

IDENTIFICATION OF IRREGULAR FREQUENCIES IN SIMPLE DIRECT INTEGRAL-EQUATION METHODS FOR SCATTERING BY HOMOGENEOUS INCLUSIONS

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A time-harmonic sound wave is scattered by a fluid inclusion immersed in a different fluid. This problem can be reduced to various pairs of coupled boundary integral equations over the interface between the inclusion and the surrounding fluid. The simplest and most commonly used pair is shown to have at most one solution if, and only if, the exterior wavenumber is not an eigenvalue of the associated interior Dirichlet problem. A similar result obtains for the corresponding inclusion problem in elastodynamics.

1. Introduction

It is well known that the problem of time-harmonic acoustic scattering by an *impenetrable* obstacle can be reduced to an integral equation over the boundary of the obstacle, S . In fact, this can be done in many ways, leading to various integral equations. The simplest of these suffer from *irregular frequencies* (or 'characteristic frequencies' or 'fictitious eigenfrequencies'), at which the integral equations can have infinitely many solutions or no solutions. These frequencies can usually be identified as a countable set of values coinciding with the eigenfrequencies of a related interior problem.

Irregular frequencies are a nuisance, both analytically and numerically. They are induced by the choice of integral equation, and can be eliminated by making a different choice. This usually means choosing a more complicated equation, or supplementing the original equations in some ways; see, e.g. [1]. In computational practice, these complications are often discarded: a simple integral equation is used. If one uses such an equation and computes, say, the scattering cross-section of the obstacle as a function of the frequency, one obtains a smooth curve, with narrow spikes around each irregular frequency. These spikes are spurious: one can interpolate graphically through them. This pragmatic approach succeeds because one is not usually interested in high frequencies, and so only a few well-spaced irregular frequencies will be encountered; moreover, the spiky response near these frequencies is quite different to the response away from them.

Suppose now that the obstacle is *penetrable*, so that waves can propagate through the interface S . The solution to this problem has a more complicated structure, often with many sharp peaks and troughs in the frequency-response curves. This response can sometimes be analysed in terms of the resonant oscillations of the obstacle. Thus, if the coupling parameter in the interface conditions (ρ in (2.3b) below) is small, then the obstacle is nearly rigid (Neumann boundary condition), and so we expect a larger response whenever the frequency is such that the interior wavenumber k_i is an eigenvalue of the interior Dirichlet problem. A similar expectation obtains if the coupling parameter is large. These remarks are the impetus behind the so-called Resonance Scattering Theory; the developments for spherical

homogeneous obstacles are reviewed in [2] and [3]. Kriegsmann et al. [4] have obtained asymptotic expansions of the solution, when ρ is small (or large), which are uniformly valid in the frequency; these hold for any obstacle, and give the correction to the corresponding 'background field', i.e. the solution when ρ is zero (or infinite).

The scattering problem for a homogeneous penetrable scatterer can also be reduced to integral equations over S . Usually, a pair of coupled integral equations is derived, although single integral equations are available [5]. Many of these have been analysed, and their irregular frequencies (if any) have been identified. However, the simplest, and most commonly used pair of equations does not seem to have been studied before. This pair is derived by applying Green's theorem both inside and outside the obstacle to the unknown field and the appropriate fundamental solution, and then letting the field points approach S (see Section 4). We call this the 'simple direct method': it is 'simple' because normal derivatives of double-layer potentials are not required; and it is 'direct' because the unknown quantities on S are physically meaningful (e.g., pressures and velocities). In this paper, we prove that the simple direct pair of integral equations does have irregular frequencies. These occur only when the exterior wavenumber k_e , coincides with an eigenvalue of the associated interior Dirichlet problem. This result is in accord with the numerical results of Morita [6].

Clearly, it is important to know where the irregular frequencies are located; they induce spurious additional spikes in a frequency-response curve which actually can be spiky. We observe that since the irregular frequencies are independent of k_i and ρ , they can be detected by comparing the frequency-response curves (as functions of k_e) for different values of k_i or ρ .

Finally, we note that all of our results can be extended to the corresponding problem in elastodynamics, in which a stress wave is scattered by a homogeneous elastic inclusion. This problem is discussed briefly in Section 5.

