Scattering of water waves by submerged plates using hypersingular integral equations

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The interaction of surface water waves with submerged plates is considered. The problem is formulated in terms of a hypersingular integral equation for the discontinuity in the potential across the plate. Once found, the discontinuity may be used for direct calculation of the reflection and transmission coefficients. A numerical solution is employed, whereby the discontinuity is approximated by a truncated series of orthogonal polynomials, multiplied by an appropriate weight function. The choice of polynomials is dictated by physical arguments. Published results are reproduced for horizontal and vertical plates. New results are presented for inclined plates, showing the variation of the reflection coefficient with angle of inclination and depth of submergence.

1 INTRODUCTION

The two-dimensional scattering of linear water waves by thin rigid plates has been treated in several ways by many authors. One reason for this attention is that thin plates have been used as simple models for certain floating breakwaters; Sobhani et al.[1] discuss this application and give further references. Another reason is that thin plates can lead to boundary-value problems that can be solved exactly; this is very unusual in linear hydrodynamics, but is also very valuable in that such solutions provide benchmarks against which approximate solutions can be assessed.

Previous work on scattering by thin plates can be classified according to whether the depth of water is finite or infinite, and whether the plate is completely submerged or pierces the free surface. Moreover, most previous work assumes that the plate is flat. In what follows, we only refer to work on scattering by a single plate of finite length.

1.1 Infinite depth

Ursell[2] solved the problem of wave scattering by a fixed, surface-piercing, vertical plate. He constructed the potential on each side of the plate by using an

expansion theorem due to Havelock.[3] Continuity of motion across the plane of the plate gave him an integral equation for the horizontal velocity, which he solved exactly. The reflection and transmission coefficients were obtained from the limiting forms of the potential at large distances from the plate.

John[4] considered surface-piercing plates making an angle of $\pi/2n$ to the horizontal, where n is an integer. He showed that this problem can be solved by complex function techniques. However, as n increases, the method quickly becomes unwieldy; in fact, it seems that even the case $n = 2$ has not been worked out in detail.

Evans[5] considered the scattering of surface waves by a fixed, vertical plate, submerged beneath the free surface. His method of solution is similar to that used by John, whereby a complex potential is introduced, from which a reduced potential may be defined. The choice of reduced potential ensures that the boundary conditions on the free surface and on the plate take the same form for this new problem. This simplification allows the reduced potential problem to be solved, from which the desired result follows by integration.

Burke[6] treated the problem of scattering by a fixed, submerged, horizontal plate, using the Wiener–Hopf technique. Unfortunately, no numerical results were given.

Finally, we mention a paper by Shaw,[7] who considered the problem of scattering by a surface-piercing plate, whose orientation is altered slightly from

the vertical, and whose shape is slightly altered from being flat. Using perturbation techniques, Shaw found that to first order, the problem is the same as that solved by Ursell.[2] A new second order correction is found, however, with corresponding corrections to the reflection and transmission coefficients.

1.2 Finite depth

For water of constant finite depth, it is conventional to divide the fluid domain into three, namely a finite (rectangular) domain containing the plate, and two semi-infinite domains. In the latter, the velocity potential can be written as a series of eigenfunctions (with unknown coefficients). In the finite domain, different methods have been used. Thus, Patarapanich[8] used the finite element method and calculated the reflection and transmission coefficients for a submerged horizontal plate. The main disadvantage of this method is that it does not readily account for the inevitable singularities at the two edges of the plate, where inviscid theory predicts infinite velocities. Moreover, care must be taken in matching with the eigenfunction expansions in the two semi-infinite domains, so as to satisfy the radiation conditions and to avoid spurious reflections. Finite elements have also been used by Sobhani et al.[1] in their study of inclined, surface-piercing plates, wherein the plate is hinged at the sea-floor and the effects of a mooring line are also included.

For submerged horizontal plates, one can also use eigenfunction expansions within the finite domain. This leads to the method of matched eigenfunction expansions. It has been used by McIver[9] for scattering by moored, horizontal plates, although she also computed the reflection and transmission coefficients for a fixed plate.

Liu and Abbaspour[10] have used a simple boundary integral equation method within the finite domain for inclined, surface-piercing plates. They partitioned the finite domain into two by introducing an additional boundary, extending from the lower edge of the plate to a point on the sea-floor. They then solved Laplace's equation in each sub-domain using Green's theorem and a simple (log R) fundamental solution. Again, this method does not account for the plate-edge singularities in a natural way: special elements are introduced so as to incorporate the expected singular behaviour.

1.3 The present paper

In this paper, we use an integral-equation method for plates submerged beneath the free surface of deep water. An application of Green's theorem to the velocity potential ϕ and an appropriate fundamental solution G shows that ϕ must be represented in the water as a distribution of normal dipoles, with strength $[\phi]$, over the surface of the plate; $[\phi]$ is the discontinuity in the potential across the plate. To find this discontinuity, the boundary condition of no flow through the plate is applied. The resulting equation can be written as a so-called hypersingular integral equation over the plate for $[\phi]$ (sections 3 and 4). This approach has several advantages over the methods discussed above. For instance, the radiation condition is automatically satisfied by the choice of G. Similarly, the behaviour of $[\phi]$ at each edge of the plate, where there are square-root zeros, can be easily enforced. Moreover, the method is applicable to curved plates as well as flat plates. In fact, apart from some simple quadratures, the only approximation required is that of a bounded function defined on a finite interval. We do this by choosing an appropriate set of orthogonal polynomials, namely Chebyshev polynomials of the second kind, and then using a collocation method on the governing integral equation (section 5). Similar methods have been used by Frenkel[11] and by Kaya and Erdogan,[12] and the method is known to be convergent.[13,14] Having computed an approximation to $[\phi]$, the reflection and transmission coefficients can then be calculated directly (section 6).

