rho|$. However, they considered neither the solvability of (4.10) for arbitrary k_e, k_i and ρ, nor the solvability of the transmission problem using (4.3). We shall correct these omissions here.

THEOREM 4.2. *Assume that* (i) $U(k_i;\, k_e;\, \rho)$ *holds and* (ii) $\operatorname{Im}(k_e)\geqq 0$. *If* u *and* $\partial u/\partial n$ *solve the system* (4.10), $u_e(P)$ *and* $u_i(P)$, *given by* (4.3), *solve the transmission problem.*

Proof. Clearly, (4.3) satisfies (2.1), (2.2) and (2.5). It remains to show that (2.3) are satisfied, i.e., that u and $\partial u/\partial n$ satisfy (4.9). To do this, consider $\tilde{u}$ and $\tilde{v}$, defined by

$$\tilde{u}(P)=\left(S_e\frac{\partial u}{\partial n}\right)(P)-(D_eu)(P)+2u_{inc}(P),\qquad P\in B_i$$

and

$$\tilde{v}(P)=\left(S_i\frac{\partial u}{\partial n}\right)(P)-\rho(D_iu)(P),\qquad P\in B_e.$$

Letting $P\to p\in S$, and using (4.10), we see that

$$\tilde{v}=\tilde{u}\quad\text{and}\quad\frac{\partial\tilde{v}}{\partial n}=\rho\frac{\partial\tilde{u}}{\partial n},$$

whence condition (i) implies that $\tilde{v}\equiv 0$ in B_e and $\tilde{u}\equiv 0$ in B_i. In particular, we have

$$\tilde{v}+\rho\tilde{u}=0\quad\text{and}\quad\frac{\partial\tilde{v}}{\partial n}+\frac{\partial\tilde{u}}{\partial n}=0$$

on S; it follows that (4.9) are satisfied.

We now prove that (4.10) is uniquely solvable.

THEOREM 4.3. *Assume that* (i) $U(k_e;\, k_i;\, \rho)$ *holds and* (ii) $U(k_i;\, k_e;\, \rho)$ *holds. Then the system* (4.10) *is uniquely solvable.*

Proof. Since $1+\rho\neq 0$, (4.10) is of the form $(I_2+A)\mathbf{u}=\mathbf{f}$, where I_2 is the 2×2 identity matrix, A is a 2×2 matrix whose components are weakly-singular integral operators and $\mathbf{u}$ and $\mathbf{f}$ are two-dimensional vectors. Such a system is governed by the standard Fredholm theory, and so it is sufficient to show that the homogeneous system ($\mathbf{f}=\mathbf{0}$) has only the trivial solution. Thus, suppose that u_0 and v_0 solve (4.10) with $u_{inc}\equiv 0$:

$$\begin{aligned}(1+\rho)u_0+(\bar{K}_e^*-\rho\bar{K}_i^*)u_0-(S_e-S_i)v_0&=0,\\ (1+\rho)v_0+\rho(N_e-N_i)u_0-(\rho K_e-K_i)v_0&=0.\end{aligned}\tag{4.11}$$

If we set $u_0 = \nu$ and $v_0 = -\mu$, we see that μ and ν satisfy the homogeneous form of (4.2), but with k_e and k_i interchanged, and $c = 1$. The result follows from an application of Theorem 4.1.

5. Single integral equations, 1. In this section, and the next, the basic idea is to use a layer *ansatz* in one region (B_e, say) and Green's theorem in the other; different combinations lead to different integral equations with different properties. In this section, we shall use a layer *ansatz* in B_e and Green's theorem in B_i, whereas in § 6, we shall do the opposite.

Assume that $u_e(P)$ can be represented as a linear combination of single-layer and double-layer potentials, with the same density $\mu(q)$:

$$u_e(P) = a(S_e\mu)(P) + b(D_e\mu)(P), \qquad P \in B_e. \tag{5.1}$$

Here, the constants a and b are at our disposal. From Green's theorem in B_i, and (2.3), we have the representation (4.3b), namely

$$-2u_i(P) = \frac{1}{\rho}\left(S_i \frac{\partial u}{\partial n}\right)(P) - (D_i u)(P), \qquad P \in B_i. \tag{5.2}$$

Letting $P \to p \in S$, (5.1) gives

$$u_e(p) = \{aS_e + b(-I + \bar{K}_e^*)\}\mu \equiv L_e\mu \tag{5.3a}$$

and

$$\frac{\partial u_e}{\partial n_p} = \{a(I + K_e) + bN_e\}\mu \equiv M_e\mu. \tag{5.3b}$$