2. Formulation of the transmission problem

In this paper, we use the same notation as in [5]. Thus, let B_i be a bounded region in either \mathbf{R}^2 or \mathbf{R}^3 , with a smooth closed boundary S and unbounded exterior B_e . We suppose that B_e and B_i are filled with different compressible fluids, with constant wavenumbers k_e and k_i , respectively. A given time-harmonic incident wave, with potential u_{inc} , is scattered by the penetrable obstacle, B_i . (As usual, a time-dependence of $e^{-i\omega t}$ is suppressed throughout.) The field scattered to infinity, u_e , and the interior field, u_i , are then seen to solve the following problem.

2.1. Transmission problem

Find functions $u_e(P)$ and $u_i(P)$ that satisfy

$$(\nabla^2 + k_e^2)u_e(P) = 0, \quad P \in B_e \quad (2.1)$$

$$(\nabla^2 + k_i^2)u_i(P) = 0, \quad P \in B_i \quad (2.2)$$

and two *transmission conditions* on the interface:

$$u(p) = u_i(p) \quad \text{and} \quad \frac{\partial u}{\partial n_p} = \rho \frac{\partial u_i}{\partial n_p}, \quad p \in S, \quad (2.3a, b)$$

where

$$u(P) = u_e(P) + u_{\text{inc}}(P), \quad P \in B_e \quad (2.4)$$

is the total potential in B_e . Also, u_e must satisfy a radiation condition at infinity.

The wavenumbers, k_e and k_i , and the coupling parameter (density ratio) ρ are given constants. In (2.3b), $\partial/\partial n_p$ denotes normal differentiation at $p \in S$ in the direction from ∂D towards B_e .

If we restrict the choice of k_e , k_i and ρ , we can prove that the transmission problem has precisely one solution. Sufficient conditions are that k_e , k_i and ρ be real and positive [5]; these conditions are assumed below.

Note that we can move the parameter ρ from (2.3b) to (2.3a) by a simple change in the dependent variables.

We shall also need to consider the following problem.

2.2. Interior Dirichlet problem (IDP)

Find a function $w(P)$ that satisfies

$$(\nabla^2 + k^2)w(P) = 0, \quad P \in B_i$$

and the Dirichlet condition

$$w(p) = 0, \quad p \in S.$$

If this problem has a non-trivial solution, we say that k^2 is an eigenvalue of the interior Dirichlet problem. It is known that these eigenvalues are all real.

3. Acoustic potentials and Green's theorem

We define single-layer and double-layer potentials by

$$(S_\alpha \mu)(P) = \int_S \mu(q) G_\alpha(P, q) ds_q, \quad P \in B_e \cup S \cup B_i \quad (3.1)$$

and

$$(D_\alpha \nu)(P) = \int_S \nu(q) \frac{\partial}{\partial n_q} G_\alpha(P, q) ds_q, \quad P \in B_e \cup B_i, \quad (3.2)$$

respectively, where

$$G_\alpha(P, Q) = \begin{cases} -\frac{1}{2}i H_0^{(1)}(k_\alpha R) & \text{in } \mathbf{R}^2, \\ -\exp(ik_\alpha R)/(2\pi R) & \text{in } \mathbf{R}^3, \end{cases} \quad (3.3)$$

$R = |r_P - r_Q|$ is the distance between the two points P and Q , $H_0^{(1)}(z)$ is a Hankel function, and $\alpha = e$ or i . The jump relations on S are

$$D_\alpha \nu = (\mp I + \bar{K}_\alpha^*) \nu \quad \text{and} \quad \frac{\partial}{\partial n_p} S_\alpha \mu = (\mp I + K_\alpha) \mu, \quad (3.4)$$

where the upper (lower) sign corresponds to $P \rightarrow p \in S$ from $B_e(B_i)$. K_α and \bar{K}_α^* are boundary integral operators, defined for $p \in S$ by

$$K_\alpha \mu = \int_S \mu(q) \frac{\partial}{\partial n_p} G_\alpha(p, q) ds_q$$

and

$$\bar{K}_\alpha^* \nu = \int_S \nu(q) \frac{\partial}{\partial n_q} G_\alpha(p, q) ds_q.$$

