

# ON THE SCATTERING OF ELASTIC WAVES BY AN ELASTIC INCLUSION IN TWO DIMENSIONS

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[Received 20 February 1989. Revises 26 June and 26 September 1989]

## SUMMARY

Plane time-harmonic elastic waves are scattered by a cylindrical elastic inclusion. This plane-strain inclusion problem is reduced to a pair of coupled singular integral equations over the interface. In fact, two different quasi-Fredholm systems of singular integral equations are obtained, one using an indirect method and one using a direct method. Both systems are shown to be uniquely solvable, and lead to simple existence proofs for the two-dimensional inclusion problem.

## 1. Introduction

CONSIDER an infinitely long cylinder embedded in an unbounded solid. Both the cylinder (the 'inclusion') and the surrounding solid (the 'matrix') are composed of homogeneous, isotropic elastic materials. Choose Cartesian coordinates  $(x, y, z) = (x_1, x_2, x_3)$  so that the  $z$ -axis is parallel to the generators of the cylinder. We consider the scattering of plane elastic waves by the inclusion, where the motion is confined to cross-section planes,  $z = \text{constant}$  (plane strain). For small time-harmonic oscillations, this leads to a vector transmission problem, which we call the *inclusion problem*; Kupradze *et al.* (1) call it the *basic contact problem*. In transmission problems, one usually specifies jumps in function values across an interface, whereas in boundary-value problems one specifies the function values themselves. Here, we assume that we have *perfect bonding* across the interface between the inclusion and the matrix ('welded contact'). Thus, we require that both the displacement vector and the traction vector be continuous across the interface. Note that antiplane motions are simpler to analyse, for they lead to a scalar transmission problem in acoustics; see (2) for references.

The literature on inclusion problems is extensive, especially on problems in elastostatics. Walpole's survey (3) allows for anisotropic materials, and concentrates on ellipsoidal inclusions. Mura (4) gives a brief review, supplementing his book (5).

We are interested in integral-equation methods for inclusion problems, both for their use in obtaining numerical solutions, and for their use in answering questions of solvability. Historically, this second use came first. Thus, for two-dimensional static problems, complex-variable methods are

available; see Parton and Perlin (6, §22) for references. Dynamic problems are considered by Kupradze *et al.* (1), but only in three dimensions. They reduce the inclusion problem to a pair of complicated coupled integral equations, over the interface, for a pair of unknown vector-valued functions.

Integral equations have also been used in numerical computations; typically, they are solved using a boundary-element method. Integral-equation methods are usually classified into two types: *direct methods*, in which the unknowns are physical, such as displacements and tractions; and *indirect methods*, in which the unknowns are not physical, such as source densities in elastic potentials (see section 3 below). For applications of these methods in elastodynamics, see (7, 8).

Rizzo and Shippy (9) have used a simple direct method for two-dimensional static inclusion problems, whilst Tan (10) and Kobayashi and Nishimura (11) have used a similar method for dynamic problems. Indirect (and direct) methods, in plane elastostatics, are described in (12, §7.5). Kupradze *et al.* (1) make extensive use of indirect methods, but do not give numerical results.

All of the work cited in the previous paragraph concerned integral equations over the interface. Other types of integral equation can be derived. Thus, for example, one can use the fundamental Green's tensor (see section 3 below) for the matrix to derive an integral equation over the whole inclusion. This approach is especially attractive for inhomogeneous inclusions; see, for example, Gubernatis *et al.* (13) or Willis (14).

In this paper, we reduce the inclusion problem to a pair of coupled singular integral equations over the interface. We do this in two ways, using an indirect method in section 5 and a direct method in section 6. In both cases, we prove that the system of integral equations is a quasi-Fredholm system, and hence that it is uniquely solvable. Both systems are new; both make use of a simple regularization of the operator  $N_\alpha$ , defined by (3.9) below as the tractions corresponding to an elastic double-layer potential; and both lead to simple existence proofs for the inclusion problem. Unfortunately, the method does not extend to the three-dimensional inclusion problem.

## 2. The inclusion problem

Let  $B_i$  denote a bounded domain, with a smooth (twice differentiable) closed boundary curve  $S$ , and a simply-connected unbounded exterior  $B_e$ . The plane domain  $B_\alpha$  is filled with homogeneous isotropic elastic material, with Lamé moduli  $\lambda_\alpha$  and  $\mu_\alpha$ , Poisson's ratio  $\nu_\alpha$ , and mass density  $\rho_\alpha$  ( $\alpha = e, i$ ). A stress wave is incident upon the inclusion; it can propagate through the interface  $S$  before being scattered to infinity. This leads to the following problem.

INCLUSION PROBLEM. Find displacement vectors  $\mathbf{u}_\alpha(P) \in C^2(B_\alpha) \cap C^1(\overline{B_\alpha})$  ( $\alpha = e, i$ ), which satisfy

$$L_e \mathbf{u}_e = \mathbf{0}, \quad P \in B_e, \quad (2.1)$$

$$L_i \mathbf{u}_i = \mathbf{0}, \quad P \in B_i, \quad (2.2)$$

and two *interface conditions*

$$\mathbf{u}(p) = \mathbf{u}_i(p), \quad p \in S, \quad (2.3a)$$

and

$$T_e \mathbf{u}(p) = T_i \mathbf{u}_i(p), \quad p \in S, \quad (2.3b)$$

where the total displacement in  $B_e$  is

$$\mathbf{u}(P) = \mathbf{u}_e(P) + \mathbf{u}_{inc}(P), \quad P \in B_e. \quad (2.4)$$

In addition,  $\mathbf{u}_e$  must satisfy radiation conditions; these are specified below in section 2.1.

The given incident wave  $\mathbf{u}_{inc}$  is assumed to satisfy (2.1) everywhere, except possibly at isolated points in  $B_e$ . The operator  $L_\alpha$  is defined by

$$L_\alpha \mathbf{u} = k_\alpha^{-2} \text{grad div } \mathbf{u} - K_\alpha^{-2} \text{curl curl } \mathbf{u} + \mathbf{u}, \quad (2.5)$$

where the wave numbers  $k_\alpha$  and  $K_\alpha$  are defined by

$$\rho_\alpha \omega^2 = (\lambda_\alpha + 2\mu_\alpha) k_\alpha^2 = \mu_\alpha K_\alpha^2,$$

the time-dependence  $e^{-i\omega t}$  is suppressed throughout and, for simplicity, we assume that  $\lambda_\alpha$ ,  $\mu_\alpha$  and  $\rho_\alpha$  are all positive constants ( $\alpha = e, i$ ). In our two-dimensional context, we interpret (2.5) as follows: set  $\mathbf{u} = (u_1, u_2)$  and then

$$(L_\alpha \mathbf{u})_1 = \frac{1}{k_\alpha^2} \frac{\partial^2 u_j}{\partial x_1 \partial x_j} - \frac{1}{K_\alpha^2} \frac{\partial}{\partial x_2} \left( \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) + u_1$$

and

$$(L_\alpha \mathbf{u})_2 = \frac{1}{k_\alpha^2} \frac{\partial^2 u_j}{\partial x_2 \partial x_j} + \frac{1}{K_\alpha^2} \frac{\partial}{\partial x_1} \left( \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) + u_2;$$

here, and below, we use the usual summation convention; for example,  $u_j v_j = u_1 v_1 + u_2 v_2$ . The *traction operator*  $T_\alpha$  is defined on  $S$  by

$$(T_\alpha \mathbf{u})_m(p) = \lambda_\alpha n_m \frac{\partial u_j}{\partial x_j} + \mu_\alpha n_j \left( \frac{\partial u_m}{\partial x_j} + \frac{\partial u_j}{\partial x_m} \right), \quad (2.6)$$

where  $\mathbf{n}(p)$  is the unit normal at  $p \in S$ , pointing into  $B_e$ .