We have used the method outlined above to compute the reflection and transmission coefficients for a single flat, inclined plate. This is for simplicity rather than any inherent limitation of the method. We find excellent agreement with published results for horizontal and vertical plates. We also give new results for inclined plates, and investigate the variation of the reflection coefficient with angle of inclination and depth of submergence. In particular, we find that the most noteworthy feature of scattering by a horizontal plate, namely the zeros of the reflection coefficient, is absent for a nearly-horizontal plate.

The method can also be applied to surface-piercing plates. However, the nature of the singularity at the point where the plate pierces the free surface is then different.[15] This leads to some complications, which we are currently studying.

2 FORMULATION

A Cartesian coordinate system is chosen, in which y is directed vertically downwards into the fluid, the undisturbed free surface lying at $y = 0$. We choose the z-axis perpendicular to the direction of propagation of the incident wavetrain. A plate, lying parallel to the incident wavecrests, is introduced below the free surface of the fluid, the submergence of the plate being independent of z. The problem is assumed to be two-dimensional, by considering a plate infinitely long in the z-direction, and the motion is taken to be simple harmonic in time. We use the assumptions of an

inviscid, incompressible fluid, and an irrotational motion, to allow the introduction of a velocity potential $\Re\{\phi(x,y)e^{-i\omega t}\}$ to describe the small fluid motions. The conditions to be satisfied by $\phi(x,y)$ are

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi(x,y) = 0 \quad \text{in the fluid,} \tag{1}$$

along with the free-surface condition

$$K\phi + \frac{\partial\phi}{\partial y} = 0 \quad \text{on } y = 0, \tag{2}$$

where $K = \omega^2/g$ and g is the acceleration due to gravity. On the plate, the normal velocity vanishes, that is

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } \Gamma, \tag{3}$$

where in general, Γ is a finite, simple, smooth arc. The choice of a linear theory of water waves enables us to split the potential into two parts,

$$\phi = \phi_{sc} + \phi_{inc}, \tag{4}$$

where ϕ_{inc} is the known incident potential and ϕ_{sc} is the scattered potential. Reformulated in terms of ϕ_{sc}, the boundary-value problem now reads

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi_{sc} = 0 \quad \text{in the fluid,} \tag{5}$$

$$K\phi_{sc} + \frac{\partial\phi_{sc}}{\partial y} = 0 \quad \text{on } y = 0, \tag{6}$$

$$\frac{\partial\phi_{sc}}{\partial n} = -\frac{\partial\phi_{inc}}{\partial n} \quad \text{on } \Gamma, \tag{7}$$

The fact that ϕ_{sc} is due to the presence of the plate indicates the need for a radiation condition on ϕ_{sc}, that waves travel outwards towards infinity. Mathematically, this may be written as

$$\frac{\partial\phi_{sc}}{\partial r} - iK\phi_{sc} \to 0 \quad \text{as } r = (x^2 + y^2)^{\frac{1}{2}} \to \infty, \tag{8}$$

In the sequel, we use capital letters P, Q to denote points in the fluid, and lower-case letters p, q to denote points on the plate Γ.

3 GREEN'S THEOREM AND THE INTEGRAL EQUATION

We formulate the problem as an integral equation by choosing an appropriate fundamental solution with an application of Green's theorem. We use the fundamental solution

$$G(x,y;\xi,\eta) = \frac{1}{2}\log\frac{(x-\xi)^2 + (y-\eta)^2}{(x-\xi)^2 + (y+\eta)^2} - 2\int_0^\infty e^{-k(y+\eta)}\cos k(x-\xi)\frac{dk}{k-K} \tag{9}$$

which satisfies eqns (5) and (6); G has a logarithmic source singularity at the point $(x,y) = (\xi,\eta)$; the integration path is indented below the pole of the integrand at $k = K$ so that G also satisfies the radiation condition at infinity. Applying Green's theorem to $\phi_{sc}(P)$ and $G(x,y;\xi,\eta) \equiv G(P,Q)$, we find

$$\phi_{sc}(P) = \frac{1}{2\pi}\int_\Gamma [\phi_{sc}(q)]\frac{\partial G(P,q)}{\partial n_q}ds_q, \tag{10}$$

where P is any point in the fluid, $\partial/\partial n_q$ represents normal differentiation at q on Γ, and $[\phi_{sc}(q)]$ is the discontinuity in the scattered potential across the plate at the point q. Since $[\phi_{inc}] = 0$, we can replace $[\phi_{sc}]$ in eqn (10) by $[\phi]$. The potential defined by eqn (10) satisfies eqns (5), (6) and (8). We finally need to impose the boundary condition on the plate. We find that

$$\frac{1}{2\pi}\frac{\partial}{\partial n_p}\int_\Gamma [\phi(q)]\frac{\partial G(p,q)}{\partial n_q}ds_q = -\frac{\partial\phi_{inc}}{\partial n_p}, \quad p \in \Gamma. \tag{11}$$

This is an integro-differential equation for $[\phi(q)]$, $q \in \Gamma$. It is to be solved subject to the conditions

$$[\phi] = 0 \quad \text{at the two edges of } \Gamma; \tag{12}$$

physically, because the plate is completely submerged, we expect the discontinuity in pressure across the plate to tend to zero as we approach each edge of the plate.