We shall also need the normal derivative of the double-layer potential, defined by

$$N_\alpha \nu = \frac{\partial}{\partial n_p} (D_\alpha \nu).$$

In order to derive integral equations, we shall use Green's theorem; this is the so-called 'direct' method. If we apply Green's theorem in B_e to u_e and G_e , we obtain the Helmholtz formula,

$$\int_S \left\{ G_e(P, q) \frac{\partial u_e}{\partial n_q} - u_e(q) \frac{\partial}{\partial n_q} G_e(P, q) \right\} ds_q = \begin{cases} 2u_e(P), & P \in B_e, \\ 0, & P \in B_i. \end{cases} \quad (3.5)$$

If we apply Green's theorem in B_i to u_{inc} and G_e , and add the result to (3.5), we obtain

$$\int_S \left\{ G_e(P, q) \frac{\partial u}{\partial n_q} - u(q) \frac{\partial}{\partial n_q} G_e(P, q) \right\} ds_q = \begin{cases} 2u_e(P), & P \in B_e, \\ -2u_{inc}(P), & P \in B_i. \end{cases} \quad (3.6a)$$

$$(3.6b)$$

Finally, if we apply Green's theorem in B_i to u_i and G_i , we obtain

$$\int_S \left\{ G_i(P, q) \frac{\partial u_i}{\partial n_q} - u_i(q) \frac{\partial}{\partial n_q} G_i(P, q) \right\} ds_q = \begin{cases} 0, & P \in B_e, \\ -2u_i(P), & P \in B_i. \end{cases} \quad (3.7a)$$

$$(3.7b)$$

These formulae yield boundary integral equations if we let P approach S .

4. A simple pair of coupled integral equations

If we use (3.6a) and (3.7b) with (2.3), we obtain the representations

$$2u_e(P) = (S_e v)(P) - (D_e u)(P), \quad P \in B_e, \quad (4.1a)$$

and

$$-2u_i(P) = \rho^{-1}(S_i v)(P) - (D_i u)(P), \quad P \in B_i, \quad (4.1b)$$

where we have set

$$v(q) \equiv \frac{\partial u}{\partial n_q}. \quad (4.2)$$

Letting $P \rightarrow p \in S$ in (4.1), using (2.3a) and (3.4), we obtain

$$(I + \bar{K}_e^*)u - S_e v = 2u_{inc}, \quad \rho(I - \bar{K}_i^*)u + S_i v = 0 \quad (4.3)$$

which is a pair of coupled boundary integral equations for $u(q)$ and $v(q)$, $q \in S$.

The system (4.3) is probably the most commonly used for the numerical solution of the transmission problem. Representative examples are Morita [6] (two dimensions with $\rho = 1$), Tobocman [7], Schuster and Smith [8] (their system S.I., with $\rho = 1$) and Seybert and Casey [9]. As we have already remarked in Section 1, other (non-simple) systems can be derived: see [8, 10, 11] for some alternative direct methods, some of which do not suffer from irregular frequencies. However, although these methods may be theoretically attractive, they are seldom used for computations; for an application of the method in [10], see [12].

It is clear from the derivation that if $u(P)$ and $u_i(P)$ solve the transmission problem (we already know that such functions exist and are unique), then their boundary values will satisfy (4.3). We start with a partial converse to this result.

Theorem 1. *Suppose that $u(q)$, $v(q)$ solve (4.3). Then, $u_e(P)$ and $u_i(P)$, defined by (4.1), solve the transmission problem, provided that k_e^2 is not an eigenvalue of the interior Dirichlet problem.*

Proof. Clearly, $u_e(P)$ satisfies (2.1) and the radiation condition, and $u_i(P)$ satisfies (2.2). On S , we have

$$\begin{aligned} 2u_e &= S_e v - (-I + \bar{K}_e^*)u = (I + \bar{K}_e^*)u - 2u_{\text{inc}} - (-I + \bar{K}_e^*)u \\ &= 2u - 2u_{\text{inc}}, \end{aligned} \quad (4.4)$$

using the first of (4.3). Similarly,

$$-2u_i = \rho^{-1} S_i v - (I + \bar{K}_i^*)u = -2u. \quad (4.5)$$

Adding (4.4) and (4.5), we see that the first transmission condition, (2.3a), is satisfied (for all values of k_e^2).