(The operators L_e and M_e occur frequently below.) Similarly, (5.2) gives (4.5) and (4.7), i.e.,

$$\rho(I - \bar{K}_i^*)u + S_i \frac{\partial u}{\partial n} = 0 \tag{5.4}$$

and

$$(I + K_i)\frac{\partial u}{\partial n} - \rho N_i u = 0, \tag{5.5}$$

where we have again used (2.3). If we substitute (5.3) into (5.4), using (2.4), we obtain

$$\{\rho(I - \bar{K}_i^*)L_e + S_i M_e\}\mu = f \tag{5.6}$$

where

$$f(p) = -\rho(I - \bar{K}_i^*)u_{inc} - S_i \frac{\partial u_{inc}}{\partial n}. \tag{5.7}$$

Similarly, (5.5) gives

$$\{-\rho N_i L_e + (I + K_i)M_e\}\mu = g \tag{5.8}$$

where

$$g(p) = \rho N_i u_{inc} - (I + K_i)\frac{\partial u_{inc}}{\partial n}. \tag{5.9}$$

Equation (5.6) is a boundary integral equation which is to be solved for $\mu(q)$. Equation (5.8) is another boundary integral equation for $\mu(q)$. Having solved either, $u_e(P)$ and $u_i(P)$ are to be constructed from (5.1) and

$$(5.10)\qquad -2u_i(P)=\frac{1}{\rho}\left(S_i\left\{\frac{\partial u_{inc}}{\partial n}+M_e\mu\right\}\right)(P)-(D_i\{u_{inc}+L_e\mu\})(P),\qquad P\in B_i,$$

respectively.

We note that if $a=1$ and $b=0$, (5.6) reduces to the equation obtained by Maystre and Vincent [15] and by Marx [13], [14] (the MVM equation); the derivation above, in this special case and with $\rho=1$, was given by Knockaert and De Zutter [11]. Later, we shall give explicit results for the MVM equation (see Theorem 5.6 below) and other special cases.

Since

$$(5.11)\qquad S_iN_e=S_i(N_e-N_i)+S_iN_i=S_i(N_e-N_i)+(\bar{K}_i^*)^2-I,$$

where we have used (3.12), we see that (5.6) is a Fredholm integral equation of the second kind, with a weakly-singular kernel, provided $b\neq 0$; if $b=0$, we obtain the MVM equation. Similarly, since

$$(5.12)\qquad N_iS_e=(N_i-N_e)S_e+N_eS_e=(N_i-N_e)S_e+K_e^2-I,$$

where we have used (3.11), we see that (5.8) is only a Fredholm integral equation of the second kind when $b=0$; if $b\neq 0$, we obtain a hypersingular equation. Our first two theorems do not depend on these classifications; they are concerned with solvability of the transmission problem and with uniqueness.

THEOREM 5.1. *Assume that* $\operatorname{Im}(k_e)\geqq 0$ *and* $\operatorname{Im}(k_i)\geqq 0$. *If* $\mu(q)$ *solves* (5.6) *or* (5.8), $u_e(P)$ *and* $u_i(P)$, *given by* (5.1) *and* (5.10), *respectively, solve the transmission problem.*

Proof. Clearly, u_e satisfies (2.1) and (2.5), and u_i satisfies (2.2). It remains to verify that (2.3) are satisfied. On S, we have

$$(5.13)\qquad \begin{aligned}2\rho(u_e+u_{inc}-u_i)&=2\rho(u_{inc}+L_e\mu)+S_i\left(\frac{\partial u_{inc}}{\partial n}+M_e\mu\right)-\rho(I+\bar{K}_i^*)(u_{inc}+L_e\mu)\\&=S_iM_e\mu+\rho(I-\bar{K}_i^*)L_e\mu-f\end{aligned}$$

and

$$(5.14)\qquad 2\left(\frac{\partial u_e}{\partial n}+\frac{\partial u_{inc}}{\partial n}-\rho\frac{\partial u_i}{\partial n}\right)=(I+K_i)M_e\mu-\rho N_iL_e\mu-g.$$

If $\mu(q)$ solves (5.6), then (5.13) shows that (2.3a) is satisfied, whereas if $\mu(q)$ solves (5.8), then (5.14) shows that (2.3b) is satisfied.

Define a function w by

$$w(P)=\left(S_i\left\{\frac{\partial u_{inc}}{\partial n}+M_e\mu\right\}\right)(P)-\rho(D_i\{u_{inc}+L_e\mu\})(P),\qquad P\in B_e.$$

$w(P)$ satisfies (2.2) in B_e and the radiation condition (2.5), with k_e replaced by k_i. Moreover, on S, we have $w(p)=0$ if $\mu(q)$ solves (5.6) or $\partial w/\partial n_p=0$ if $\mu(q)$ solves (5.8). In either case, we can deduce that $w\equiv 0$ in B_e. Then, in the first case, $\partial w/\partial n_p=0$ implies that (2.3b) is satisfied, whereas in the second case, $w(p)=0$ implies that (2.3a) is satisfied.