We shall use the following notation: capital letters  $P, Q$  denote points of  $B_e \cup B_i$ ; lower-case letters  $p, q$  denote points of  $S$ ;  $\mathbf{r}_P$  is the position vector of  $P$  with respect to the origin  $O$ , which is chosen at some point in  $B_i$ ; and  $r_P = |\mathbf{r}_P|$ .

### 2.1 Radiation conditions and uniqueness

The formulation of radiation conditions is given in (1, pp. 124–130) for three dimensions; for two dimensions, see, for example, Barratt (15), Tan (10) or Hudson (16, §6.9). One formulation (there are others) is the following: write

$$\mathbf{u}_e = \mathbf{u}_e^P + \mathbf{u}_e^S,$$

where

$$\mathbf{u}_e^P = -k_\alpha^{-2} \text{grad div } \mathbf{u}_e \quad \text{and} \quad \mathbf{u}_e^S = \mathbf{u}_e - \mathbf{u}_e^P;$$

then we require that

$$r_P^{\frac{1}{2}} \left( \frac{\partial \mathbf{u}_e^P}{\partial r_P} - i k_e \mathbf{u}_e^P \right) \rightarrow \mathbf{0} \quad \text{as } r_P \rightarrow \infty \quad (2.7a)$$

and

$$r_P^{\frac{1}{2}} \left( \frac{\partial \mathbf{u}_e^S}{\partial r_P} - i K_e \mathbf{u}_e^S \right) \rightarrow \mathbf{0} \quad \text{as } r_P \rightarrow \infty. \quad (2.7b)$$

These are the radiation conditions. It is common to require also that both  $\mathbf{u}_e^P \rightarrow \mathbf{0}$  and  $\mathbf{u}_e^S \rightarrow \mathbf{0}$  as  $r_P \rightarrow \infty$ . However, it is straightforward to show that these conditions are implied by (2.7).

We shall require the following theorem.

**UNIQUENESS THEOREM 1.** *The inclusion problem has at most one solution.*

This theorem is proved in (1, p. 137; 17, p. 61). It also holds for certain complex wave numbers and for certain inhomogeneous inclusions (18, 19). In fact, we shall also require a slight generalization of Uniqueness Theorem 1, in which the interface conditions (2.3) are replaced by

$$\mathbf{u}(p) = \kappa_1 \mathbf{u}_i(p), \quad p \in S, \quad (2.8a)$$

and

$$T_e \mathbf{u}(p) = \kappa_2 T_i \mathbf{u}_i(p), \quad p \in S, \quad (2.8b)$$

where  $\kappa_1$  and  $\kappa_2$  are non-zero constants.

**UNIQUENESS THEOREM 2.** *The modified inclusion problem, in which the interface conditions (2.3) are replaced by (2.8), has at most one solution.*

This theorem is proved in (17, p. 61).

### 3. Elastic potentials

We introduce two fundamental Green's tensors,  $\mathbf{G}_\alpha(P; Q)$  ( $\alpha = e, i$ ):

$$(\mathbf{G}_\alpha(P; Q))_{ij} = \frac{1}{\mu_\alpha} \left\{ \Psi_\alpha \delta_{ij} + \frac{1}{K_\alpha^2} \frac{\partial^2}{\partial x_i \partial x_j} (\Psi_\alpha - \Phi_\alpha) \right\}, \quad (3.1)$$

where

$$\Phi_\alpha = -\frac{1}{2} i H_0^{(1)}(k_\alpha R), \quad \Psi_\alpha = -\frac{1}{2} i H_0^{(1)}(K_\alpha R), \quad (3.2)$$

$H_0^{(1)}(z)$  is a Hankel function and  $R = |\mathbf{r}_P - \mathbf{r}_Q|$ . Now  $\mathbf{G}_e$  ( $\mathbf{G}_i$ ) satisfies (2.1) ((2.2)) everywhere, except at  $P = Q$ . Also,  $\mathbf{G}_e$  satisfies the radiation conditions (2.7).

Next, we define elastic single-layer and double-layer potentials by

$$(S_\alpha \mathbf{f})(P) = \int_S \mathbf{f}(q) \cdot \mathbf{G}_\alpha(q; P) ds_q \quad (3.3)$$

and

$$(D_\alpha \mathbf{f})(P) = \int_S \mathbf{f}(q) \cdot T_\alpha^q \mathbf{G}_\alpha(q; P) ds_q, \quad (3.4)$$

respectively, where  $T_\alpha^q$  means  $T_\alpha$  applied at  $q \in S$ . Now  $(S_\alpha \mathbf{f})(P)$  is continuous in  $P$  as  $P$  crosses  $S$ , whereas both  $D_\alpha$  and  $T_\alpha^p S_\alpha$  exhibit jumps given by

$$D_\alpha \mathbf{f} = (\mp I + \overline{K}_\alpha^*) \mathbf{f} \quad (3.5)$$

and

$$T_\alpha^p S_\alpha \mathbf{f} = (\pm I + K_\alpha) \mathbf{f}, \quad (3.6)$$

respectively, where, in each case, the upper (lower) sign corresponds to  $P \rightarrow p \in S$  from  $B_e$  ( $B_i$ ), and  $I$  is the  $2 \times 2$  identity matrix. Here,  $K_\alpha$  and  $\overline{K}_\alpha^*$  are *singular* integral operators, defined, for  $p \in S$ , by

$$K_\alpha \mathbf{f} = \int_S \mathbf{f}(q) \cdot T_\alpha^p \mathbf{G}_\alpha(q; p) ds_q \quad (3.7)$$

and

$$\overline{K}_\alpha^* \mathbf{f} = \int_S \mathbf{f}(q) \cdot T_\alpha^q \mathbf{G}_\alpha(q; p) ds_q. \quad (3.8)$$

More precisely, if we write

$$(K_\alpha \mathbf{f})_i = \int_S f_j(q) K_{ji}(q; p) ds_q,$$

then  $K_{11}$  and  $K_{22}$  are continuous functions for  $p \in S$  and  $q \in S$ , but  $K_{12}$  and  $K_{21}$  have Cauchy singularities; that is, the corresponding integrals must be interpreted as Cauchy principal-value integrals; see (A.4).