For the second transmission condition, (2.3b), we have to show that

$$2 \left\{ \frac{\partial u_e}{\partial n} + \frac{\partial u_{\text{inc}}}{\partial n} - \rho \frac{\partial u_i}{\partial n} \right\} = (K_e + K_i)v - (N_e + \rho N_i)u + 2 \frac{\partial u_{\text{inc}}}{\partial n} = 0. \quad (4.6)$$

To do this, we construct functions \tilde{u} and \tilde{v} by

$$\tilde{u}(P) = (S_e v)(P) - (D_e u)(P) + 2u_{\text{inc}}(P), \quad P \in B_i, \quad (4.7a)$$

and

$$\tilde{v}(P) = (S_i v)(P) - \rho(D_i u)(P), \quad P \in B_e. \quad (4.7b)$$

Then,

$$(\nabla^2 + k_e^2)\tilde{u}(P) = 0, \quad P \in B_i, \quad (\nabla^2 + k_i^2)\tilde{v}(P) = 0, \quad P \in B_e,$$

and \tilde{v} satisfies a radiation condition (with wavenumber k_i). The integral equations (4.3) imply that

$$\tilde{u}(p) = 0 \quad \text{and} \quad \tilde{v}(p) = 0, \quad p \in S. \quad (4.8)$$

The second of these, together with uniqueness for the exterior Dirichlet problem then imply that $\tilde{v}(P) \equiv 0$ for $P \in B_e$. In particular, we have, on S ,

$$\frac{\partial \tilde{v}}{\partial n} = 0 = (I + K_i)v - \rho N_i u. \quad (4.9)$$

The first of (4.8), together with the assumption that k_e^2 is not an eigenvalue of the IDP, imply that $\tilde{u}(P) \equiv 0$ for $P \in B_i$. Hence, on S ,

$$\frac{\partial \tilde{u}}{\partial n} = 0 = (-I + K_e)v - N_e u + 2 \frac{\partial u_{\text{inc}}}{\partial n}. \quad (4.10)$$

Adding (4.9) and (4.10), we see that (4.6) is satisfied, i.e. the second transmission condition is satisfied. This completes the proof.

Our next result identifies all the irregular frequencies of the system (4.3).

Theorem 2. *The pair of integral equations (4.3) has one solution if, and only if, k_e^2 is not an eigenvalue of the interior Dirichlet problem.*

Proof. Suppose that $u_0(q)$ and $v_0(q)$ solve the homogeneous form of (4.3), i.e.

$$(I + \bar{K}_e^*)u_0 - S_e v_0 = 0, \quad \rho(I - \bar{K}_i^*)u_0 + S_i v_0 = 0. \quad (4.11a, b)$$

Assume that u_0 and v_0 are not both identically zero. We show that this can only occur if k_e^2 is an eigenvalue of the IDP, using the same argument as in the proof of Theorem 1. Define $u_e(P)$, $u_i(P)$, $\tilde{u}(P)$ and $\tilde{v}(P)$ by (4.1) and (4.7), but with $u(q)$ and $v(q)$ replaced by $u_0(q)$ and $v_0(q)$, respectively, and with $u_{inc}(P) \equiv 0$ in (4.7). We deduce that, on S ,

$$u_e(p) = u_i(p) = u_0(p), \quad \tilde{u}(p) = 0 \quad \text{and} \quad \tilde{v}(p) = 0. \quad (4.12)$$

The last equation gives $\tilde{v}(P) \equiv 0$ for $P \in B_e$, whence

$$\frac{\partial \tilde{v}}{\partial n} = 0 = (I + K_i)v_0 - \rho N_i u_0. \quad (4.13)$$

Similarly, if k_e^2 is not an eigenvalue of the IDP, we can deduce that $\tilde{u}(P) \equiv 0$ for $P \in B_i$, whence

$$\frac{\partial \tilde{u}}{\partial n} = 0 = (-I + K_e)v_0 - N_e u_0. \quad (4.14)$$