The next theorem is concerned with uniqueness for (5.6) and (5.8), i.e., with nontrivial solutions of the homogeneous forms of (5.6) and (5.8); these are

$$\rho(I-\bar{K}_i^*)L_e\mu_0+S_iM_e\mu_0=0 \tag{5.15}$$

and

$$\rho N_iL_e\mu_0-(I+K_i)M_e\mu_0=0. \tag{5.16}$$

We show that uniqueness depends on the eigenvalues of the following.

Associated interior problem. Find a function $v(P)$ which satisfies

$$(\nabla^2+k_e^2)v(P)=0, \qquad P\in B_i$$

and

$$av(p)+b\frac{\partial v}{\partial n_p}=0, \qquad p\in S.$$

If this problem has a nontrivial solution, we say that k_e^2 is an eigenvalue of the associated interior problem. If $b=0$, the associated interior problem reduces to the interior Dirichlet problem; in this case, it is known that all eigenvalues are real. If $b\neq 0$, the associated interior problem reduces to the interior Robin problem with impedance $\lambda=a/b$; in this case, a simple application of Green's theorem gives

$$\operatorname{Im}(\lambda)\int_S |v|^2\,ds=2\operatorname{Re}(k_e)\operatorname{Im}(k_e)\int_{B_i}|v|^2\,dV, \tag{5.17}$$

which shows, for example, that if k_e is real and $\operatorname{Im}(\lambda)\neq 0$, then $v\equiv 0$.

THEOREM 5.2. *Assume that $U'(k_e;k_i;\rho)$ holds. Then the homogeneous equations* (5.15) *and* (5.16) *have a nontrivial solution if and only if k_e^2 is an eigenvalue of the associated interior problem.*

Proof. Suppose that $\mu_0\not\equiv 0$ solves (5.15) or (5.16). Construct v_e and v_i by

$$v_e(P)=a(S_e\mu_0)(P)+b(D_e\mu_0)(P), \qquad P\in B_e$$

and

$$-2v_i(P)=\rho^{-1}(S_iM_e\mu_0)(P)-(D_iL_e\mu_0)(P), \qquad P\in B_i.$$

By Theorem 5.1, these functions solve the homogeneous transmission problem, whence $v_e\equiv 0$ in B_e and $v_i\equiv 0$ in B_i. Now construct

$$v(P)=a(S_e\mu_0)(P)+b(D_e\mu_0)(P), \qquad P\in B_i.$$

On S, we have

$$v_e-v=-2b\mu \tag{5.18}$$

and

$$\frac{\partial v_e}{\partial n}-\frac{\partial v}{\partial n}=2a\mu. \tag{5.19}$$

But $v_e(p)=\partial v_e/\partial n_p=0$, whence

$$av(p)+b\frac{\partial v}{\partial n_p}=0, \qquad p\in S. \tag{5.20}$$

Then, either v is an eigenfunction of the associated interior problem or $v(P)\equiv 0$. But this latter possibility can be eliminated, since it implies that $v(p)=\partial v/\partial n_p=0$, whence $\mu=0$ by (5.18) or (5.19), contrary to hypothesis.

Suppose now that k_e^2 is an eigenvalue of the associated interior problem. Let $v_0(P) \not\equiv 0$, $P \in B_i$, be a corresponding eigenfunction. Then, Green's theorem gives

$$(I - \bar{K}_e^*) v_0 + S_e \frac{\partial v_0}{\partial n} = 0$$

and

$$(I + K_e) \frac{\partial v_0}{\partial n} - N_e v_0 = 0.$$

Using the boundary condition (5.20), these give $L_e v_0 = 0$ and $M_e v_0 = 0$, respectively. Hence, $v_0(p)$ is a nontrivial solution of both (5.15) and (5.16), if $b \neq 0$; if $b = 0$, $\partial v_0 / \partial n_p$ is a nontrivial solution.

If the homogeneous integral equations have a nontrivial solution, we say that k_e^2 is an *irregular value.*

THEOREM 5.3. *Assume that* (i) $b \neq 0$ (ii) $U'(k_e; k_i; \rho)$ *holds and* (iii) k_e^2 *is not an irregular value. Then, the integral equation* (5.6) *is uniquely solvable for any f.*

Proof. By condition (i), (5.6) is a Fredholm integral equation of the second kind, and so it is sufficient to show that the homogeneous equation, namely (5.15), has only the trivial solution; conditions (ii) and (iii) guarantee this by Theorem 5.2.

Next, we consider those equations that are *not* Fredholm integral equations of the second kind.

THEOREM 5.4. *Assume that* (i) $b \neq 0$, (ii) $U'(k_e; k_i; \rho)$ *holds and* (iii) k_e^2 *is not an irregular value. Then, the integral equation* (5.8) *is uniquely solvable.*

Proof. Given $u_{inc}(P)$, Theorem 5.3 asserts that we can determine μ uniquely by solving (5.6). We then construct $u_e(P)$ and $u_i(P)$, using (5.1) and (5.10). By Theorem 5.1, these functions solve the transmission problem. In particular, the transmission condition (2.3b) implies that μ solves (5.8). Thus, we have proved the existence of a solution to (5.8), albeit for the particular g given by (5.9). Uniqueness follows from Theorem 5.2.