Let us now interchange the order of integration and normal differentiation at p in eqn (11). Although this leads to a non-integrable integrand, Martin and Rizzo[16] have shown that this procedure is legitimate, provided that the integral is then interpreted as a finite-part integral. By adopting this procedure, we find

$$\frac{1}{2\pi}\oint_\Gamma [\phi(q)]\frac{\partial}{\partial n_p\partial n_q}G(p,q)ds_q = -\frac{\partial\phi_{inc}}{\partial n_p}, \quad p \in \Gamma. \tag{13}$$

which is to be solved for $[\phi]$, subject to eqn (12). The cross on the integral sign indicates that it is to be interpreted as a two-sided finite-part integral of order two. The definition of such integrals is given in the Appendix.

Hypersingular integral equations, such as eqn (13), are unfamiliar. However, they arise naturally for many problems involving thin bodies upon which a Neumann boundary condition is imposed. They can be treated directly (as herein), or they can be rewritten in a more familiar form by a process of regularisation; this may lead to a different integro-differential equation or to an equation involving tangential derivatives of $[\phi]$ (which are singular at the edges of Γ). However, the hypersingular integral equation (13) is quite general: it is valid for water of constant finite depth and in three dimensions, with an appropriate choice for G. Therefore, it is worthwhile to develop general methods for their treatment, rather than relying on special methods that only work for special geometries.

4 THE KERNEL

It is convenient to first find a general expression for the kernel of eqn (13), which may then be used for any choice of Γ. We do this by denoting the unit normals at p and $q \in \Gamma$ by $\mathbf{n}(p) = (n_1^p, n_2^p)$ and $\mathbf{n}(q) = (n_1^q, n_2^q)$, respectively, and applying the formula

$$\frac{\partial^2 G}{\partial n_p \partial n_q} = n_1^p n_1^q \frac{\partial^2 G}{\partial x \partial \xi} + n_1^p n_2^q \frac{\partial^2 G}{\partial x \partial \eta}$$
$$+ n_2^p n_1^q \frac{\partial^2 G}{\partial y \partial \xi} + n_2^p n_2^q \frac{\partial^2 G}{\partial y \partial \eta}. \qquad (14)$$

After differentiating, and rearranging, we find that

$$\frac{\partial^2 G}{\partial n_p \partial n_q} = -\frac{\mathcal{N}}{R^2} + \frac{2\Theta}{R^4} + 2(n_1^p n_2^q - n_2^p n_1^q)$$
$$\times \left\{ K \frac{\partial \Phi_0}{\partial X} - \frac{XY}{(X^2 + Y^2)^2} \right\} - \mathcal{N}\mathcal{K}(X, Y) \qquad (15)$$

where

$$\mathcal{K}(X, Y) = \frac{Y^2 - X^2}{(X^2 + Y^2)^2} + \frac{2KY}{X^2 + Y^2} + 2K^2 \Phi_0(X, Y)$$

$$\Phi_0(X, Y) = \int_0^\infty e^{-kY} \cos kX \frac{dk}{k - K} \qquad (16)$$

$X = x - \xi$, $Y = y + \eta$, $\mathcal{N} = \mathbf{n}(p) \cdot \mathbf{n}(q)$, $\Theta = (\mathbf{n}(p) \cdot \mathbf{R}) (\mathbf{n}(q) \cdot \mathbf{R})$, $\mathbf{R} = (x - \xi, y - \eta)$ and $R = |\mathbf{R}|$. (Equation (15) corrects a formula given by Martin;[15] the error is only significant for non-flat plates). We note that Φ_0 can be computed using an expansion derived by Yu and Ursell:[17]

$$\Phi_0(X, Y) = -e^{-KY}\{(\log KR - i\pi + \gamma) \cos KX$$
$$+ \beta \sin KX\} + \sum_{m=1}^{\infty} \frac{(-KR)^m}{m!}$$
$$\times \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{m} \right) \cos m\beta \qquad (17)$$

where $\gamma = 0.5772\ldots$ is Euler's constant and β is defined by $X = R \sin \beta$ and $Y = R \cos \beta$.

