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ON HYPERSINGULAR BOUNDARY INTEGRAL EQUATIONS FOR CERTAIN PROBLEMS IN MECHANICS

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Introduction

Improper or weakly singular integrals, once thought to be a handicap in computations, are now accepted as a source of effectiveness and stability in the numerical solution of many problems in mechanics. Such integrals naturally arise, for example, in the Boundary Integral Equation (BIE) method, and the literature contains many examples of successful computations based on Boundary Element-type (BEM) solutions of the BIEs (e.g. [1,2]). For some vector problems, the BIE/BEM process gives rise to a stronger type of singular integral which exists in the sense of the Cauchy Principal Value (CPV) (cf. [3]). This integral also has been treated numerically with success. Less commonly, but with growing frequency it seems, the gradient or normal derivative of such boundary integrals is taken, especially in the formulation of mechanics problems involving cracks. Then integrals more (hyper) singular than the CPV can explicitly arise. Usually, however, rather than confront such hypersingular integrals directly, a process of regularization (e.g. [4,5,6,7]) is employed to lower the singularity of the integrands. Such regularization usually carries a formulational complexity and computational cost if it is even possible. However, the alternatives to regularization seem to be divergent integrals or numerical computation with integrals more singular than the CPV with, perhaps, even questionable definition.

The purpose of this brief paper is to examine a reasonable alternative to the mentioned regularization. Indeed, we consider the essential analytical issues surrounding the occurrence of hypersingular boundary integrals in the context of a crack problem. Then we suggest certain computational strategies. The relevant integrals are contrasted with truly divergent integrals and conditions under which they exist in the finite-part sense of Hadamard [8] are stated. The relation of such integrals to both one-sided and two-sided CPV integrals is examined as well. The critical role of smoothness of relevant functions in the integrands for existence of these integrals is emphasized. Our goal is to encourage understanding and use of finite-part integrals in mechanics. We wish to discourage the practice of going to considerable lengths to avoid them as seems presently the case. How to compute them, analytically or numerically, is a rich area of research. Indeed, we recommend the works [9,10,11,12,13,14,15] which are probably not sufficiently familiar to the mechanics community. While we confine attention here to an easy-to-describe (scalar) sound-scattering cracklike problem in two dimensions, extension of the present ideas to (vector) problems for static loading or wave scattering by crack surfaces in three dimensions is relatively clear.

Analysis

Consider the scattering of a plane time-harmonic sound wave in a compressible fluid by a rigid thin plate Γ. We seek the complex potential $\phi(x,y;k)$ which satisfies

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + k^2\phi = 0 \qquad \text{in the fluid,} \tag{1}$$

and
$$\frac{\partial\phi}{\partial n^\pm} = 0 \qquad \text{on } \Gamma^\pm \tag{2}$$

Here, $k \equiv \omega/c$ where c is the wavespeed and ω is the wave frequency (both real), $\partial/\partial n^\pm$ denotes the normal derivative into the fluid at a point on $\Gamma^\pm$ where $\Gamma^\pm$ are the two sides of an open smooth arc Γ, $\phi \equiv \phi^i + \phi^s$ wherein ϕ^i is the given incident wave and ϕ^s is the scattered wave satisfying a radiation condition at infinity, and x,y are Cartesian coordinates.

To solve this problem, consider the fundamental solution G defined by ($H_o^{(1)}$ is the Hankel function)

$$G(P,Q) \equiv G(x,y;\xi,\eta) = -\frac{1}{2}\, iH_o^{(1)}(k|P-Q|) \tag{3}$$

and let

$$[\phi(q)] = \phi(q^+) - \phi(q^-) \tag{4}$$

where $q^\pm$ are corresponding points of $\Gamma^\pm$. Then, an application of Green's Theorem to ϕ^s and G, with application of the boundary conditions (2) (letting P $\to p^\pm$ in $\Gamma^\pm$) gives

$$\frac{\partial}{\partial n_p}\int_\Gamma [\phi(q)]\,\frac{\partial}{\partial n_q}G(p,q)\,ds_q = 2\,\frac{\partial\phi^i}{\partial n_p}(p), \qquad p \in \Gamma \tag{5}$$

where p can be thought of as being on either side ($^\pm$) of Γ since both limits are identical and the superscripts ($^\pm$) can be deleted.