We now consider some special cases of the integral equations (5.6) and (5.8). Suppose first that we set $a = 0$ and $b = 1$ in (5.1), (5.6), (5.8) and (5.10); this gives

(5.21a) $$u_e(P) = (D_e \mu)(P), \qquad P \in B_e$$

and

(5.21b)
$$-2u_i(P) = \frac{1}{\rho}\left(S_i\left\{\frac{\partial u_{inc}}{\partial n} + N_e \mu\right\}\right)(P) - (D_i\{u_{inc} - (I - \bar{K}_e^*)\mu\})(P), \qquad P \in B_i,$$

where $\mu(q)$ solves

(5.22) $$\{\rho(I - \bar{K}_i^*)(I - \bar{K}_e^*) - S_i N_e\}\mu = -f$$

or

(5.23) $$\{\rho N_i(I - \bar{K}_e^*) + (I + K_i) N_e\}\mu = g.$$

Since $1 + \rho \neq 0$, (5.22) is a Fredholm integral equation of the second kind and (5.23) is a hypersingular equation. Specializing Theorems 5.1, 5.3 and 5.4, we obtain the following.

THEOREM 5.5. *Assume that* (i) $U'(k_e; k_i; \rho)$ *holds and* (ii) k_e^2 *is not an eigenvalue of the interior Neumann problem. Then, given* $u_{inc}(P)$, *both of the integral equations*

(5.22) *and* (5.23) *are uniquely solvable. Moreover, the representations* (5.21) *solve the transmission problem.*

Suppose now that we set $a=1$ and $b=0$; this gives

$$u_e(P)=(S_e\mu)(P) \qquad P\in B_e \tag{5.24a}$$

and

$$-2u_i(P)=\frac{1}{\rho}\left(S_i\left\{\frac{\partial u_{inc}}{\partial n}+(I+K_e)\mu\right\}\right)(P)-(D_i\{u_{inc}+S_e\mu\}), \qquad P\in B_i, \tag{5.24b}$$

where $\mu(q)$ solves

$$\{S_i(I+K_e)+\rho(I-\bar{K}_i^*)S_e\}\mu=f \tag{5.25}$$

or

$$\{(I+K_i)(I+K_e)-\rho N_iS_e\}\mu=g. \tag{5.26}$$

Equation (5.25) is a Fredholm integral equation of the first kind with a weakly-singular kernel; this is the MVM equation. Since $1+\rho\neq 0$, (5.26) is a Fredholm integral equation of the second kind. Both can be analyzed by trivial alterations to Theorems 5.3 and 5.4.

THEOREM 5.6. *Assume that* (i) $U'(k_e;\,k_i;\,\rho)$ *holds, and* (ii) k_e^2 *is not an eigenvalue of the interior Dirichlet problem. Then, given* $u_{inc}(P)$, *both the* MVM *equation* (5.25) *and* (5.26) *are uniquely solvable. Moreover, the representations* (5.24) *solve the transmission problem.*

It is convenient to describe one of DeSanto's two single integral equations [5] here. He represents $u_e(P)$ as a single-layer potential, (5.24a), and then substitutes this into (4.4)+(4.5), i.e., into the first of (4.10); the MVM equation is obtained by substituting into (4.5) alone.

6. Single integral equations, 2. In this section, we shall use a layer *ansatz* in B_i and Green's theorem in B_e. The latter gives the representation (4.3a), namely

$$2u_e(P)=\left(S_e\frac{\partial u}{\partial n}\right)(P)-(D_eu)(P), \qquad P\in B_e. \tag{6.1}$$

Letting $P\to p\in S$, (6.1) and (2.3) give (cf. (4.4) and (4.6))

$$(I+\bar{K}_e^*)u_i-\rho S_e\frac{\partial u_i}{\partial n}=2u_{inc} \tag{6.2}$$

and

$$\rho(I-K_e)\frac{\partial u_i}{\partial n}+N_eu_i=2\frac{\partial u_{inc}}{\partial n}. \tag{6.3}$$

Multiply (6.2) by a and (6.3) by b, where a and b are disposable constants; add to obtain

$$-\rho\bar{L}_e^*\frac{\partial u_i}{\partial n}+\bar{M}_e^*u_i=h \tag{6.4}$$

where the operators L_e and M_e are defined by (5.3), and

$$h(p)=2au_{inc}(p)+2b\frac{\partial u_{inc}}{\partial n_p}, \qquad p\in S. \tag{6.5}$$

If we suppose that $u_i(P)$ can be represented as a single-layer potential,

$$u_i(P)=(S_i\nu)(P), \qquad P\in B_i, \tag{6.6}$$

we obtain

$$u_i(p)=S_i\nu \quad \text{and} \quad \frac{\partial u_i}{\partial n_p}=(-I+K_i)\nu, \quad p\in S. \tag{6.7}$$

Substituting these into (6.4), we obtain

$$\{\rho\bar{L}_e^*(I-K_i)+\bar{M}_e^*S_i\}\nu=h. \tag{6.8}$$

This is a boundary integral equation which is to be solved for $\nu(q)$. Having found ν, $u_e(P)$ and $u_i(P)$ are to be constructed from

$$2u_e(P)=-\rho(S_e(I-K_i)\nu)(P)-(D_eS_i\nu)(P), \qquad P\in B_e \tag{6.9}$$

and (6.6), respectively.