5 METHOD OF SOLUTION

An appropriate way of tackling eqn (13) is to define the plate in terms of a single parameter, thereby reducing the problem to the solution of a one-dimensional hypersingular integral equation. In this paper, we consider only flat plates of length $2a$, inclined at an arbitrary angle α to the vertical. Henceforth, we take $a = 1$ without any loss of generality. A suitable parametrisation of the plate is given by

$$\xi(t) = t \sin \alpha \qquad \eta(t) = d + t \cos \alpha \qquad -1 \leq t \leq 1$$

where $q \equiv (\xi, \eta)$ and $|\alpha| \leq \pi/2$; d is the submergence of the mid-point of the plate, and satisfies $d > \cos \alpha$ to ensure that the plate is completely submerged. The point $p \equiv (x, y)$ on Γ has the same parametrisation, but with t replaced by s. It follows that $\mathbf{n}(p) = \mathbf{n}(q) = (-\cos \alpha, \sin \alpha)$, and then eqn (15) simplifies to

$$\frac{\partial^2 G}{\partial n_p \partial n_q} = -\frac{1}{(s - t)^2} - \mathcal{K}(X, Y). \qquad (18)$$

If we introduce the new unknown function $f(t) = [\phi(q(t))]$, representing the discontinuity in ϕ across the plate at the point q, eqn (13) becomes

$$\oint_{-1}^{1} \frac{f(t)}{(s - t)^2} dt + \int_{-1}^{1} f(t) \mathcal{K}(X, Y) dt = h(s),$$
$$-1 < s < 1, \qquad (19)$$

where

$$h(s) = 2\pi K \exp[-K(d + s \cos \alpha) + i(Ks \sin \alpha - \alpha - \pi/2)] \qquad (20)$$

$X = (s - t) \sin \alpha$ and $Y = (s + t) \cos \alpha + 2d$. The function $h(s)$ is appropriate when

$$\phi_{\text{inc}} = e^{-Ky + iKx} \qquad (21)$$

corresponding to an incident wave travelling towards $x = +\infty$.

Note that we have isolated the hypersingular part in the first term on the left-hand side of eqn (19); the other term is an ordinary non-singular integral. Using the relationship of eqn (A3) in the Appendix, we can also write eqn (19) as

$$-\frac{d}{ds} \int_{-1}^{1} \frac{f(t)}{s - t} dt + \int_{-1}^{1} f(t) \mathcal{K}(X, Y) dt = h(s),$$
$$-1 < s < 1, \qquad (22)$$

which is another integro-differential equation for $f \equiv [\phi]$. In fact, for *flat* plates, we can derive eqn (22) directly. Write $G = G_0 + G_1$, where $G_0 = \frac{1}{2} \log\{(x - \xi)^2 + (y - \eta)^2\}$ and $G_1 = G - G_0$. Then, the left-hand side of eqn (11) is proportional to

$$\frac{\partial}{\partial n_p} \int_\Gamma [\phi] \frac{\partial G_0}{\partial n_q} ds_q + \int_\Gamma [\phi] \frac{\partial^2 G_1}{\partial n_p \partial n_q} ds_q.$$

Formally, the first term is

$$-\int_\Gamma [\phi] \frac{\partial^2 G_0}{\partial n_q^2} ds_q = \int_\Gamma [\phi] \frac{\partial^2 G_0}{\partial t_q^2} ds_q = -\frac{\partial}{\partial t_p} \int_\Gamma [\phi] \frac{\partial G_0}{\partial t_q} ds_q$$

where $\partial/\partial t_q$ denotes tangential differentiation at q, and we have used Laplace's equation. It is difficult to justify this formula, and it does not generalise to curved plates.

Let us return to eqn (19) (or eqn (22)). It can be shown[15] that any solution of eqn (19) that satisfies $f(\pm 1) = 0$ actually behaves as $f(t) \sim \sqrt{1 \mp t} f_\pm$ as $t \to \pm 1$, where $f_\pm$ are constants. We build this into

our numerical procedure for solving eqn (19) by first writing

$$f(t) = \sqrt{1 - t^2}\, g(t). \tag{23}$$

This ensures that the edge conditions are satisfied for any bounded function g. We then consider the hypersingular part of eqn (19), that is we look at

$$\oint_{-1}^{1} \frac{\sqrt{1 - t^2}\, g(t)}{(s - t)^2}\, dt = h(s). \tag{24}$$

This is known as the dominant equation. The magnitude of the hypersingular part of eqn (19) will, in general, be larger than the other integral in that equation. For this reason, and because the hypersingular requirement of eqn (19) is hard to enforce using standard quadrature methods, we look at some exact solutions of the dominant equation. In particular, we consider the case where $g(t)$ is a Chebyshev polynomial of the second kind, U_n, defined by

$$U_n(\cos\theta) = \frac{\sin(n + 1)\theta}{\sin\theta}. \tag{25}$$

If we combine formula (22.13.4) from Abramowitz and Stegun,[18] namely

$$\int_{-1}^{1} \frac{\sqrt{1 - t^2}\, U_n(t)}{s - t}\, dt = \pi T_{n+1}(s),$$

where T_n is a Chebyshev polynomial of the first kind, with eqn (A3), we obtain

$$\oint_{-1}^{1} \frac{\sqrt{1 - t^2}\, U_n(t)}{(s - t)^2}\, dt = -\pi(n + 1)U_n(s) \tag{26}$$

whereby $g(t) = U_n(t)$ is the exact bounded solution of eqn (24) when $h(s) = -\pi(n + 1)U_n(s)$. Since the Chebyshev polynomials of the second kind form a complete set over the interval $[-1, 1]$, we approximate $g(t)$ by a series,

$$g(t) \cong \sum_{n=0}^{N} a_n U_n(t) \equiv g_N(t) \tag{27}$$

say, where N is finite and the unknown coefficients a_n are to be found. Substituting eqn (27) into eqn (19), we find

