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Boundary Integral Equations for the Scattering of Elastic Waves by Elastic Inclusions with Thin Interface Layers

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Elastic waves are scattered by an elastic inclusion. The interface between the inclusion and the surrounding material is imperfect: the displacement and traction vectors on one side of the interface are assumed to be linearly related to both the displacement vector and the traction vector on the other side of the interface. The literature on such inclusion problems is reviewed, with special emphasis on the development of interface conditions modeling different types of interface layer. Inclusion problems are formulated mathematically, and uniqueness theorems are proved. Finally, various systems of boundary integral equations over the interface are derived.

KEY WORDS: Elastic waves; interface layers; inclusions; boundary integral equations; uniqueness.

1. INTRODUCTION

Consider a bounded obstacle embedded in an unbounded solid. Both the obstacle (the "inclusion") and the surrounding solid (the "matrix") are composed of homogeneous, isotropic elastic materials. We consider the scattering of elastic waves by the inclusion. For small time-harmonic oscillations, this leads to a vector transmission problem, which we call the *inclusion problem*, in which conditions are specified on the smooth interface, S, between the matrix and the inclusion.

Usually, the matrix and the inclusion are assumed to be welded together, i.e., the displacement and traction vectors are both continuous across S, which is then called a *perfect interface*. The opposite extreme is when there is no interaction ("complete debonding"). Intermediate situations arise when the two solids can slip or separate, or when there is a thin layer of a different material (such as glue or lubricant) between the solids. In this paper, we are especially interested in those intermediate situations that can be modeled by simple linear modifications to the perfect-interface continuity conditions.

To begin, we step back and consider *plane* interfaces between two elastic solids (Section 2). We give a brief review of the literature (updating an earlier review),[1] showing the development of various model interfaces. For example, one model assumes that the discontinuity in the displacement (traction) vector across the interface depends linearly on the average of the traction (displacement) vectors on the two sides of the interface (see (7) below). Another assumes that the displacements and tractions on one side of the interface are related to both the displacements and the tractions on the other side of the interface (see (8) below). These two models include most of the phenomenological models of imperfect interfaces in the literature. They are used subsequently for problems with bounded inclusions. Such problems are formulated in Section 3 and relevant literature is reviewed in Section 4.

In Section 5, we consider the question of *uniqueness*: given an interface model, does the inclusion problem have at most one solution? Conditions are found which are sufficient to guarantee uniqueness. Some models suffer from non-uniqueness; an example is the "lubricated interface" (see (13) below), for which Jones frequencies might occur (Section 5.2).

In Section 6, we describe methods for reducing in-

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clusion problems to boundary integral equations over the interface S. We limit ourselves to singular integral equations (with Cauchy principal-value integrals), as these are amenable to standard boundary-element methods. However, our aim is also to derive systems of integral equations for which a theoretical framework can be given, so as to establish solvability. These mathematical aspects will be discussed elsewhere.

2. IMPERFECT PLANE INTERFACES: A REVIEW

Consider a plane interface $x_3 = z = 0$ between two elastic solids. Let $u_i^{\pm}$ and $\tau_{ij}^{\pm}$ be the components of the displacement vector and stress tensor, respectively, in $\pm z > 0$. The traction vectors on $z = 0$ are given by $t_i^{\pm} = \tau_{i3}^{\pm}$. If $z = 0$ is a perfect interface, we have

$$[\mathbf{t}] = \mathbf{0} \text{ and } [\mathbf{u}] = \mathbf{0} \qquad (1)$$

where square brackets denote discontinuities across the interface:

$$[\mathbf{t}] = \mathbf{t}^+ - \mathbf{t}^- \quad \text{and} \quad [\mathbf{u}] = \mathbf{u}^+ - \mathbf{u}^-, \quad \text{evaluated on } z = 0.$$

The perfect-interface conditions (1) were first modified by Newmark in 1943.[2] He was concerned with the transmission of static loads between straight beams, and explicitly allowed slipping to occur, replacing (1) by

$$[\mathbf{t}] = \mathbf{0}, \quad [u_3] = 0 \quad \text{and} \quad [u_\alpha] = Mt_\alpha \qquad (2)$$

where $\alpha = 1$ or 2, and M is a positive constant. Note that $M = 0$ corresponds to a perfect interface, whereas $M = \infty$ corresponds to a "lubricated interface," i.e., one where there is a thin layer of inviscid fluid between the two solid.