In all of the above formulae, it is sufficient that the density  $\mathbf{f}$  be Hölder continuous on  $S$  (1, Chapter 5):  $\mathbf{f} \in C^{0,\beta}$ ,  $0 < \beta \leq 1$ . However, we shall also require the tractions corresponding to the elastic double-layer potential, defined by

$$N_\alpha \mathbf{f} = \mu_\alpha^{-1} T_\alpha^p D_\alpha \mathbf{f}. \quad (3.9)$$

The existence of  $N_\alpha \mathbf{f}$  requires that  $\mathbf{f}$  be smoother: a sufficient condition is that the tangential derivative of  $\mathbf{f}(q)$  be Hölder continuous for  $q \in S$  ( $\mathbf{f} \in C^{1,\beta}$ ), and then the right-hand side of (3.9) is continuous across  $S$  (1, p. 320). Although  $N_\alpha$  is a hypersingular operator, we show in the next section that it can be regularized in a simple manner.

#### 4. Regularization

We have

$$N_\alpha \mathbf{f} = \frac{1}{\mu_\alpha} T_\alpha^p \int_S \mathbf{f}(q) \cdot T_\alpha^q \mathbf{G}_\alpha(q; p) ds_q. \quad (4.1)$$

The fundamental solution  $\mathbf{G}_\alpha \sim \log R$  as  $R \rightarrow 0$ , whence we expect that  $T_\alpha^p T_\alpha^q \mathbf{G}_\alpha \sim R^{-2}$  as  $R \rightarrow 0$  (these results are confirmed in Appendix A). Thus, we cannot immediately write (4.1) as an integral operator by applying  $T_\alpha^p$  to the integrand. However, we shall show that the combination

$$(1 - \nu_i)N_i - (1 - \nu_e)N_e \quad (4.2)$$

is a singular integral operator.

In Appendix A, we calculate  $T_\alpha^p T_\alpha^q \mathbf{G}_\alpha$ , and extract the singular terms; the result can be written as

$$\begin{aligned} \pi[T_\alpha^p T_\alpha^q \mathbf{G}_\alpha(q; p)]_{mn} &= 2\mu_\alpha R^{-2} \{ [(2A_\alpha - 1)R_{,m}R_{,n} - A_\alpha \delta_{mn}] \mathbf{n}(p) \cdot \mathbf{n}(q) \\ &\quad + (3A_\alpha - 1)n_m(q)n_n(p) - A_\alpha n_m(p)n_n(q) \} + o(R^{-1}) \end{aligned} \quad (4.3)$$

as  $R \rightarrow 0$ , where  $R_{,m} = \partial R / \partial x_m$  and

$$A_\alpha = \frac{1 - 2\nu_\alpha}{2(1 - \nu_\alpha)}. \quad (4.4)$$

The non-integrable singularity  $R^{-2}$  is apparent, since the expression in braces is  $O(1)$  as  $R \rightarrow 0$ . We can simplify this expression, using the following lemma.

LEMMA 1. *Let  $S$  be a twice-differentiable curve. Then*

$$\mathbf{n}(p) \cdot \mathbf{n}(q) = 1 \quad (4.5)$$

and

$$R_{,m}R_{,n} = \delta_{mn} - \frac{1}{2}\{n_m(p)n_n(q) + n_n(p)n_m(q)\}, \quad (4.6)$$

both with an error of  $o(R)$  as  $R \rightarrow 0$ .

Lemma 1 is proved in Appendix B. Note that (4.5) also holds if  $S$  is a smooth surface in three dimensions, whereas (4.6) does not.

Using Lemma 1, (4.3) becomes

$$\begin{aligned} \pi[T_\alpha^p T_\alpha^q \mathbf{G}_\alpha(q; p)]_{mn} &= \mu_\alpha R^{-2} \{ 2(A_\alpha - 1)\delta_{mn} \\ &\quad + (4A_\alpha - 1)[n_m(q)n_n(p) - n_n(q)n_m(p)] \} + o(R^{-1}). \end{aligned}$$

Substituting for  $A_\alpha$  from (4.4), we obtain

$$\begin{aligned} \pi\mu_\alpha^{-1}(1 - \nu_\alpha)[T_\alpha^p T_\alpha^q \mathbf{G}_\alpha(q; p)]_{mn} \\ = -\delta_{mn}R^{-2} + (1 - 3\nu_\alpha)[n_m(q)n_n(p) - n_n(q)n_m(p)]R^{-2} + o(R^{-1}). \end{aligned}$$

The first term on the right-hand side is not integrable, whereas the second

term is  $O(R^{-1})$  and leads to Cauchy singular integrals. Forming the combination (4.2), we obtain

$$[(1 - \nu_i)N_i - (1 - \nu_e)N_e]\mathbf{f} = \int_S \mathbf{f}(q) \cdot \hat{N}(q; p) ds_q \\ = \hat{N}\mathbf{f},$$

say, where

$$\pi \hat{N}_{mn}(q; p) = 3(\nu_e - \nu_i)[n_m(q)n_n(p) - n_n(q)n_m(p)]R^{-2} + o(R^{-1}).$$

Thus, we have a simple regularization of  $N_\alpha$  (in two dimensions):  $\hat{N}$  is a singular integral operator.

Note that  $\hat{N}$  is a weakly-singular operator if  $\nu_e = \nu_i$ ; this special case is well known to give rise to a simpler inclusion problem (1, Chapter 12, §5; 6, §36).

### 5. Integral equations: indirect method

We look for a solution of the inclusion problem in the form

$$\mathbf{u}_\alpha(P) = (D_\alpha \mathbf{f}_\alpha)(P) + (S_\alpha \mathbf{g}_\alpha)(P), \quad P \in B_\alpha, \quad (5.1)$$

where  $\mathbf{f}_\alpha \in C^{1,\beta}$ ,  $\mathbf{g}_\alpha \in C^{0,\beta}$  and  $\alpha = e, i$ . Applying the interface conditions (2.3), making use of the jump relations (3.5) and (3.6), we obtain

$$\left. \begin{aligned} (I + \overline{K_i^*})\mathbf{f}_i - (-I + \overline{K_e^*})\mathbf{f}_e + S_i \mathbf{g}_i - S_e \mathbf{g}_e &= \mathbf{u}_{\text{inc}}, \\ \mu_i N_i \mathbf{f}_i - \mu_e N_e \mathbf{f}_e + (-I + K_i)\mathbf{g}_i - (I + K_e)\mathbf{g}_e &= T_e \mathbf{u}_{\text{inc}}. \end{aligned} \right\} \quad (5.2)$$

This is a pair of coupled integral equations for the four unknown vector-valued densities,  $\mathbf{f}_\alpha(q)$  and  $\mathbf{g}_\alpha(q)$  ( $\alpha = e, i$ ), defined for  $q \in S$ . Thus, we can impose two constraints in order to get two equations in two unknowns; this can be done in various ways.