In Equation (5) we seek the jump $[\phi(q)]$ among the class of functions $C_0^{1,\alpha}(\bar{\Gamma})$, $0<\alpha\leq 1$. This means that $[\phi]$ has one Hölder-continuous tangential derivative on Γ and vanishes at the end points $\partial\Gamma$ of Γ ($\bar{\Gamma} = \Gamma\cup\partial\Gamma$). If $[\phi(q)]$ can be found as the solution of (5), it will have the expected square-root behavior, i.e.

$$[\phi(q)] \sim s^{1/2} \text{ as } s \to 0 \tag{6}$$

where s is the arclength on Γ of q from an edge $\partial\Gamma$. Moreover, the field ϕ^s will be given by the representation integral

$$\phi^s(P) = -\frac{1}{2}\int_\Gamma [\phi(q)] \frac{\partial}{\partial n_q} G(P,q)\, ds_q \tag{7}$$

obtained as an intermediate step in deriving (5).

To solve (5) for $[\phi(q)]$, it is tempting to simply take the normal derivative under the integral sign, but this leads to a nonintegrable integrand. Instead it is common to regularize (5) by one of various schemes transferring one of the normal derivatives of G to a tangential derivative of $[\phi(q)]$, e.g. [4]

$$\begin{aligned}\rlap{\,\,-}\int_\Gamma \left\{ \frac{\partial}{\partial s_q}[\phi(q)] \frac{\partial}{\partial s_p} G(p,q) + k^2\, n(q)\cdot n(p)\, G(p,q)\, [\phi(q)] \right\} ds_q \\ = 2\frac{\partial\phi^i}{\partial n_p}(p), \qquad p \in \Gamma,\end{aligned} \tag{8}$$

in which the dash through the integral sign signifies a CPV. Equation (8) is unattractive because of the appearance of $\partial[\phi]/\partial s_q$ rather than $[\phi]$ itself, the latter being the desired quantity, and the former being unbounded at $\partial\Gamma$. In addition to these unattractive features, the regularized counterpart of (8) for vector problems in three dimensions (cf. [16]) is considerably more complicated than is the vector counterpart of (5) when compared with (5) itself. Also, numerical procedures, whenever $\partial[\phi]/\partial s_q$ appears explicitly as a function of q, can require extraordinary care (cf. [17]). An alternative approach involves the following concept.

Finite-Part Integral

A two-sided finite-part integral of order 2, for $f\in C^{1,\alpha}$ for x,t points on the line a,b, is defined as (cf. [18,19]):

$$\rlap{\,\,=}\int_a^b \frac{f(t)\,dt}{(x-t)^2} \equiv \lim_{\varepsilon\to 0}\left\{ \int_a^{x-\varepsilon} \frac{f(t)\,dt}{(x-t)^2} + \int_{x+\varepsilon}^b \frac{f(t)\,dt}{(x-t)^2} - \frac{2f(x)}{\varepsilon} \right\} \tag{9}$$

A one-sided finite-part integral of order μ, where $1<\mu<2$, with the same smoothness requirement on t, is similarly defined as:

$$\neq\!\!\!\!\!\!\int_a^x \frac{f(t)\,dt}{(x-t)^\mu} \equiv \lim_{\varepsilon\to 0} \left\{ \int_a^{x-\varepsilon} \frac{f(t)\,dt}{(x-t)^\mu} - \frac{f(x)}{(\mu-1)\varepsilon^{\mu-1}} \right\} \tag{10}$$

A similar definition exists for the one-sided finite-part integral from x to b by simply interchanging t and x in the denominator of the terms in (10) with an obvious change of limits. Based on these definitions and that of the CPV, it is possible to prove that

$$\frac{d}{dx} -\!\!\!\!\!\!\int_a^b \frac{f(t)\,dt}{(x-t)} = -\neq\!\!\!\!\!\!\int_a^b \frac{f(t)\,dt}{(x-t)^2}\,. \tag{11}$$