Newmark's theory is for two beams in two dimensions, but it has been generalized to

$$[\mathbf{t}] = \mathbf{0}, \quad [u_3] = 0 \quad \text{and} \quad [u_\alpha] = \sum_{\beta=1}^{2} M_{\alpha\beta} t_\beta \qquad (3)$$

where $M_{\alpha\beta}$ can be a function of t_γ and $[u_\gamma]$ (giving a nonlinear model). Toledano and Murakami[3] have reviewed applications of the model (3); usually, $M_{\alpha\beta}$ is assumed to be a positive diagonal matrix.

Similar boundary conditions have been used by Murty[4] to model the propagation of waves through a "loosely-bonded interface." Banghar *et al.*[5] have derived the conditions (2) by assuming that there is a thin interface layer of viscous fluid. Now, the parameter M is given by

$$M = \frac{ih}{\omega\eta} \qquad (4)$$

where h is the thickness of the layer, η is the shear viscosity coefficient of the fluid, and a harmonic time-dependence of $e^{-i\omega t}$ is implied.

Jones and Whittier[6] have modeled wave propagation through a "flexibly-bonded interface" by allowing both slip and separation. They replaced (1) with

$$[\mathbf{t}] = \mathbf{0} \quad \text{and} \quad [\mathbf{u}] = F.\mathbf{t} \qquad (5)$$

where $F = \text{diag}\,\{M_s, M_s, M_n\}$ is a constant diagonal matrix. Several authors have used (5).[7–14] Mal and Xu[11] and Pilarski and Rose[12] have given the formulae

$$M_s = \frac{h}{\mu} \quad \text{and} \quad M_n = \frac{h}{\lambda + 2\mu} \qquad (6)$$

where λ and μ are the Lamé moduli of a thin elastic layer modeling the bond. Other formulae, for thin layers made of other materials, have been given by Persson and Olsson.[13] Klarbring[15] has derived (5) and (6) for a plane interface between two *bounded* elastic bodies.

Baik and Thompson[16,17] have argued that inertial effects should be included. Thus, they replaced (5) with

$$[\mathbf{t}] = G.\langle\mathbf{u}\rangle \quad \text{and} \quad [\mathbf{u}] = F.\langle\mathbf{t}\rangle \qquad (7)$$

where angled brackets denote (vector) averages across the interface:

$$\langle\mathbf{t}\rangle = \frac{1}{2}(\mathbf{t}^+ + \mathbf{t}^-) \quad \text{and}$$
$$\langle\mathbf{u}\rangle = \frac{1}{2}(\mathbf{u}^+ \pm \mathbf{u}^-) \quad \text{evaluated on } z = 0.$$

The matrix G was assumed to be diagonal with negative real elements. We call (7) the Baik–Thompson model.

It is well known that the problem of wave propagation through a layer of viscoelastic solid between two semi-infinite elastic solids can be analyzed exactly.[18] However, the extraction of approximate conditions, connecting the displacements and tractions across a thin layer, has not usually been the aim of such analyses.[19] A recent exception is the paper by Rokhlin and Wang.[20] They start with the plane problem for incident plane waves, and then approximate its known exact solution, assuming that the wavelength in the layer is much greater than h. This results in interface conditions of the form

$$[\mathbf{t}] = G.\mathbf{u}^- + B.\mathbf{t}^- \text{ and } [\mathbf{u}] = F.\mathbf{t}^- + A.\mathbf{u}^- \qquad (8)$$

where A, B, F, and G are 2×2 matrices: F and G are real diagonal matrices, and $A = B^T$ (the transpose of B) is skew-diagonal with negative imaginary elements; in particular, $F = \text{diag}\,\{M_s, M_n\}$, where M_s and M_n are given by (6). We call (8) the Rokhlin–Wang model. Rokhlin and Wang[20] show further that, in certain circumstances, the coupling terms embodied in A and B can be neglected, leaving

$$[\mathbf{t}] = G.\mathbf{u}^- \quad \text{and} \quad [\mathbf{u}] = F.\mathbf{t}^- \tag{9}$$

which is similar to the Baik–Thompson model, (7). Finally, neglecting inertial effects is equivalent to setting $G = 0$, whence (9) reduces to (5).