Since the operator  $N_\alpha$  is troublesome, one choice is

$$\mathbf{f}_e(q) = \mathbf{f}_i(q) = \mathbf{0}, \quad q \in S.$$

This implies the representations as elastic single-layer potentials,

$$\mathbf{u}_\alpha(P) = (S_\alpha \mathbf{g}_\alpha)(P), \quad P \in B_\alpha \quad (5.3)$$

for  $\alpha = e, i$ , where  $\mathbf{g}_e$  and  $\mathbf{g}_i$  solve

$$\left. \begin{aligned} S_i \mathbf{g}_i - S_e \mathbf{g}_e &= \mathbf{u}_{\text{inc}}, \\ (I - K_i)\mathbf{g}_i + (I + K_e)\mathbf{g}_e &= -\mu_e \mathbf{t}_{\text{inc}} \end{aligned} \right\} \quad (5.4)$$

and

$$\mu_e \mathbf{t}_{\text{inc}}(q) = T_e \mathbf{u}_{\text{inc}}(q), \quad q \in S. \quad (5.5)$$

Note that (5.4)<sub>1</sub> only involves weakly-singular integral operators, whereas (5.4)<sub>2</sub> involves singular integral operators. This makes the system difficult to analyse. We are also unaware of any published numerical results obtained with (5.4), although its static version is used in (12, §7.5). Further remarks on (5.4) are made at the end of this section.

A second choice is motivated by the regularization described in section 4. Thus, we make the choices

$$\mathbf{f}_e(q) = (1 - \nu_e)\mathbf{f}(q), \quad \mathbf{f}_i(q) = \tau(1 - \nu_i)\mathbf{f}(q)$$

and

$$\mathbf{g}_e(q) = \mathbf{g}_i(q) = \mu_e \mathbf{g}(q),$$

where  $\tau = \mu_e/\mu_i > 0$ . This implies the representations

$$\mathbf{u}_e(P) = (1 - \nu_e)(D_e \mathbf{f})(P) + \mu_e(S_e \mathbf{g})(P), \quad P \in B_e, \quad (5.6a)$$

and

$$\mathbf{u}_i(P) = \tau(1 - \nu_i)(D_i \mathbf{f})(P) + \mu_e(S_i \mathbf{g})(P), \quad P \in B_i, \quad (5.6b)$$

where the densities  $\mathbf{f}(q)$  and  $\mathbf{g}(q)$  solve

$$\left\{ \begin{aligned} [\tau(1 - \nu_i)(I + \overline{K}_i^*) + (1 - \nu_e)(I - \overline{K}_e^*)]\mathbf{f} + \mu_e(S_i - S_e)\mathbf{g} &= \mathbf{u}_{\text{inc}}, \\ -\hat{N}\mathbf{f} + [(I - K_i) + (I + K_e)]\mathbf{g} &= -\mathbf{t}_{\text{inc}}. \end{aligned} \right\} \quad (5.7)$$

This is a system of four coupled singular integral equations for the four components of the two densities  $\mathbf{f}(q)$  and  $\mathbf{g}(q)$ . It is the elastodynamic analogue of the system derived by Kress and Roach (20) for the scalar transmission problem.

In order to analyse the system (5.7), we first construct the corresponding  $4 \times 4$  symbol matrix  $\sigma$  (see Appendix C for a sketch of the classical theory). With

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

we find that

$$\sigma = \begin{pmatrix} \sigma_1 & 0 \\ -M & \sigma_2 \end{pmatrix},$$

where

$$\begin{aligned} \sigma_1 &= \tau(1 - \nu_i)(I + i\theta A_i J) + (1 - \nu_e)(I - i\theta A_e J), \\ \sigma_2 &= 2I + i\theta(A_e - A_i)J, \end{aligned}$$

$M$  is the  $2 \times 2$  symbol matrix of the singular integral operator  $\hat{N}$  (its elements are not required here), and  $A_\alpha$  is defined by (4.4); the  $2 \times 2$  symbol matrices of  $K_\alpha$  and  $\overline{K}_\alpha^*$  are given in Appendix C; and compact operators, such as  $S_\alpha$ , have zero symbols. Note that  $\sigma$  is a function of  $\theta$ , where  $\theta$  can take only the values  $+1$  and  $-1$ .

Next, we prove the following result.

**LEMMA 2.** *Suppose that  $-1 < \nu_\alpha \leq \frac{1}{2}$  ( $\alpha = e, i$ ) and  $\tau = \mu_e/\mu_i > 0$ . Then the symbol matrix of the system (5.7),  $\sigma$ , is regular; that is,*

$$\det(\sigma) \neq 0.$$

*Proof.* We show that  $\det(\sigma) > 0$ . We have

$$\det(\sigma) = \Delta_1 \Delta_2,$$



where

$$\Delta_1 = \det(\sigma_1) = [\tau(1 - \nu_i) + (1 - \nu_e)]^2 - [\tau(1 - \nu_i)A_i - (1 - \nu_e)A_e]^2 \quad (5.8)$$

and

$$\Delta_2 = \det(\sigma_2) = 4 - (A_e - A_i)^2. \quad (5.9)$$

Now, since  $-1 < \nu_\alpha \leq \frac{1}{2}$ , we see that  $1 - \nu_\alpha > 0$  and  $0 \leq A_\alpha < \frac{3}{4}$ . Thus

$$|\tau(1 - \nu_i)A_i - (1 - \nu_e)A_e| < \frac{3}{4}[\tau(1 - \nu_i) + (1 - \nu_e)]$$

whence  $\Delta_1 > 0$ . Similarly,  $|A_e - A_i| < \frac{3}{4}$ , whence  $\Delta_2 > 0$ .

As  $\det(\sigma)$  does not depend on  $\theta$ , we deduce immediately (since  $\Phi(t) = 1$  in (C.6)) that the system (5.7) has *index zero*. This means that (5.7) is a *quasi-Fredholm* system of singular integral equations; that is, all the usual Fredholm theorems hold, just as if (5.7) was a system of Fredholm integral equations of the second kind, with weakly-singular kernels. In particular, we can deduce the existence of a unique solution to (5.7) merely by showing that the corresponding homogeneous system, namely

$$\left\{ \begin{aligned} &[\tau(1 - \nu_i)(I + \overline{K_i^*}) + (1 - \nu_e)(I - \overline{K_e^*})]\mathbf{f} + \mu_e(S_i - S_e)\mathbf{g} = \mathbf{0}, \\ &-\hat{N}\mathbf{f} + [(I - K_i) + (I + K_e)]\mathbf{g} = \mathbf{0} \end{aligned} \right\} \quad (5.10)$$

has only the trivial solution (see Appendix C).