$$\sum_{n=0}^{N} a_n A_n(s) = h(s) \qquad -1 < s < 1 \tag{28}$$

where

$$A_n(s) = -\pi(n + 1)U_n(s) + \int_{-1}^{1} \sqrt{1 - t^2}\, U_n(t)\mathcal{K}(X, Y)dt \tag{29}$$

and $h(s)$ is defined by eqn (20). To find the unknown coefficients, we choose a family of functions $\psi_j(s)$, where $j = 0, 1, \ldots N$, called trial functions. Multiplying both sides of eqn (28) by $\psi_j(s)$ and integrating from -1 to 1 leads to the Petrov–Galerkin system

$$A\mathbf{a} = \mathbf{h} \tag{30}$$

where

$$A_{jn} = \int_{-1}^{1} A_n(s)\psi_j(s)ds$$

$$h_j = \int_{-1}^{1} h(s)\psi_j(s)ds.$$

One choice for $\psi_j(s)$ is $\sqrt{1 - s^2}\, U_j(s)$, leading to a classical Galerkin method; Golberg[13,14] has analysed the convergence of this method. A more pragmatic choice, which avoids double integrals, is $\psi_j(s) = \delta(s - s_j)$, where $s_j (j = 0, 1, \ldots, N)$ are points, with $|s_j| < 1$. This yields

$$\sum_{n=0}^{N} a_n A_n(s_j) = h(s_j) \qquad j = 0, 1, \ldots, N \tag{31}$$

which is a straightforward collocation scheme, with collocation points s_j. A suitable set of collocation points is

$$s_j = \cos\left(\frac{(j + 1)\pi}{N + 2}\right) \qquad j = 0, 1, \ldots, N \tag{32}$$

these being the zeros of $U_{N+1}(s)$. This is expected to be a good choice, since, if the coefficients a_n are decaying rapidly, the error in eqn (27) is roughly proportional to $U_{N+1}(s)$; see p. 228 of Atkinson.[19] Another possible choice is

$$s_j = \cos\left(\frac{(2j + 1)\pi}{2N + 2}\right) \qquad j = 0, 1, \ldots, N \tag{33}$$

these being the zeros of $T_{N+1}(s)$. Golberg[13,14] has shown that eqns (32) and (33) are both good choices: he has proved that they both yield uniformly-convergent methods,

$$\max_{-1 \leq t \leq 1} |g(t) - g_N(t)| \to 0 \qquad \text{as } N \to \infty.$$

The rate of convergence depends on the smoothness of the kernel $\mathcal{K}$ in eqn (19) and is exponential here, since, for submerged flat plates, $\mathcal{K}$ is infinitely differentiable. In all our numerical computations, we have used the set given by eqn (33).

The problem now requires evaluation of the non-singular integrals of $A_n(s_j)$. For computational purposes, we substitute $\Phi_0(X, Y)$ from eqn (17) into eqn (29). If we consider a vertical plate ($\alpha = 0$), we find that the simplifications enable all but one of the seven integrals arising in that problem to be evaluated analytically. For a plate at any other inclination, however, the symmetry of the problem is lost and the problem becomes more difficult.

The reasons for wishing to evaluate the integrals analytically are less to do with accuracy, but rather the quicker computation achieved with a closed formula, as opposed to using some quadrature scheme, such as Simpson's rule. In practice, it was found that the computer time needed for a vertical plate, in which, as already stated, the integrals were found analytically, was much less than for the equivalent method using

Simpson's rule. Unfortunately, for a plate with a general angle of inclination, it appears that the only straight-forward integration is

$$\int_{-1}^{1} \sqrt{1-t^2}\, U_n(t) \exp\left[-K(s_j + t)\cos\alpha\right]$$

$$\times \cos\left[K(s_j - t)\sin\alpha\right] dt$$

$$= \Re\left\{ \exp(-Ks_j e^{-i\alpha}) \int_{-1}^{1} \sqrt{1-t^2}\, U_n(t) \right.$$

$$\left. \times \exp(-Kt\, e^{i\alpha})\, dt \right\}$$

$$= \Re\{\exp(-Ks_j e^{-i\alpha})(L_n - L_{n+2})\} \qquad (34)$$

where

$$L_n = (\pi/2)(-1)^n I_n(Ke^{i\alpha}) \qquad (35)$$

and I_n is a modified Bessel function; here, we have used eqn (25) and a standard integral representation for I_n. We note that eqn (34) represents the only imaginary entry of the matrix **A**.