3. INCLUSIONS PROBLEMS

Let B_i denote a bounded domain, with a smooth closed boundary S and simply-connected exterior, B_e. We seek displacements $\mathbf{u}_e(P)$ and $\mathbf{u}_i(P)$ so that

$$L_e\mathbf{u}_e(P) = \mathbf{0}, \quad P \in B_e \quad \text{and}$$
$$L_i\mathbf{u}_i(P) = \mathbf{0}, \quad P \in B_i$$

where

$$\mathbf{u}(P) = \mathbf{u}_e(P) + \mathbf{u}_{\text{inc}}(P) \text{ for } P \in B_e$$

$\mathbf{u}_{\text{inc}}$ is the given incident wave and $\mathbf{u}_e$ satisfies a radiation condition at infinity. In addition, we shall impose certain continuity conditions across S; these are specified below. The operator L_a is defined by

$$L_a\mathbf{u} = k_a^{-2}\,\text{grad div}\,\mathbf{u} - K_a^{-2}\,\text{curl curl}\,\mathbf{u} + \mathbf{u}$$

where the wavenumbers k_a and K_a are defined by

$$\rho_a\omega^2 = (\lambda_a + 2\mu_a)k_a^2 = \mu_a K_a^2 \quad \text{and} \quad a = e \text{ or } i \tag{10}$$

The density of the solid in B_a is ρ_a, λ_a and μ_a are the Lamé moduli, and the time-dependence $e^{-i\omega t}$ is suppressed throughout. The traction operator T_a is defined on S by

$$(T_a\mathbf{u})_m(p) = \lambda_a n_m \frac{\partial u_k}{\partial x_k} + \mu_a n_\ell \left(\frac{\partial u_m}{\partial x_\ell} + \frac{\partial u_\ell}{\partial x_m}\right)$$

where $\mathbf{n}(p)$ is the unit normal at $p \in S$, pointing into B_e.

If S is a perfect interface, we impose

$$[\mathbf{t}] = \mathbf{0} \quad \text{and} \quad [\mathbf{u}] = \mathbf{0} \tag{11}$$

where $\mathbf{t} = T_e\mathbf{u}$ and $\mathbf{t}_i = T_i\mathbf{u}_i$ are traction vectors and square brackets denote discontinuities across the interface:

$$[\mathbf{t}] = \mathbf{t} - \mathbf{t}_i \quad \text{and} \quad [\mathbf{u}] = \mathbf{u} - \mathbf{u}_i, \quad \text{evaluated on } S.$$

The corresponding inclusion problem has been studied extensively (see, e.g., Kupradze *et al.*[21] and Mura[22]) Martin[23] has given a simplified treatment of the two-dimensional problem, and further references.

Modifications to the perfect-interface conditions (11), analogous to those described in Section 2, are reviewed next.

4. INCLUSIONS WITH IMPERFECT INTERFACES: A REVIEW

Suppose that the interface S is imperfect. For example, the inclusion might be surrounded by a thin layer of a different elastic material. For simple geometries, such problems can be treated exactly.[24,25] The first approximate treatment, using continuity conditions across S similar to those described in Section 2, was given by Mal and Bose.[26] They considered spherical inclusions with the following interface conditions:

$$[\mathbf{t}] = \mathbf{0}, \quad [u_n] = 0, \quad \text{and} \quad [u_\alpha] = Mt_\alpha. \tag{12}$$

Here, we decompose vectors as

$$\mathbf{u}(p) = u_1\mathbf{s}_1 + u_2\mathbf{s}_2 + u_n\mathbf{n} \text{ for } p \in S$$

where $\mathbf{s}_1$ and $\mathbf{s}_2$ are unit vectors in the tangent plane at p, satisfying $\mathbf{s}_1\cdot\mathbf{s}_2 = 0$ and $\mathbf{n} = \mathbf{s}_1 \times \mathbf{s}_2$. The parameter M can be a complex function of ω: $M = 0$ for a perfect interface; M is given by (4) for a thin layer of viscous fluid; and $M = \infty$ for a lubricated interface, whence

$$[\mathbf{t}] = \mathbf{0}, \quad t_\alpha = 0 \quad [u_n] = 0. \tag{13}$$

Similar interface conditions have been used in models of composite materials, where identical inclusions are arranged periodically prior to analysis using homogenization techniques. Let ϵ be a length scale associated with the periodicity. For elastostatics, Lene and Leguillon[27] used $M = k\epsilon$, where $k > 0$. For time-harmonic waves, Santosa and Symes[28] used $M = i\epsilon/(\omega c)$, where c is a "viscous constant" (Eq. 4).