In order to do this, we first consider the inhomogeneous system (5.7). Usually, the functions  $\mathbf{u}_{\text{inc}}(p)$  and  $\mathbf{t}_{\text{inc}}(p)$  are very smooth; we shall always assume that  $\mathbf{u}_{\text{inc}} \in C^{1,\beta}$  and  $\mathbf{t}_{\text{inc}} \in C^{0,\beta}$ . Suppose that we look for solutions of (5.7) with  $\mathbf{f}$  and  $\mathbf{g}$  both in  $C^{0,\beta}$  (see Appendix C). We must deduce that  $\mathbf{f}$  is necessarily smoother ( $\mathbf{f} \in C^{1,\beta}$ ), for then any solution of (5.7) will generate a solution of the inclusion problem, using the representations (5.6). Now, from (5.7)<sub>1</sub>, we have

$$[\tau(1 - \nu_i)(I + iA_iJS) + (1 - \nu_e)(I - iA_eJS)]\mathbf{f} = \mathbf{h},$$

say, where  $\mathbf{h} \in C^{1,\beta}$ ,  $\mathbf{S}$  is defined by (C.2), and we have noted that weakly-singular operators (such as  $S_\alpha$ ) map  $C^{0,\beta} \rightarrow C^{1,\beta}$ . Solving this system, explicitly, for the two components of  $\mathbf{f}$ , using (C.3), we find that

$$\Delta_1 \mathbf{f} \in C^{1,\beta},$$

where  $\Delta_1$  is defined by (5.8). Since  $\Delta_1 > 0$ , we deduce that  $\mathbf{f} \in C^{1,\beta}$ , as required.

Now, in order to prove existence, we adapt the arguments in (20) and suppose that  $\mathbf{f}(q)$  and  $\mathbf{g}(q)$  solve the homogeneous system (5.10). Substitute  $\mathbf{f}$  and  $\mathbf{g}$  into the representations (5.6); since  $\mathbf{f} \in C^{1,\beta}$  and  $\mathbf{g} \in C^{0,\beta}$ , the displacements  $\mathbf{u}_e(P)$  and  $\mathbf{u}_i(P)$  satisfy

$$\mathbf{u}_e(p) = \mathbf{u}_i(p) \quad \text{and} \quad T_e \mathbf{u}_e(p) = T_i \mathbf{u}_i(p), \quad p \in S,$$

by (5.10); that is, they solve the homogeneous inclusion problem (with

$\mathbf{u}_{\text{inc}} = \mathbf{0}$ ). Uniqueness Theorem 1 then implies that

$$\mathbf{u}_\alpha(P) = \mathbf{0}, \quad P \in B_\alpha \quad (\alpha = e, i).$$

In particular,

$$\mathbf{u}_\alpha(p) = \mathbf{0} \quad \text{and} \quad T_\alpha \mathbf{u}_\alpha(p) = \mathbf{0}, \quad p \in S \quad (\alpha = e, i).$$

Define displacements  $\mathbf{U}_e$  and  $\mathbf{U}_i$  by

$$\mathbf{U}_e(P) = (1 - \nu_e)(D_e \mathbf{f})(P) + \mu_e(S_e \mathbf{g})(P), \quad P \in B_i,$$

and

$$\mathbf{U}_i(P) = \tau(1 - \nu_i)(D_i \mathbf{f})(P) + \mu_e(S_i \mathbf{g})(P), \quad P \in B_e.$$

We have

$$L_e \mathbf{U}_e = \mathbf{0}, \quad P \in B_i, \quad L_i \mathbf{U}_i = \mathbf{0}, \quad P \in B_e;$$

also,  $\mathbf{U}_i$  satisfies the radiation conditions (with  $k_e$  and  $K_e$  replaced by  $k_i$  and  $K_i$ , respectively). Letting  $P \rightarrow p \in S$ , we have

$$\begin{aligned} \mathbf{U}_e(p) &= (1 - \nu_e)(I + \overline{K_e^*})\mathbf{f} + \mu_e S_e \mathbf{g} \\ &= 2(1 - \nu_e)\mathbf{f}(p) \end{aligned}$$

since

$$\mathbf{u}_e(p) = \mathbf{0} = (1 - \nu_e)(-I + \overline{K_e^*})\mathbf{f} + \mu_e S_e \mathbf{g}.$$

Similarly,

$$\mathbf{U}_i(p) = -2\tau(1 - \nu_i)\mathbf{f}(p), \quad (5.11)$$

$$T_e \mathbf{U}_e(p) = -2\mu_e \mathbf{g}(p), \quad (5.12)$$

and

$$T_i \mathbf{U}_i(p) = 2\mu_e \mathbf{g}(p).$$

Hence,

$$\mathbf{U}_i(p) = \kappa_1 \mathbf{U}_e(p), \quad p \in S,$$

and

$$T_i \mathbf{U}_i(p) = -T_e \mathbf{U}_e(p), \quad p \in S,$$

where

$$\kappa_1 = \frac{-\tau(1 - \nu_i)}{1 - \nu_e} \neq 0. \quad (5.13)$$

Thus,  $\mathbf{U}_e(P)$  and  $\mathbf{U}_i(P)$  solve a modified homogeneous inclusion problem, where the materials of the matrix and the inclusion are interchanged. Uniqueness Theorem 2 for this problem implies that

$$\mathbf{U}_e(P) = \mathbf{0}, \quad P \in B_i, \quad \mathbf{U}_i(P) = \mathbf{0}, \quad P \in B_e.$$

Then, (5.11) and (5.12) imply that  $\mathbf{f}(q) = \mathbf{0}$  and  $\mathbf{g}(q) = \mathbf{0}$ , respectively. Thus, we have proved the following theorem.

**THEOREM 5.1.** *The system of singular integral equations (5.7) has precisely one solution  $(\mathbf{f}(q), \mathbf{g}(q))$ , with  $\mathbf{f} \in C^{1,\beta}$  and  $\mathbf{g} \in C^{0,\beta}$ . Moreover, the corresponding displacement fields (5.6) solve the inclusion problem.*

We conclude this section with some remarks on the system (5.4). Its symbol matrix is

$$\sigma = \begin{pmatrix} 0 & 0 \\ I - i\theta A_e J & I + i\theta A_e J \end{pmatrix};$$

clearly,  $\det(\sigma) \equiv 0$ ; that is,  $\sigma$  is degenerate, and so the classical theory, as used to analyse the system (5.7), does not give any useful information here. We are not aware of any theoretical results on the solvability of (5.4).

### 6. Integral equations: direct method

If we apply Betti's reciprocal theorem (Green's theorem) in  $B_e$  to  $\mathbf{u}_e$  and  $\mathbf{G}_e$ , we obtain

$$2\mathbf{u}_e(P) = (S_e(T_e \mathbf{u}_e))(P) - (D_e \mathbf{u}_e)(P), \quad P \in B_e. \quad (6.1)$$