We note that if $a=1$, $b=0$ and $\rho=1$, (6.8) reduces to the equation obtained by Knockaert and De Zutter [11] (the KDeZ equation); see Theorem 6.5 below.

As an alternative, we suppose that $u_i(P)$ can be represented as a double-layer potential

$$u_i(P)=(D_i\omega)(P), \qquad P\in B_i, \tag{6.10}$$

whence

$$u_i(p)=(I+\bar{K}_i^*)\omega \quad \text{and} \quad \frac{\partial u_i}{\partial n_p}=N_i\omega, \quad p\in S. \tag{6.11}$$

Substituting these into (6.4), we obtain

$$\{-\rho\bar{L}_e^*N_i+\bar{M}_e^*(I+\bar{K}_i^*)\}\omega=h. \tag{6.12}$$

This is a boundary integral equation which is to be solved for $\omega(q)$. Having found ω, $u_e(P)$ and $u_i(P)$ are to be constructed from

$$2u_e(P)=\rho(S_eN_i\omega)(P)-(D_e(I+\bar{K}_i^*)\omega)(P), \qquad P\in B_e \tag{6.13}$$

and (6.10), respectively.

The next theorem is concerned with the solvability of the transmission problem.

THEOREM 6.1. *Assume that* (i) $\operatorname{Im}(k_e)\geqq 0$ *and* (ii) k_e^2 *is not an eigenvalue of the associated interior problem. Then*: *if* $\nu(q)$ *solves* (6.8), $u_e(P)$ *and* $u_i(P)$, *given by* (6.9) *and* (6.6), *respectively, solve the transmission problem*; *and, if* $\omega(q)$ *solves* (6.12), $u_e(P)$ *and* $u_i(P)$, *given by* (6.13) *and* (6.10), *respectively, solve the transmission problem.*

Proof. In either case, the assumed representations satisfy (2.1), (2.2) and (2.5). Suppose that $\omega(q)$ solves (6.12), which we rewrite here as

$$\begin{aligned} &a\{(I+\bar{K}_e^*)(I+\bar{K}_i^*)\omega-\rho S_eN_i\omega-2u_{inc}\}\\ &\qquad +b\left\{N_e(I+\bar{K}_i^*)\omega+\rho(I-K_e)N_i\omega-2\frac{\partial u_{inc}}{\partial n}\right\}=0. \end{aligned} \tag{6.14}$$

Then, using (6.10) and (6.13), and letting $P\to p\in S$, we obtain

$$2(u_e+u_{inc}-u_i)=\rho S_eN_i\omega-(I+\bar{K}_e^*)(I+\bar{K}_i^*)\omega+2u_{inc} \tag{6.15a}$$

and

$$2\left(\frac{\partial u_e}{\partial n}+\frac{\partial u_{inc}}{\partial n}-\rho\frac{\partial u_i}{\partial n}\right)=\rho(-I+K_e)N_i\omega-N_e(I+\bar{K}_i^*)\omega+2\frac{\partial u_{inc}}{\partial n}. \tag{6.15b}$$

We have to show that the right-hand sides of (6.15) vanish.

Define a function in B_i by

$$v(P) = (D_e(I+\bar{K}_i^*)\omega)(P) - \rho(S_e N_i \omega)(P) - 2u_{inc}(P), \qquad P \in B_i.$$

Letting $P \to p \in S$, and comparing with (6.14), we find that

$$av(p) + b\frac{\partial v}{\partial n_p} = 0, \qquad p \in S.$$

Since $v(P)$ satisfies (2.1) in B_i, condition (ii) implies that $v \equiv 0$ in B_i. In particular, v and $\partial v/\partial n$ both vanish on S, and hence (6.15) show that (2.3) are satisfied. A similar argument can be given when $\nu(q)$ solves (6.8).

Using (3.10), we see that (6.8) and (6.12) are the Hermitian adjoints of (5.6) and (5.8), respectively. In particular, (6.8) is a Fredholm integral equation of the second kind, provided $b \neq 0$, and (6.12) is only a Fredholm integral equation of the second kind when $b = 0$. For these cases, we can use the Fredholm Alternative to obtain corresponding versions of Theorems 5.2 and 5.3. Thus, we have the following theorem.