But, we have

$$2 \frac{\partial u_e}{\partial n} = (I + K_e)v_0 - N_e u_0$$

and

$$-2\rho \frac{\partial u_i}{\partial n} = (-I + K_i)v_0 - \rho N_i u_0,$$

whence comparison with (4.13) and (4.14) shows that

$$\frac{\partial u_e}{\partial n} = \rho \frac{\partial u_i}{\partial n} = v_0. \quad (4.15)$$

Thus, $u_e(P)$ and $u_i(P)$ solve the homogeneous transmission problem (with $u_{inc} \equiv 0$). The uniqueness theorem for this problem implies that $u_e \equiv 0$ and $u_i \equiv 0$, whence (4.12)₁ and (4.15) imply that $u_0 = v_0 = 0$, which is contrary to our assumptions.

We have just shown that non-uniqueness for the system (4.11) (or (4.3)) implies that k_e^2 is an eigenvalue of the IDP. We now prove the converse. At such a value of k_e^2 , we know that there is a function $\mu_0(q)$, not identically zero, satisfying

$$(I + K_e)\mu_0 = 0 \quad \text{and} \quad S_e \mu_0 = 0$$

(see, e.g. Theorem 3.2.1 and Corollary 3.2.1 in [13]). Clearly, μ_0 also satisfies

$$\{(I + K_i)(I + K_e) - \rho N_i S_e\} \mu_0 = 0, \quad (4.16)$$

which is a homogeneous Fredholm integral equation of the second kind (it is (5.26) in [5]). Hence, there exists a non-trivial solution, ν_0 , of the corresponding adjoint equation, namely

$$\{I + \bar{K}_e^*(I + \bar{K}_i^*) - \rho S_e N_i\} \nu_0 = 0 \quad (4.17)$$

((6.21) in [5]). Now set

$$u_0 = (I + \bar{K}_i^*)f \quad \text{and} \quad v_0 = -\rho N_i f,$$

for some $f(q)$. It follows that (4.11b) is satisfied *identically*, for any f , since

$$S_\alpha N_\alpha = (\bar{K}_\alpha^*)^2 - I.$$

Moreover, (4.11a) is also seen to be satisfied if we choose

$$f(q) = \nu_0(q),$$

by comparison with (4.17). Thus, we have found a non-trivial solution to the system (4.11). This concludes the proof of Theorem 2.

5. Scattering of elastic waves by an elastic inclusion

The methods described above for the scalar transmission problem are easily extended to the corresponding vector problem in plane elasticity. In this two-dimensional *inclusion problem*, the regions B_e and B_i are filled with different homogeneous elastic materials. The conditions on the (perfect) interface S are

$$u(p) = u_i(p) \quad \text{and} \quad T_e u(p) = T_i u_i(p), \quad p \in S,$$

where $u = u_e + u_{\text{inc}}$ is the total displacement in B_e , u_i is the displacement in B_i , u_{inc} is the given incident wave and T_α is the traction operator corresponding to the material in B_α . For more details on this problem, see, e.g. ([14], Chpt. 12), ([15], p. 462) or [16].

The inclusion problem can be reduced to a system similar to (4.3), namely

$$(I + \bar{K}_e^*)u - S_e t = 2u_{\text{inc}}, \quad (I - \bar{K}_i^*)u + S_i t = 0 \quad (5.1)$$

where $u(q)$ and $t(q) = T_e u$ are unknown vector-valued functions of position $q \in S$; here, S_α and \bar{K}_α^* are the elastodynamic analogues of the operators defined in Section 3 [16, 17]. The system (5.1) has been used by several authors; see, e.g. [18–20].

The irregular frequencies for (5.1) can be characterized using the following interior problem. Suppose that the region B_i is filled with the elastic material from the exterior region, B_e , and that the displacement vector vanishes everywhere on the boundary of B_i , S . This object can oscillate freely at an infinite discrete set of values of the frequency ω . These frequencies coincide with all the irregular frequencies of the system (5.1). This result can be proved by a simple modification of the proof in Section 4. Note that the elastic versions of (4.16) and (4.17) are quasi-Fredholm singular integral equations, as can be readily shown using results in [16]. This is the only part of the proof where the restriction to two dimensions is required.

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