Similarly, applying Green's theorem in  $B_i$  to  $\mathbf{u}_{\text{inc}}$  and  $\mathbf{G}_e$ , and adding the result to (6.1), we obtain

$$2\mathbf{u}_e(P) = (S_e(T_e \mathbf{u}))(P) - (D_e \mathbf{u})(P), \quad P \in B_e, \quad (6.2)$$

where we have used (2.4). Also, applying Green's theorem in  $B_i$  to  $\mathbf{u}_i$  and  $\mathbf{G}_i$ , we obtain

$$\begin{aligned} -2\mathbf{u}_i(P) &= (S_i(T_i \mathbf{u}_i))(P) - (D_i \mathbf{u}_i)(P) \\ &= (S_i(T_e \mathbf{u}))(P) - (D_i \mathbf{u})(P), \quad P \in B_i, \end{aligned} \quad (6.3)$$

after using the interface conditions (2.3). Letting  $P \rightarrow p \in S$  in (6.2) and (6.3), we obtain

$$(I + \overline{K_e^*})\mathbf{u} - \mu_e S_e \mathbf{t} = 2\mathbf{u}_{\text{inc}}, \quad p \in S, \quad (6.4)$$

and

$$(I - \overline{K_i^*})\mathbf{u} + \mu_e S_i \mathbf{t} = \mathbf{0}, \quad p \in S, \quad (6.5)$$

where we have set

$$\mu_e \mathbf{t}(q) = T_e \mathbf{u}(q), \quad q \in S. \quad (6.6)$$

Similarly, if we calculate the tractions on  $S$  corresponding to (6.2) and (6.3), we obtain

$$N_e \mathbf{u} + (I - K_e) \mathbf{t} = 2\mathbf{t}_{\text{inc}}(p), \quad p \in S, \quad (6.7)$$

and

$$-N_i \mathbf{u} + \tau(I + K_i) \mathbf{t} = \mathbf{0}, \quad p \in S, \quad (6.8)$$

where  $\tau = \mu_e/\mu_i > 0$  and  $\mathbf{t}_{\text{inc}}$  is given by (5.5). Thus, we have four integral equations on  $S$  in the two unknowns  $\mathbf{u}(q)$  and  $\mathbf{t}(q)$ ; we must choose two equations, or two linear combinations of the four equations.

The usual choice (for numerical computations) is (6.4) and (6.5), but this system has a degenerate symbol; see the discussion at the end of section 5.

Another choice is (6.4) + (6.5) and  $(1 - \nu_e)(6.7) + (1 - \nu_i)(6.8)$ , namely

$$\left\{ \begin{aligned} [\tau(1 - \nu_i)(I + K_i) + (1 - \nu_e)(I - K_e)]\mathbf{t} - \hat{N}\mathbf{u} &= 2(1 - \nu_e)\mathbf{t}_{\text{inc}}, \\ \mu_e(S_i - S_e)\mathbf{t} + [(I - \overline{K}_i^*) + (I + \overline{K}_e^*)]\mathbf{u} &= 2\mathbf{u}_{\text{inc}}. \end{aligned} \right\} \quad (6.9)$$

This is the elastodynamic analogue of the system studied by Kittappa and Kleinman (21) for the scalar transmission problem. Comparison of (6.9) with (5.7) shows that (6.9) is a quasi-Fredholm system of singular integral equations.

Suppose that we can solve (6.9) for  $\mathbf{u}(q)$  and  $\mathbf{t}(q)$ . We then use (6.2) and (6.3) to construct displacement fields

$$2\mathbf{u}_e(P) = \mu_e(S_e\mathbf{t})(P) - (D_e\mathbf{u})(P), \quad P \in B_e, \quad (6.10a)$$

and

$$-2\mathbf{u}_i(P) = \mu_e(S_i\mathbf{t})(P) - (D_i\mathbf{u})(P), \quad P \in B_i. \quad (6.10b)$$

We have the following result.

**THEOREM 6.1.** *Suppose that  $\mathbf{u}$  and  $\mathbf{t}$  are Hölder-continuous solutions of the system (6.9). Then,  $\mathbf{u}_e(P)$  and  $\mathbf{u}_i(P)$ , given by (6.10), solve the inclusion problem.*

*Proof.* The representations (6.10) certainly define solutions of (2.1), (2.2) and (2.7). It remains to show that (6.10) satisfy (2.3). First, we apply the argument following (5.10) to (6.9)<sub>2</sub>. This implies that  $\Delta_2\mathbf{u} \in C^{1,\beta}$ , where  $\Delta_2$  is defined by (5.9). Since  $\Delta_2 > 0$ , we deduce that  $\mathbf{u} \in C^{1,\beta}$ , and so we can define the quantity  $N_\alpha\mathbf{u}$ . We have, from (6.10),

$$2(\mathbf{u}_e + \mathbf{u}_{\text{inc}} - \mathbf{u}_i) = 2\mathbf{u}_{\text{inc}} + \mu_e(S_e + S_i)\mathbf{t} - (\overline{K}_e^* + \overline{K}_i^*)\mathbf{u}, \quad p \in S, \quad (6.11a)$$

and

$$2(T_e\mathbf{u}_e + \mu_e\mathbf{t}_{\text{inc}} - T_i\mathbf{u}_i) = 2\mu_e\mathbf{t}_{\text{inc}} + \mu_e(K_e + K_i)\mathbf{t} - (\mu_e N_e + \mu_i N_i)\mathbf{u}, \quad p \in S. \quad (6.11b)$$

Let

$$\tilde{\mathbf{U}}(P) = \mu_e(S_e\mathbf{t})(P) - (D_e\mathbf{u})(P) + 2\mathbf{u}_{\text{inc}}(P), \quad P \in B_i,$$

and

$$\tilde{\mathbf{V}}(P) = \mu_e(S_i\mathbf{t})(P) - (D_i\mathbf{u})(P), \quad P \in B_e.$$

Letting  $P \rightarrow p \in S$ , and using (6.9), we find that

$$\tilde{\mathbf{V}} = \tilde{\mathbf{U}} \quad \text{and} \quad T_i\tilde{\mathbf{V}} = -\kappa_1^{-1}T_e\tilde{\mathbf{U}},$$

where  $\kappa_1$  is defined by (5.13). Uniqueness Theorem 2 implies that  $\tilde{\mathbf{U}} \equiv \mathbf{0}$  in  $B_i$  and  $\tilde{\mathbf{V}} \equiv \mathbf{0}$  in  $B_e$ . In particular,

$$\tilde{\mathbf{U}} + \tilde{\mathbf{V}} = \mathbf{0} \quad \text{and} \quad T_i\tilde{\mathbf{V}} + T_e\tilde{\mathbf{U}} = \mathbf{0}$$

on  $S$ , and these imply that the right-hand sides of (6.11) vanish, as required.

If we use this theorem in a straightforward adaptation of the proof of Theorem 5.1, we obtain the next theorem.

**THEOREM 6.2.** *The system of singular integral equations (6.9) has precisely one solution  $(\mathbf{u}(q), \mathbf{t}(q))$ , with  $\mathbf{u} \in C^{1,\beta}$  and  $\mathbf{t} \in C^{0,\beta}$ . Moreover, the corresponding displacement fields (6.10) solve the inclusion problem.*

## 7. Discussion

We have described two methods for reducing the elastodynamic inclusion problem, in two dimensions, to a pair of coupled singular integral equations over the interface between the inclusion and the surrounding matrix. Each system is uniquely solvable at all frequencies. Each system seems to be new, although each has a counterpart for the transmission problem of acoustics (2).