Aboudi[29,30] has used flexibly-bonded interfaces in a different model of composites, with

$$[\mathbf{t}] = \mathbf{0}, \quad [u_n] = M_n t_n, \quad \text{and} \quad [u_\alpha] = M_t t_\alpha \tag{14}$$

He identified M_n and M_t as h/E and h/μ, respectively, where the thin elastic interface layer has thickness h, Young's modulus E and shear modulus μ (Eq. 6).

Kitahara *et al.*[31] consider an inclusion in "spring contact" with the exterior solid. This is intended to model a thin compliant interface layer and leads to

$$[\mathbf{t}] = \mathbf{0} \text{ and } [\mathbf{u}] = 2F.\mathbf{t} \tag{15}$$

where the matrix F (called the "flexibility matrix")[11] is a given positive diagonal matrix and the factor 2 is inserted for algebraic convenience. Later, we shall allow F to be a full matrix, with elements that vary with position p on S.

Interface conditions of the type (15) have been reviewed by Hashin[32]. For a thin elastic interface layer, he showed that $2F = \mathrm{diag}\{M_s, M_s, M_n\}$, where M_s and M_n are given by (6). The derivation assumes that S is smooth, and that both the inclusion and the matrix are much stiffer than the layer. Hashin[32] has also given similar results for inhomogeneous layers, in which the material properties vary through the layer (see also Datta *et al.*)[33]

More complicated interface conditions were obtained by Datta *et al.*,[34,35] by including all terms to $O(h)$ for an elastic interface layer of thickness h. Nonlocal terms, involving various tangential derivatives, are present (see also Nayfeh and Nassar).[36] The simplest (local) conditions discussed by Datta *et al.*[34,35] are

$$[\mathbf{t}] = -\rho h\omega^2\mathbf{u} \quad \text{and} \quad [\mathbf{u}] = \mathbf{0} \tag{16}$$

where the layer has density ρ. A generalization of (16) is

$$[\mathbf{t}] = 2G.\mathbf{u} \quad \text{and} \quad [\mathbf{u}] = \mathbf{0} \tag{17}$$

where the elements of the given matrix G could vary with position p on S.

As a further generalization, we consider a model that includes both (15) and (17), namely, the Baik–Thompson model Eq. (7).

$$[\mathbf{t}] = 2G.\langle\mathbf{u}\rangle \quad \text{and} \quad [\mathbf{u}] = 2F.\langle\mathbf{t}\rangle \tag{18}$$

where angled brackets denote (vector) average across the interface:

$$\langle\mathbf{t}\rangle = \frac{1}{2}(\mathbf{t} + \mathbf{t}_i) \quad \text{and}$$

$$\langle\mathbf{u}\rangle = \frac{1}{2}(\mathbf{u} + \mathbf{u}_i) \quad \text{evaluated on } S.$$

Finally, we shall also consider the Rokhlin–Wang model (Eq. 8)

$$[\mathbf{t}] = G.\mathbf{u}_i + B.\mathbf{t}_i \quad \text{and} \quad [\mathbf{u}] = F.\mathbf{t}_i + A.\mathbf{u}_i \tag{19}$$

where A and B are given matrices.