6 REFLECTION AND TRANSMISSION COEFFICIENTS

When an incident wave is scattered by a fixed body, some of the wave energy will be transmitted past the body, and some will be reflected back. To quantify this, we introduce the complex numbers $\mathscr{R}$ and $\mathscr{T}$, which are known as the reflection and transmission coefficients, respectively. The magnitudes of $\mathscr{R}$ and $\mathscr{T}$ are related to the amplitude of the reflected and transmitted waves, respectively. Similarly, the arguments of $\mathscr{R}$ and $\mathscr{T}$ correspond to a phase shift in the scattered waves. For an incident wave given by eqn (21), $\mathscr{R}$ and $\mathscr{T}$ are defined by the asymptotic behaviour of ϕ as $|x| \to \infty$. More precisely, we have

$$\phi(x, y) \sim \begin{cases} \mathscr{T}\, e^{-Ky+iKx} & \text{as } x \to +\infty \\ \mathscr{R}\, e^{-Ky-iKx} & \text{as } x \to -\infty \end{cases} \qquad (36)$$

From eqn (4), we can define $\mathscr{R}$ and $\mathscr{T}$ solely in terms of the scattered potential,

$$\phi_{sc}(x, y) \sim \begin{cases} (\mathscr{T} - 1)\, e^{-Ky+iKx} & \text{as } x \to +\infty \\ \mathscr{R}\, e^{-Ky-iKx} & \text{as } x \to -\infty. \end{cases} \qquad (37)$$

Now, ϕ_{sc} is given by eqn (10). Since

$$G(x, y; \xi, \eta) \sim -2\pi i e^{-K(y+\eta)\pm iK(x-\xi)} \quad \text{as } x \to \pm\infty \qquad (38)$$

the integral representation of eqn (10) gives

$$\phi_{sc}(x, y) \sim -i e^{-Ky \pm iKx} \int_{\Gamma} [\phi(\xi, \eta)] \frac{\partial}{\partial n_q} e^{-K\eta \pm iK\xi} ds_q$$

$$\text{as } x \to \pm\infty. \qquad (39)$$

Simple comparison of eqns (37) and (39) now yields the formulae

$$\mathscr{R} = -i \int_{\Gamma} [\phi(q)] \frac{\partial}{\partial n_q} e^{-K\eta + iK\xi}\, ds_q \qquad (40)$$

and

$$\mathscr{T} - 1 = -i \int_{\Gamma} [\phi(q)] \frac{\partial}{\partial n_q} e^{-K\eta - iK\xi} ds_q. \qquad (41)$$

From these formulae, we see that once the discontinuity in ϕ across the plate has been found, the values of $\mathscr{R}$ and $\mathscr{T}$ may be found directly, without having to find ϕ_{sc} everywhere in the fluid first. Thus, by substituting eqn (27) and parametrising as before, we find that $\mathscr{R}$ and $\mathscr{T}$ are given by

$$\mathscr{R} = -K\, e^{-Kd-i\alpha} \sum_{n=0}^{N} a_n[\bar{L}_n - \bar{L}_{n+2}] \qquad (42)$$

and

$$\mathscr{T} - 1 = K\, e^{-Kd+i\alpha} \sum_{n=0}^{N} a_n[L_n - L_{n+2}] \qquad (43)$$

respectively, where L_n is defined by eqn (35) and $\bar{L}_n$ is the complex conjugate of L_n. Note that, as the plate submergence d is increased, $\mathscr{R} \to 0$ and $\mathscr{T} \to 1$. This behaviour is to be expected on physical grounds.

It is known that $\mathscr{R}$ and $\mathscr{T}$ satisfy certain relations, for any scatterer. Let $\mathscr{R}_+$ and $\mathscr{T}_+$ be the reflection and transmission coefficients when the incident wave propagates towards $x = +\infty$. Similarly, define $\mathscr{R}_-$ and $\mathscr{T}_-$ for incident waves propagating towards $x = -\infty$. Then

$$\mathscr{T}_+ = \mathscr{T}_- = \mathscr{T}, \quad \text{say}, \qquad (44)$$

$$|\mathscr{R}_+| = |\mathscr{R}_-| = |\mathscr{R}|, \quad \text{say}, \qquad (45)$$

$$|\mathscr{R}|^2 + |\mathscr{T}|^2 = 1 \qquad (46)$$

and

$$\mathscr{T}\bar{\mathscr{R}}_+ + \bar{\mathscr{T}}\mathscr{R}_- = 0. \qquad (47)$$

These relationships are well known,[20] and can be used as an independent check on the method of solution employed.

7 RESULTS

Figure 1 shows graphs of $|\mathscr{R}|$, for a vertical plate, plotted against Kb, where K is the wavenumber and $b = d + a$ is the distance from the undisturbed free surface to the lowest point on the plate. Graphs are given for three values of $\mu = (d - a)/(d + a)$, the submergence ratio of the top edge of the plate to the bottom edge; this ratio occurs in the exact analytic solution obtained by Evans.[5] Comparison of our results (with $N = 15$) with his shows excellent agreement for all values of μ shown. However, as the plate approaches the free surface ($\mu \to 0$) more
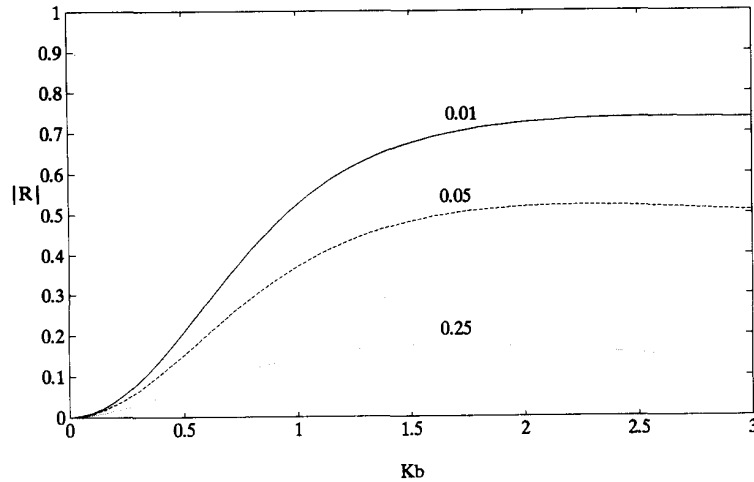
Fig. 1. Reflection coefficient for a vertical plate, for various μ.

polynomials need to be used to ensure convergence until finally the method used becomes unsuitable in the form used.