The method used a simple regularization: the quantity (4.2) is a singular integral operator. This result is not true in three dimensions. Kupradze *et al.* (1, Chapter 12, §2) treat the three-dimensional inclusion problem by introducing the *generalized traction operator*  $\tilde{T}$ , defined by (cf. (2.6))

$$(\tilde{T}\mathbf{u})_m(p) = (\lambda + \mu(1 - \beta))n_m \frac{\partial u_j}{\partial x_j} + \mu n_j \left( \frac{\partial u_m}{\partial x_j} + \beta \frac{\partial u_j}{\partial x_m} \right),$$

where  $\beta$  is a constant; we have  $\tilde{T} = T$  when  $\beta = 1$ . The analysis in (1) is very complicated, as it uses generalized traction operators applied to generalized elastic double-layer potentials  $\tilde{D}_\alpha(P)$  in both  $B_e$  and  $B_i$ ; here,  $\tilde{D}_\alpha(P)$  is defined by (3.4), with  $T_\alpha^q$  replaced by  $\tilde{T}_\alpha^q$ . We should like to find a simpler solution. One possibility was sketched in (22). Let  $\mu_i \tilde{N}_i = T_i^p \tilde{D}_i$ , and consider  $\tilde{N}_i - N_e$ . We can show that

$$\mu_i^{-1} T_i^p \tilde{T}_i^q \mathbf{G}_i(p; q) - \mu_e^{-1} T_e^p T_e^q \mathbf{G}_e(p; q) = o(R^{1-n}) \quad \text{as } R \rightarrow 0,$$

if  $\beta$  is chosen as

$$\beta = 1 - \frac{2(\nu_e - \nu_i)}{(1 - \nu_e)(1 - 2\nu_i)};$$

this result holds in  $n$  dimensions, where  $n = 2$  or  $n = 3$ . However, this is not sufficient to obtain a useful regularization: the price one pays for introducing  $\tilde{D}_i$  is that, unlike  $N_i \mathbf{f}$ ,  $\tilde{N}_i \mathbf{f}$  is discontinuous across  $S$ , with a jump involving various derivatives of  $\mathbf{f}$  (1, p. 322). This fact was overlooked in (22). Note that  $\beta = 1$  when  $\nu_e = \nu_i$ , whence  $\tilde{N}_i - N_e$  is a regularization (even in three dimensions). As we have already noted, the special inclusion problem for equal Poisson's ratios is well known to be simpler than the general case.

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#### APPENDIX A

##### Singularities

Let  $P$  and  $Q$  be at  $(x'_1, x'_2)$  and  $(x_1, x_2)$ , respectively. Set  $X_j = x_j - x'_j$  and  $X_{ij} = X_i X_j$ . Then

$$\mu[G(Q; P)]_{ij} = \Psi \delta_{ij} + K^{-2}[\delta_{ij} D(\Psi - \Phi) + X_{ij} D^2(\Psi - \Phi)],$$

where  $D$  denotes the operator  $R^{-1}d/dR$ . Applying the traction operator (2.6) at  $q$ , we obtain

$$\begin{aligned} [T^*G(q; P)]_{mk} &= \gamma n_m^q X_k D\Phi + [\theta \delta_{mk} + n_k^q X_m] D\Psi \\ &\quad + 2K^{-2}\{[n_m^q X_k + n_k^q X_m + \theta \delta_{mk}] D^2(\Psi - \Phi) \\ &\quad + \theta X_{mk} D^3(\Psi - \Phi)\}, \end{aligned} \quad (\text{A.1})$$

where  $\gamma = \nu/(1 - \nu)$ ,  $\theta = n_j^q X_j$  and  $n_j^q = n_j(q)$ . Applying the traction operator at  $p$ ,

we obtain

$$\begin{aligned} \mu^{-1}[T^p T^q \mathbf{G}(q; p)]_{mn} = & K^2 \gamma^2 n_m^q n_n^p \Phi - 2[\mathcal{N} \delta_{mn} + n_m^p n_n^q] D \Psi - 4 \gamma n_m^q n_n^p D \Phi \\ & - [\mathcal{N} X_{mn} + \Theta \delta_{mn} + B_{mn}^{pq} + B_{nm}^{qp}] D^2 \Psi - 2 \gamma [B_{mn}^{pq} + B_{nm}^{qp}] D^2 \Phi \\ & - 4 K^{-2} \{[\mathcal{N} \delta_{mn} + n_m^p n_n^q + n_m^q n_n^p] D^2 (\Psi - \Phi) \\ & + [\mathcal{N} X_{mn} + \Theta \delta_{mn} + B_{mn}^{pq} + B_{nm}^{qp} + B_{mn}^{qp} + B_{nm}^{pq}] D^3 (\Psi - \Phi) \\ & + \Theta X_{mn} D^4 (\Psi - \Phi)\}, \end{aligned} \quad (\text{A.2})$$

where

$$\mathcal{N} = n_j^p n_j^p, \quad \Theta = n_j^p n_k^q X_{jk}, \quad B_{mn}^{pq} = n_m^p n_n^q X_{mn}.$$

The above formulae are exact; they also hold in three dimensions if one sets

$$\Phi = -\frac{e^{ikR}}{2\pi R} \quad \text{and} \quad \Psi = -\frac{e^{iKR}}{2\pi R}.$$

We now examine the singular behaviour as  $R = |\mathbf{r}_P - \mathbf{r}_Q| = (X_{jj})^{\frac{1}{2}} \rightarrow 0$ . From (3.2) we have

$$\pi \Phi = \{1 - \frac{1}{4} k^2 R^2 + O(R^4)\} \log R \quad \text{as } R \rightarrow 0,$$

whence

$$\begin{aligned} \pi D \Phi &= R^{-2} + O(\log R), & \pi D^2 \Phi &= -2R^{-4} - \frac{1}{2} k^2 R^{-2} + O(\log R), \\ \pi D^3 \Phi &= 8R^{-6} + k^2 R^{-4} + O(R^{-2}), & \pi D^4 \Phi &= -48R^{-8} - 4k^2 R^{-6} + O(R^{-4}) \end{aligned}$$

as  $R \rightarrow 0$ , with similar estimates for  $\Psi$ . It is well known from classical potential theory (see, for example, (23, §199)) that

$$\theta R^{-2} = n_j^q \frac{\partial}{\partial x_j} (\log R) = o(1) \quad \text{as } R \rightarrow 0. \quad (\text{A.3})$$

Thus,  $\theta D \Psi$ ,  $\theta D^2 (\Psi - \Phi)$  and  $\theta X_{mk} D^3 (\Psi - \Phi)$  are all  $o(1)$  as  $R \rightarrow 0$  ( $X_j = O(R)$  and  $X_{jk} = O(R^2)$ ), whence

$$\pi [T^q \mathbf{G}(q; p)]_{mk} = A R^{-2} (n_k^q X_m - n_m^q X_k) + o(1), \quad (\text{A.4})$$

where

$$A = \frac{1 - 2\nu}{2(1 - \nu)}. \quad (\text{A.5})$$

The formula (A.4) is well known in two-dimensional elastostatics; see, for example, (24) or (6, p. 214). It shows that the integral operators  $K_\alpha$  and  $K_\alpha^*$ , defined by (3.7) and (3.8), respectively, involve singular integrals in contradistinction to classical potential theory.