5. UNIQUENESS THEOREMS

Consider the problem of scattering by an inclusion with an imperfect interface. We can prove uniqueness theorems for interfaces characterized by (15), (17), or (18); for its generality, we use (18) here, and always assume that all the elements of the matrices F and G are finite. We adapt standard arguments given by Kupradze *et al.*[21] Thus, surround S with a large sphere S_R of radius R. For $P \in B_e$, write

$$\mathbf{u}(P) = \mathbf{u}^{(p)} + \mathbf{u}^{(s)}$$

where

$$\mathbf{u}^{(p)} = -k_e^{-2}\,\mathrm{grad\,div}\,\mathbf{u}, \qquad \mathbf{u}^{(s)} = \mathbf{u} - \mathbf{u}^{(p)}$$

and $\mathbf{u}_{\mathrm{inc}} \equiv 0$. Then, an application of Betti's reciprocal theorem to $\mathbf{u}$ and its complex conjugate, $\bar{\mathbf{u}}$, in the region between S and S_R gives

$$k_e(\lambda_e + 2\mu_e)\lim_{R\to\infty}\int_{S_R}|\mathbf{u}^{(p)}|^2\,ds + K_e\mu_e\lim_{R\to\infty}\int_{S_R}|\mathbf{u}^{(s)}|^2\,ds + J = 0 \tag{20}$$

where

$$J = \frac{1}{2i}\int_S(\mathbf{u}.\bar{\mathbf{t}} - \bar{\mathbf{u}}.\mathbf{t})ds = \mathcal{I}m\int_S \mathbf{u}.\bar{\mathbf{t}}\,ds \tag{21}$$

$\mathcal{I}m$ denotes imaginary part and the radiation condition has been used (see Kupradze *et al.*,[21] Chap. 3, Section 2). Similarly, an application of Betti's theorem in B_i to $\mathbf{u}_i$ and $\bar{\mathbf{u}}_i$ gives

$$0 = \frac{1}{2i}\int_S(\mathbf{u}_i\cdot\bar{\mathbf{t}}_i - \bar{\mathbf{u}}_i\cdot\mathbf{t}_i)ds = \mathcal{I}m\int_S \mathbf{u}_i\cdot\bar{\mathbf{t}}_i\,ds. \tag{22}$$

If we can show that $J \geq 0$, we can deduce from (20) that $\mathbf{u}^{(p)} \equiv \mathbf{0}$ and $\mathbf{u}^{(s)} \equiv \mathbf{0}$, whence $\mathbf{u} \equiv \mathbf{0}$ in B_e; in particular, we have $\mathbf{u} = \mathbf{0}$ and $\mathbf{t} = \mathbf{0}$ on S. Next, we use the given interface conditions. If these imply that $\mathbf{u}_i = \mathbf{0}$ and $\mathbf{t}_i = \mathbf{0}$ on S, we can deduce further that $\mathbf{u}_i \equiv \mathbf{0}$ in B_i, as required. Let us now carry out this program for the Baik–Thompson model and for the Rokhlin–Wang model.

5.1. The Baik–Thompson Model

Consider the Baik–Thompson model (18). Since

$$\mathbf{u}\cdot\bar{\mathbf{t}} - \mathbf{u}_i\cdot\bar{\mathbf{t}}_i = \frac{1}{2}\{(\bar{\mathbf{t}} + \bar{\mathbf{t}}_i)\cdot(\mathbf{u} - \mathbf{u}_i) + (\mathbf{u} + \mathbf{u}_i)\cdot(\bar{\mathbf{t}} - \bar{\mathbf{t}}_i)\}$$

subtracting (22) from (21) gives

$$\frac{1}{2}J = \mathcal{I}m\int_S\{\langle\bar{\mathbf{t}}\rangle\cdot F\cdot\langle\mathbf{t}\rangle + \langle\mathbf{u}\rangle\cdot\langle\overline{G}\cdot\langle\bar{\mathbf{u}}\rangle\}\,ds$$

after using (18). Thus, $J \geq 0$, provided that

$$F_{jk} = \overline{F}_{kj} \text{ for } j \neq k \text{ and } \mathscr{I}m(F_{kk}) \geq 0 \text{ (no sum)} \quad (23)$$

and

$$G_{jk} = \overline{G}_{kj} \text{ for } j \neq k \text{ and } \mathscr{I}m(G_{kk}) \leq 0 \text{ (no sum)} \quad (24)$$

So, if the elements of F and G are finite and satisfy (23) and (24), respectively (for all $p \in S$ if F and G vary with p), we have proved that $\mathbf{u} \equiv \mathbf{0}$ in B_e, whence $\mathbf{u} = \mathbf{0}$ and $\mathbf{t} = \mathbf{0}$ on S. Then, the interface conditions (18) give

$$-\mathbf{t}_i = G\cdot\mathbf{u}_i \quad \text{and} \quad -\mathbf{u}_i = F\cdot\mathbf{t}_i$$

on S. Hence

$$(I - GF)\cdot\mathbf{t}_i = 0 \quad \text{and} \quad (I - FG)\cdot\mathbf{u}_i = \mathbf{0}$$

Since the matrices FG and GF have the same eigenvalues, we see that, provided

$$1 \text{ is not an eigenvalue of the matrix } FG \quad (25)$$

it follows that $\mathbf{u}_i = \mathbf{0}$ and $\mathbf{t}_i = \mathbf{0}$, whence $\mathbf{u}_i \equiv \mathbf{0}$ in B_i.