THEOREM 6.2. *Assume that* (i) $b \neq 0$ *and* (ii) $U'(k_e; k_i; \rho)$ *holds. Then, the homogeneous form of* (6.8) $(h=0)$ *has a nontrivial solution if and only if* k_e^2 *is an eigenvalue of the associated interior problem. Moreover, if* k_e^2 *is not one of these eigenvalues, then the inhomogeneous equation* (6.8) *is uniquely solvable for any* h.

We can also prove a similar result for (6.12) when $b = 0$; see Theorem 6.5 below. Let us now consider (6.12) when $b \neq 0$; in this case, (6.12) is a hypersingular equation. Note that the arguments used in the proof of Theorem 5.4 are not immediately applicable here.

THEOREM 6.3. *Assume that* (i) $b \neq 0$, (ii) $U'(k_e; k_i; \rho)$ *holds, and* (iii) k_e^2 *is not an eigenvalue of the associated interior problem. Then, given* $u_{inc}(P)$, *the integral equation* (6.12) *is uniquely solvable.*

Proof. First, we prove existence. By Theorems 6.1 and 6.2, we know that the solution of the transmission problem can be represented by (6.6) and (6.9); in particular, on S, we have (6.7). If $u_i(P)$ can be represented by (6.10), its Cauchy data are given by (6.11). Comparing (6.7) and (6.11), we obtain

$$\text{(6.16)} \qquad \begin{aligned} (I+\bar{K}_i^*)\omega &= u_i(p) = S_i \nu, \\ N_i \omega &= \frac{\partial u_i}{\partial n_p} = (K_i - I)\nu. \end{aligned}$$

Here, $\nu(q)$ is the unique solution of (6.8). Thus, the right-hand sides of (6.16) are known. But it is known [9, Thms. 5.1 and 5.3] that the system (6.16) always has precisely one solution $\omega(q)$, for any k_i and for any $\nu(q)$. Having determined $\omega(q)$ and $u_i(P)$, (2.3) gives $u_e(p)$ and $\partial u_e/\partial n_p$, and then Green's theorem, (3.13), gives $u_e(P)$. Then, for any given $u_{inc}(P)$, it follows that $\omega(q)$ satisfies (6.12).

We now prove uniqueness. Suppose that $\omega_0(q)$ solves the homogeneous form of (6.12), namely

$$\text{(6.17)} \qquad -\rho \bar{L}_e^* N_i \omega_0 + \bar{M}_e^*(I+\bar{K}_i^*)\omega_0 = 0.$$

Construct $v_e(P)$ and $v_i(P)$ by

$$2v_e(P) = \rho(S_e N_i \omega_0)(P) - (D_e(I+\bar{K}_i^*)\omega_0)(P), \qquad P \in B_e$$

and

$$v_i(P) = (D_i \omega_0)(P), \qquad P \in B_i.$$

By Theorem 6.1, these functions satisfy the homogeneous transmission problem ($u_{inc}=0$), and hence $v_e \equiv 0$ in B_e and $v_i \equiv 0$ in B_i. Now define

$$v(P)=(D_i\omega_0)(P), \qquad P \in B_e.$$

Then, $\partial v/\partial n_p = N_i\omega_0 = \partial v_i/\partial n_p = 0$, whence $v \equiv 0$ in B_e. The jump conditions (3.4) then imply that $\omega_0 = 0$, and this completes the proof.

We conclude this section with some special cases of the integral equations (6.8) and (6.12). First, we set $a=0$ and $b=1$ to give

$$\{-\rho(I-K_e)(I-K_i)+N_eS_i\}\nu = 2\frac{\partial u_{inc}}{\partial n} \tag{6.18}$$

and

$$\{\rho(I-K_e)N_i + N_e(I+\bar{K}_i^*)\}\omega = 2\frac{\partial u_{inc}}{\partial n}. \tag{6.19}$$

Since $1+\rho \neq 0$, (6.18) is a Fredholm integral equation of the second kind and (6.19) is a hypersingular equation. Specializing Theorems 6.1, 6.2, and 6.3, we obtain the following.

THEOREM 6.4. *Assume that* (i) $U'(k_e;\, k_i;\, \rho)$ *holds, and* (ii) k_e^2 *is not an eigenvalue of the interior Neumann problem. Then, given* $u_{inc}(P)$, *both of the equations* (6.18) *and* (6.19) *are uniquely solvable. Moreover, the solution of the transmission problem is given by* (6.6) *and* (6.9) *if* ν *solves* (6.18), *and by* (6.10) *and* (6.13) *if* ω *solves* (6.19).

Suppose now that we set $a=1$ and $b=0$; this gives

$$\{(I+\bar{K}_e^*)S_i + \rho S_e(I-K_i)\}\nu = 2u_{inc} \tag{6.20}$$

and

$$\{(I+\bar{K}_e^*)(I+\bar{K}_i^*) - \rho S_eN_i\}\omega = 2u_{inc}. \tag{6.21}$$

Equation (6.20) is a Fredholm equation of the first kind with a weakly-singular kernel; this is the KDeZ equation. Since $1+\rho \neq 0$, (6.21) is a Fredholm integral equation of the second kind; it can be analyzed immediately, as it is the Hermitian adjoint of (5.26). The proof of Theorem 6.3 can be modified in order to treat (6.20); to prove existence, we use [9, Thms. 5.4, 5.6]. Thus, we have the following.