Generalizing these definitions to sufficiently smooth curved lines and normal derivatives instead of x derivatives, one can prove [20] the following theorem. For the stated smoothness requirement on $[\phi(q)]$ and twice continuously differentiable Γ, then

$$\frac{\partial}{\partial n_p} \int_\Gamma [\phi(q)]\, \frac{\partial}{\partial n_q} G\,(p,q)\, ds_q = \neq\!\!\!\!\!\!\int_\Gamma [\phi(q)]\, \frac{\partial^2}{\partial n_p\, \partial n_q} G\,(p,q)\, ds_q \tag{12}$$

such that (5) becomes

$$\neq\!\!\!\!\!\!\int_\Gamma [\phi(q)]\, \frac{\partial^2}{\partial n_p\, \partial n_q} G\,(p,q)\, ds_q = 2\, \frac{\partial \phi^i}{\partial n_p}(p) \qquad p \text{ in } \overline{\Gamma}. \tag{13}$$

Thus we may indeed take the normal derivative at p under the integral sign in (5), but the resulting integral in (13) is very special. It must be interpreted in the sense of the Hadamard finite part (cf. (9) and (10)) according to the location of p on Γ or the end points $\partial\Gamma$.

Numerical Treatment

Equation (13) is a hypersingular integral equation for the unknown function $[\phi]$ in which the integral is a Hadamard finite-part integral of even order 2 if $p \in \Gamma$ and of fractional order 3/2 if $p \in \partial\Gamma$ (cf. [19]). On Γ, it is essential that $[\phi]$ be at least $C^{1,\alpha}$ at p for the very existence of the finite-part integral. Thus, with any boundary element approach to the solution of (13), one is faced with two choices: (a) represent $[\phi]$ using shape functions with sufficient smoothness on and between elements so that collocation points may be placed anywhere, or (b) place collocation points only where sufficient smoothness exists, i.e. away from element edges with the usual isoparametric elements (e.g. [1]). Choice (a) is the most attractive conceptually and is not prohibitive for line boundaries Γ such as a crack in two dimensions. However, for similar problems in three dimensions, choice (b) seems more attractive from an implementation standpoint. We have some preliminary computational experience with a test problem for a line crack Γ in two dimensions for the limiting case of zero frequency. Good results with acceptable effort have been obtained based on choice (a) using Overhauser splines and choice (b) using quadratic isoparametric line elements but collocating only at element interiors (cf. [21,22]). We also have some new results for scattering of surface water waves by a submerged plate in two dimensions and for scattering of an acoustic wave by a penny-shaped rigid plate in three dimensions. Specific computational strategies and numerical results will be presented elsewhere for these problems. However, we observe here that, in every case, it was possible to evaluate finite part integrals analytically as they occur over entire crack surfaces, or elements on those surfaces. Of note, by way of contrast, are the procedures of Lin and Keer [23], who identify and compute a finite-part integral after discretizing a plane crack surface into subdomains (elements), and those of Budreck and Achenbach [24], who likewise discretize first, then regularize, but do not identify finite-part integrals as such. Regardless, special quadrature rules for finite-part integrals, such as those of Kutt [25,26], although sometimes effective, appear not to be necessary in the BEM implementation. Numerical quadrature seems to be needed only for regular or nonhypersingular integrals.

Discussion

Whenever the normal derivative of an integral like (5) is taken, leading to a nonintegrable second derivative of the kernel function G, the common practice of regularization, to transfer one of the normal derivatives of G to a tangential derivative of the density function (here $[\phi]$) as in (8), ought to be reexamined. The alternative of taking the proper limit and identifying the Hadamard finite-part integrals as in (13) should at least be considered and, we submit, is to be preferred in most cases. It should be emphasized that the choice is not between regularization and a divergent integral, as might first appear, but rather between regularization and the finite part of the divergent integral. This latter strategy is conceptually more attractive and clear in its smoothness requirements such that the finite-part integrals exist. Moreover, a variety of computational strategies are available which offer rich ground for research.

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