For the case of a horizontal plate, we have an extra parameter to consider, in that the wavelength of the waves passing above the plate will be different from those on either side of the plate. Figure 2 shows graphs of $|\mathscr{R}|$ plotted against $2a/\lambda'$, where $2a$ is the length of the plate and λ' is the wavelength above the plate, for various values of d/λ; here d is the submergence of the plate and λ is the wavelength of the incident waves. Although Patarapanich's results[8] are for a horizontal plate in water of finite depth, he does give results for the case in which the depth of the water is twice the wavelength of the incident waves, which he calls 'deep' water inasmuch as the effect of the lower boundary on the motion of the fluid is assumed to be small. The graph in Fig. 2 for which $d/\lambda = 0.15$ has been chosen specifically for comparison with the deep-water graphs

given by Patarapanich[8] in his Figure 4. We see very similar behaviour to his results, except for a slight shift towards the origin of the peaks and troughs of $|\mathscr{R}|$, with a corresponding reduction in the maximum value of $|\mathscr{R}|$ at the peaks. This discrepancy is possibly due to finite-depth effects. We also found that our results agree with those obtained by McIver.[9] We remark that the main feature of the results for horizontal plates is the occurrence of zeros of the reflection coefficient as a function of frequency.

Figure 3 contains the results (with $N = 15$) for a plate inclined at an angle of $\pi/4$ to the vertical, and three depths of submergence. (Results have also been obtained for a plate inclined at an angle of $-\pi/4$ to the vertical; the relationships (44)–(47) were found to be well satisfied.) The behaviour of $|\mathscr{R}|$ is seen to resemble, somewhat, the behaviour of $|\mathscr{R}|$ for a vertical plate; not surprisingly, $|\mathscr{R}|$ reduces with depth of submergence, and approaches zero as the waves become shorter.
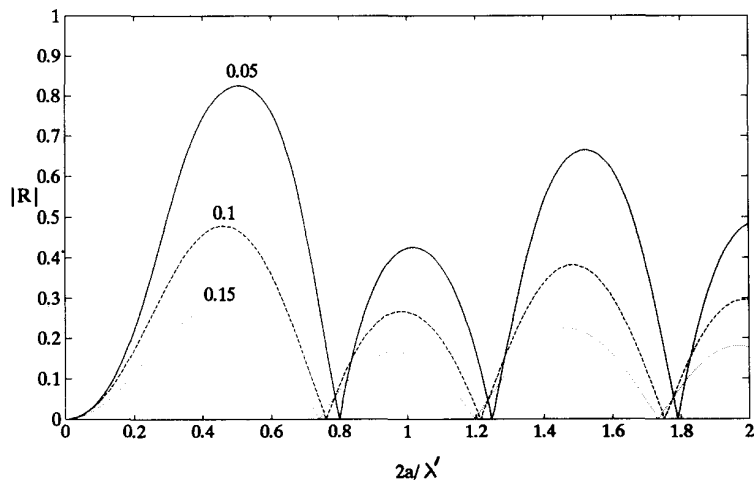

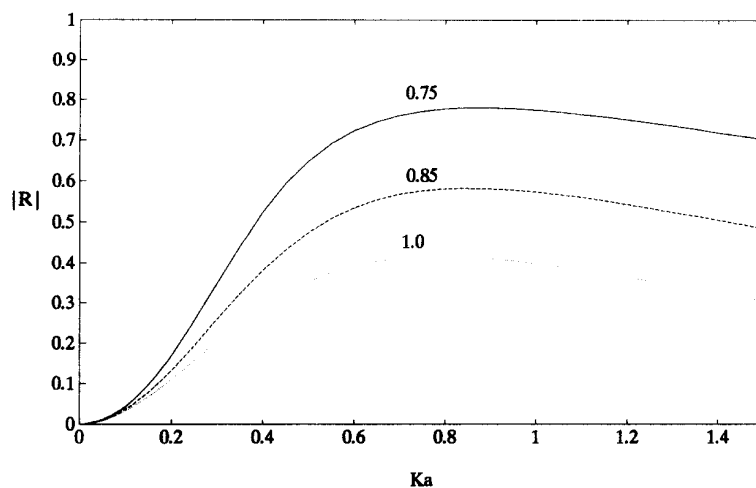**Fig. 2.** Reflection coefficient for a horizontal plate, for various d/λ.