THEOREM 6.5. *Assume that* (i) $U'(k_e;\, k_i;\, \rho)$ *holds, and* (ii) k_e^2 *is not an eigenvalue of the interior Dirichlet problem. Then, given* $u_{inc}(P)$, *both the* KDeZ *equation* (6.20) *and* (6.21) *are uniquely solvable. Moreover, the solution of the transmission problem is given by* (6.6) *and* (6.9) *if* ν *solves* (6.20), *and by* (6.10) *and* (6.13) *if* ω *solves* (6.21).

Finally, let us describe DeSanto's second integral equation [5]. He represents $u_i(P)$ as a single-layer potential (6.6), and then substitutes this into (4.4)+(4.5); the KDeZ equation is obtained by substituting into (4.4) alone.

7. Single integral equations without irregular frequencies. In order to obtain integral equations that are uniquely solvable for all values of k_e^2, we merely choose the constants a and b so that k_e^2 is not an eigenvalue of the associated interior problem. By (5.17), this can be achieved by setting

$$a=-i\eta \quad \text{and} \quad b=1,$$

where η is any real number satisfying

$$\eta \neq 0 \quad \text{and} \quad \eta\,\mathrm{Re}\,(k_e) \geqq 0. \tag{7.1}$$

Then, provided $1+\rho \neq 0$ and $U'(k_e; k_i; \rho)$ holds, the integral equations (5.6), (5.8), (6.8) and (6.12) are uniquely solvable for all values of k_e^2 and any given $u_{inc}(P)$. For example, if we restrict ourselves to Fredholm integral equations of the second kind, then we have two, namely (5.6) and (6.8). Thus, (5.1) becomes

$$u_e(P) = (D_e\mu)(P) - i\eta(S_e\mu)(P), \qquad P \in B_e \tag{7.2}$$

where $\mu(q)$ solves (5.6), which we rewrite as

$$-(1+\rho)\mu + \bar{\mathbf{K}}^*\mu - i\eta\bar{\mathbf{S}}^*\mu = f \tag{7.3}$$

where

$$\mathbf{K} = K_i(\rho I + K_i) + \rho K_e(I - K_i) + (N_e - N_i)S_i$$

and

$$\mathbf{S} = \rho S_e(I - K_i) + (I + \bar{K}_e^*)S_i$$

are compact operators; f and $u_i(P)$ are given by (5.7) and (5.10), respectively. Similarly, (6.8) can be rewritten as

$$-(1+\rho)\nu + \mathbf{K}\nu - i\eta\mathbf{S}\nu = 2\frac{\partial u_{inc}}{\partial n} - 2i\eta u_{inc} \tag{7.4}$$

where

$$u_i(P) = (S_i\nu)(P), \qquad P \in B_i,$$

and $u_e(P)$ is given by (6.9). Equation (7.4) is the Hermitian adjoint of (7.3).

The representation of $u_e(P)$ as a combined single-layer and double-layer potential, (7.2), has been used previously for the exterior problems of acoustics. In particular, for scattering by a sound-soft obstacle ($u=0$ on S), we obtain

$$-\mu + \bar{K}_e^*\mu - i\eta S_e\mu = -u_{inc}, \tag{7.5}$$

an integral equation with the same structure as (7.3). Equation (7.5) is uniquely solvable for all k_e, if η satisfies (7.1); for a proof, and relevant references, see [2, § 3.6, p. 91].

Similarly, the idea of using a linear combination of (3.16) and (3.18) has been used previously by Burton and Miller [1]; for a sound-soft scatterer, this yields

$$(-I + K_e - i\eta S_e)\frac{\partial u}{\partial n} = -2\frac{\partial u_{inc}}{\partial n} + 2i\eta u_{inc}, \tag{7.6}$$

which is the Hermitian adjoint of (7.5), and so is uniquely solvable subject to the same conditions on η [2, § 3.9, p. 103].

8. Discussion. In this paper, we have studied integral-equation methods for the transmission problem of acoustics. In § 4, we described indirect and direct methods, leading to pairs of coupled integral equations. The simplest of these suffer from irregular frequencies [16]. However, there are three known pairs that do not. These all involve the operators N_e and N_i, corresponding to the normal derivative of the double-layer potentials, although these operators occur only in the compact combination $N_e - N_i$ in the second-kind pairs (4.2) and (4.10).