Proceeding similarly for (A.2), we note that  $\Theta D^2 \Psi$ ,  $\Theta D^3 (\Psi - \Phi)$  and  $\Theta X_{mn} D^4 (\Psi - \Phi)$  are all  $o(1)$  as  $R \rightarrow 0$ , whereas  $B_{mn}^{pq} D^2 \Psi$  and  $B_{mn}^{pq} D^3 (\Psi - \Phi)$  are both  $o(R^{-1})$  as  $R \rightarrow 0$ ; that is, they are both weakly singular (as are terms involving  $\log R$ , such as  $\Phi$  and  $D \Phi$ ). The remaining terms can be rearranged to give the following formula:

$$\begin{aligned} \pi \mu^{-1}[T^p T^q \mathbf{G}(q; p)]_{mn} = & -2R^{-2} \{A \mathcal{N} \delta_{mn} + (1 - 2A) \mathcal{N} X_{mn} R^{-2} \\ & + A n_m^p n_n^q + (1 - 3A) n_m^q n_n^p\} + o(R^{-1}) \end{aligned}$$

as  $R \rightarrow 0$ , where  $A$  is given by (A.5).

## APPENDIX B

*Proof of Lemma 1*

Parametrize  $S$ , so that  $p$  and  $q$  are at  $(x(p), y(p))$  and  $(x(q), y(q))$ , respectively. Let

$$v = (x'^2 + y'^2)^{1/2} \quad \text{and} \quad \kappa = (x'y'' - x''y')v^{-4},$$

where all quantities are evaluated at  $p$ ; we assume that  $v \neq 0$ ;  $v\kappa$  is the (signed) curvature of  $S$  at  $p$ . We have

$$n_1^p = y'/v, \quad n_2^p = -x'/v, \quad R = v|q - p| + O(|q - p|^2).$$

Define  $\hat{n}_j = n_j^q - n_j^p$ , whence

$$\mathcal{N} = n_j^p n_j^q = 1 + n_j^p \hat{n}_j,$$

since  $n_j^p n_j^p = 1$ . Straightforward calculation shows that

$$\begin{aligned} n_1^p \hat{n}_1 &= +(q - p)x'y'\kappa, & n_1^p \hat{n}_2 &= +(q - p)y'^2\kappa, \\ n_2^p \hat{n}_2 &= -(q - p)x'y'\kappa, & n_2^p \hat{n}_1 &= -(q - p)x'^2\kappa, \end{aligned}$$

all with an error of  $o(R)$  as  $R \rightarrow 0$ . Equation (4.5) follows immediately. We also have

$$\begin{aligned} X_1^2/R^2 &= x'^2/v^2 - (q - p)x'y'\kappa, \\ X_2^2/R^2 &= y'^2/v^2 + (q - p)x'y'\kappa, \\ X_{12}/R^2 &= x'y'/v^2 + \frac{1}{2}(q - p)(x'^2 - y'^2)\kappa, \end{aligned}$$

again with errors of  $o(R)$  as  $R \rightarrow 0$ . Equation (4.6) follows, once we write its right-hand side as

$$\delta_{mn} - n_m^p n_n^p - \frac{1}{2}(n_m^p \hat{n}_n + n_n^p \hat{n}_m).$$

## APPENDIX C

*Systems of singular integral equations on a closed curve*

We sketch the theory; see (25, Chapter 6, §8), (26, Chapter 19) or (27, §8) for details. Consider the system

$$\sum_{k=1}^n \{a_{jk}(t)u_k(t) + b_{jk}(t)(Su_k)(t) + (T_{jk}u_k)(t)\} = f_j(t) \quad (\text{C.1})$$

for  $j = 1, 2, \dots, n$  and  $t \in S$ , where  $T_{jk}$  are compact (weakly-singular) operators and  $S$  is a closed contour in the complex plane. Also

$$(Su)(t) = \frac{1}{\pi i} \int_S \frac{u(z)}{z - t} dz = \frac{1}{\pi i} \int_S \frac{u(z(q))}{v(q)(q - p)} ds_q + \text{compact}, \quad (\text{C.2})$$

where we have parametrized  $S$  using  $z = z(q) = x(q) + iy(q)$ ,  $t = z(p)$  and  $v(q) = \{(x'(q))^2 + (y'(q))^2\}^{1/2}$ . Note that

$$S^2 u = u. \quad (\text{C.3})$$

Let  $A = (a_{jk})$ ,  $B = (b_{jk})$  and write the system (C.1) as  $L\mathbf{u} = \mathbf{f}$ . The symbol (matrix) of  $L$ ,  $\sigma(L)$ , is defined by

$$\sigma(L) = A(t) + B(t)\theta, \quad (\text{C.4})$$



where  $t \in S$  and  $\theta = \pm 1$ . Note that compact terms in  $L$  do not contribute to  $\sigma(L)$ . We call  $\sigma$  *regular* (or *normal* or *non-degenerate*) if, for all  $t \in S$  and  $\theta = \pm 1$ ,  $\det(\sigma) \neq 0$ . Suppose that  $\sigma$  is regular and that

$$\Phi(t) = \det[(A+B)^{-1}] \det[A-B]; \quad (\text{C.5})$$

then, the *index* of  $L$ ,  $\text{Ind}(L)$ , is calculated using

$$\begin{aligned} 2\pi \text{Ind}(L) &= [\arg \Phi(t)]_S \\ &= \text{change in } \arg \Phi \text{ as } t \text{ makes one positive circuit of } S. \end{aligned} \quad (\text{C.6})$$

The main result of the theory is that if  $\text{Ind}(L) = 0$ , then the Fredholm theorems hold exactly; see (26; §56). In particular, we have the following.

**THEOREM.** *Suppose that the system (C.1) has index zero, and that  $f_j(t) \in C^{0,\alpha}$ ,  $0 < \alpha \leq 1$ ,  $j = 1, 2, \dots, n$ . Then, if the homogeneous form of (C.1) (that is, (C.1) with  $f_j = 0$ ) has only the trivial solution, the inhomogeneous system (C.1) has a unique solution  $u_j(t) \in C^{0,\alpha}$ ,  $j = 1, 2, \dots, n$ .*

In order to apply the theory, we first have to compute the symbol matrix. For example, from (A.4), we have

$$\begin{aligned} \pi[T_\alpha^* G_\alpha(q; p)]_{12} &= A_\alpha R^{-2} (n_2^q X_1 - n_1^q X_2) + o(1) \\ &\sim A_\alpha [v(q)(q-p)]^{-1} \end{aligned}$$

as  $|q-p| \rightarrow 0$ , where  $A_\alpha$  is defined by (4.4) and we have used formulae from Appendix B. Hence, comparison of (C.1) and (3.8) gives

$$\sigma(\overline{K_\alpha^*}) = i\theta A_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Similarly, we find that

$$\sigma(K_\alpha) = -(\sigma(\overline{K_\alpha^*}))^T = \sigma(\overline{K_\alpha^*}),$$

where the superscript  $T$  denotes matrix transpose.