Fig. 3. Reflection coefficient for a plate inclined at 45°, for various submergence depths d/a.
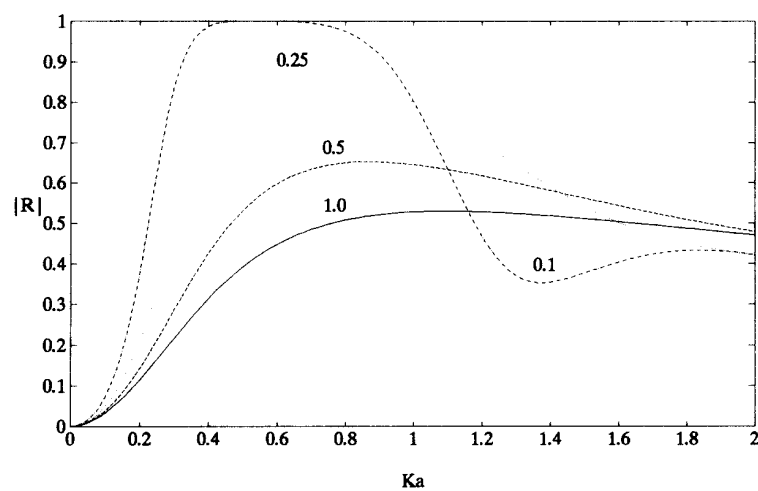

Fig. 4. Reflection coefficient for an inclined plate whose top edge is at a fixed depth below the free surface, for various inclinations δ.
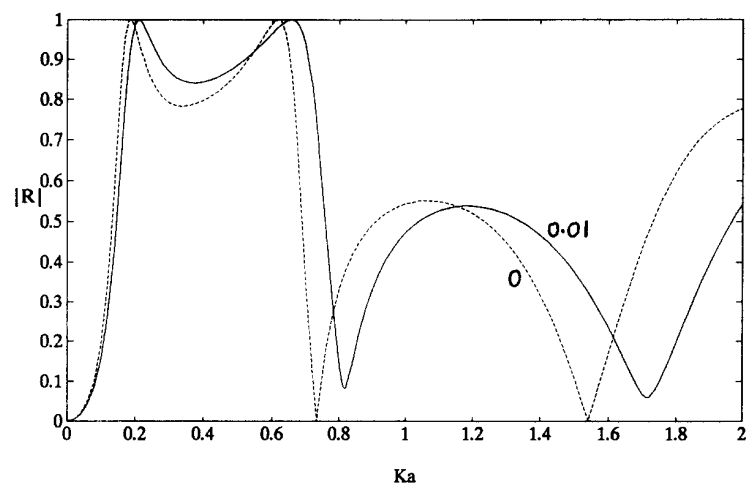

Fig. 5. Reflection coefficient for a horizontal plate ($\delta = 0$) and for a nearly-horizontal plate ($\delta = 0.01$).

Let $\delta = 1 - 2\alpha/\pi$, so that $\delta = 0$ corresponds to a horizontal plate (Fig. 2) and $\delta = 1$ corresponds to a vertical plate (Fig. 1). Figure 4 contains the results (with $N = 16$) for four values of δ and one depth of submergence, the latter chosen so that the upper plate edge is at a distance c below the free surface and $c/a = 0.1$ (whence $d = c + a\cos\alpha$). From these curves, we can begin to see the transition between vertical and horizontal plates. In particular, for the smallest value of δ shown, namely $\delta = 0.1$ ($\alpha = 81°$), $|\mathscr{R}| \simeq 1$ for a substantial range of wavelengths, $0.4 < Ka < 0.7$, and then has a local minimum at $Ka \simeq 1.4$. In Fig. 5, we give results for smaller δ, namely $\delta = 0.01$ ($\alpha = 89.1°$) and $\delta = 0$. These show a shift to the right as δ increases, resulting in large changes in $|\mathscr{R}|$ near the zeros of $\mathscr{R}$ for $\delta = 0$. Moreover, it seems that these zeros for $\delta = 0$ disappear as soon as δ becomes positive (although we do not have a proof of this result). Thus, these zeros for horizontal plates probably cannot be exploited in practice, for they are destroyed by small changes in the angle of inclination. Finally, we note that it may be possible to analyse the problem for small δ using a regular perturbation scheme, as used by Shaw[7] for small α, but we have not explored this avenue.

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APPENDIX

Let f be a Hölder-continuous function, $f \in C^{0,\beta}$. Then, the Cauchy principal-value integral of f is defined by

$$\int_{-1}^{1} \frac{f(t)}{s - t} dt = \lim_{\epsilon \to 0} \left\{ \int_{-1}^{s-\epsilon} \frac{f(t)}{s - t} dt + \int_{s+\epsilon}^{1} \frac{f(t)}{s - t} dt \right\}. \quad (A1)$$

Suppose that f is smoother, so that $f' \in C^{0,\beta}$, that is, $f \in C^{1,\beta}$. Then, we can define the two-sided Hadamard finite-part integral of order two by

$$\int_{-1}^{1} \frac{f(t)}{(s - t)^2} dt = \lim_{\epsilon \to 0} \left\{ \int_{-1}^{s-\epsilon} \frac{f(t)}{(s - t)^2} dt + \int_{s+\epsilon}^{1} \frac{f(t)}{(s - t)^2} dt - \frac{2f(s)}{\epsilon} \right\}. \quad (A2)$$

If $f \in C^{1,\beta}$, these two integrals are related by

$$-\frac{d}{ds} \int_{-1}^{1} \frac{f(t)}{s - t} dt = \int_{-1}^{1} \frac{f(t)}{(s - t)^2} dt. \quad (A3)$$