In §§ 5, 6 and 7, we have given a systematic derivation of single integral equations, using the indirect method in one region and the direct method in the other. In § 5, we used a combined single-layer and double-layer potential in B_e, and Green's theorem (Helmholtz formula) in B_i. We obtained two different integral equations, by using the

Helmholtz formula, or its normal derivative, on S. We obtained the two Hermitian adjoints of these equations in § 6, using a single-layer or a double-layer potential in B_i, together with Green's theorem in B_e; specifically, we used a linear combination of the Helmholtz formula and its normal derivative, both evaluated on S. We also investigated two special cases of each equation; these eight equations, together with the associated representations for u_e and u_i are summarized in Table 1. For each pair of representations, the unknown density can be determined from either of two equations (except when k_e^2 is an irregular value; see Theorems 5.5, 5.6, 6.4 and 6.5). Computationally, it may be preferable to choose a Fredholm integral equation of the second kind, although none of these have yet been used. Indeed, only the MVM equation, (5.25), has been used previously [7], [17, Chap. 3]. Note that if it is only the exterior field that is of interest, then one should choose a formulation that has a simple representation for $u_e(P)$. Moreover, if one also desires a Fredholm integral equation of the second kind and does not want to compute quantities such as $N_i u_{inc}$, then (5.22) is the appropriate choice, together with the double-layer representation (5.21a) for u_e. If both fields are of interest, then similar considerations lead to the choice of (6.18), together with the representations (6.6) and (6.9).

In § 7, we derived two single Fredholm integral equations of the second kind, which are uniquely solvable for all values of k_e^2. (Other equations could also be derived.) However, they are both more complicated than all those in Table 1, and so there is scope for developing alternative methods. One possibility is to replace G_e by a modified Green's function. This technique has been used extensively for other scattering problems (see, e.g., [2, § 3.6, pp. 93–97]) as well as for the indirect formulation of the transmission problem described in § 4.1 of [10]. Our work using a modified G_e to obtain single integral equations will be described elsewhere.

The methods used in this paper to treat the scalar transmission problem should extend to vector problems. Thus, it would seem to be computationally worthwhile to develop single integral equations for the scattering of electromagnetic waves by a

TABLE 1
Solutions of the transmission problem in terms of associated boundary integral equations.

Boundary integral equations	Representations of the solutions	
(5.26) $(1+\rho)\mu+\mathbf{L}\mu=g$	(5.24a) $u_e=S_e\mu$	in B_e
(5.25) $\bar{\mathbf{S}}^*\mu=f$ (MVM)	(5.24b) $u_i=-(\frac{1}{2}/\rho)S_i(v_{inc}+\mu+K_e\mu)+\frac{1}{2}D_i(u_{inc}+S_e\mu)$	in B_i
(5.22) $(1+\rho)\mu-\bar{\mathbf{K}}^*\mu=-f$	(5.21a) $u_e=D_e\mu$	in B_e
(5.23) $\bar{\mathbf{N}}^*\mu=g$	(5.21b) $u_i=-(\frac{1}{2}/\rho)S_i(v_{inc}+N_e\mu)+\frac{1}{2}D_i(u_{inc}-\mu+\bar{K}_e^*\mu)$	in B_i
(6.18) $(1+\rho)\nu-\mathbf{K}\nu=-2v_{inc}$	(6.9) $u_e=-\frac{1}{2}\rho S_e(I-K_i)\nu-\frac{1}{2}D_eS_i\nu$	in B_e
(6.20) $\mathbf{S}\nu=2u_{inc}$ (KDeZ)	(6.6) $u_i=S_i\nu$	in B_i
(6.21) $(1+\rho)\omega+\bar{\mathbf{L}}^*\omega=2u_{inc}$	(6.13) $u_e=\frac{1}{2}\rho S_eN_i\omega-\frac{1}{2}D_e(I+\bar{K}_i^*)\omega$	in B_e
(6.19) $\mathbf{N}\omega=2v_{inc}$	(6.10) $u_i=D_i\omega$	in B_i

The functions f, g and v_{inc}, the compact operators $\mathbf{K}, \mathbf{L}$ and $\mathbf{S}$, and the unbounded operator $\mathbf{N}$, are defined as follows: $v_{inc}=\partial u_{inc}/\partial n$,

$$f=-\rho(I-\bar{K}_i^*)u_{inc}-S_iv_{inc}, \qquad g=\rho N_iu_{inc}-(I+K_i)v_{inc},$$
$$\mathbf{K}=K_i(\rho I+K_i)+\rho K_e(I-K_i)+(N_e-N_i)S_i,$$
$$\mathbf{L}=K_i(\rho I+K_e)+K_e(I-\rho K_e)+\rho(N_e-N_i)S_e,$$
$$\mathbf{N}=\rho(I-K_e)N_i+N_e(I+\bar{K}_i^*), \qquad \mathbf{S}=\rho S_e(I-K_i)+(I+\bar{K}_e^*)S_i.$$

homogeneous dielectric obstacle (see [6], [13] and [14]) and for the scattering of elastic waves by a homogeneous inclusion. These problems are currently under investigation.

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