Summarizing, if the matrices F and G are finite and satisfy (23), (24) and (25), this is sufficient to ensure that the corresponding inclusion problem has at most one solution.

5.2 Jones Frequencies

The proof in Section 5.1 fails for lubricated interfaces, defined by (13). We obtain $J = 0$, whence $\mathbf{u}(P) \equiv \mathbf{0}$ for $P \in B_e$ (and so the exterior field is unique). The interface conditions then give

$$\mathbf{t}_i = \mathbf{0} \quad \text{and} \quad \mathbf{n}\cdot\mathbf{u}_i = 0 \quad (26)$$

on S. It is well known that, for any S, there is an infinite set of frequencies at which there is a non-trivial displacement field $\mathbf{u}_i$ in B_i that satisfies $(26)_1$; these are just the free oscillations of the inclusion in the absence of the matrix. Do any of these fields also satisfy $(26)_2$? It turns out that some of them do, but only for some geometries and some frequencies: we call them *Jones frequencies*, as they were first discussed by D. S. Jones[37] in the present context.

It is known that Jones frequencies exist for spheres. Thus, Lamb and Chree found that an elastic sphere could sustain "torsional oscillations," in which the radial component of the displacement is identically zero (see, e.g., Eringen, and Suhubi, Section 8.14).[38] The corresponding frequencies of oscillation (i.e., the Jones frequencies) are given as the roots of

$$(n - 1)j_n(K_i a) - K_i a j_{n+1}(K_i a) = 0$$

for $n = 1,2,\ldots$, where a is the radius of the sphere, j_n is a spherical Bessel function and the wavenumber K_i is defined by (10); some of these frequencies are listed by Eringen and Suhubi,[38] in their Table 8.14.1a.

Jones frequencies also exist for any axisymmetric body; such bodies can sustain torsional oscillations in which only the azimuthal component of displacement is non-zero. Rand[39] has computed some of the oscillation frequencies for prolate spheroids.

Intuitively, we do not expect Jones frequencies to exist for an "arbitrary" body. This has been proved recently by Hargé.[40]

5.3 The Rokhlin–Wang Model

Consider the Rokhlin–Wang model (19). Direct calculation gives

$$\begin{aligned}\mathbf{u}\cdot\overline{\mathbf{t}} - \overline{\mathbf{u}}\cdot\mathbf{t} = {} & \mathbf{u}_i\cdot\overline{\mathbf{t}}_i - \overline{\mathbf{u}}_i\cdot\mathbf{t}_i + \mathbf{u}_i\cdot E\cdot\overline{\mathbf{t}}_i - \overline{\mathbf{u}}_i\cdot\overline{E}\cdot\mathbf{t}_i \\ & + \mathbf{t}_i\cdot(F^T - \overline{F})\cdot\overline{\mathbf{t}}_i + \mathbf{u}_i\cdot(\overline{G} - G^T)\cdot\overline{\mathbf{u}}_i \\ & + \mathbf{u}_i\cdot(C - \overline{C}^T)\cdot\overline{\mathbf{u}}_i + \mathbf{t}_i\cdot(\overline{D}^T - D)\cdot\overline{\mathbf{t}}_i\end{aligned}$$

where

$$C = A^T\overline{G}, \quad D = B^T\overline{F} \quad \text{and} \quad E = A^T + \overline{B} - G^T\overline{F} + A^T\overline{B}.$$

Integrating over S, using (21) and (22), we see that $J \geq 0$ provided that F satisfies (23), G satisfies (24),

$$C_{jk} = \overline{C}_{kj} \text{ for } j \neq k \text{ and } \mathscr{I}m(C_{kk}) \geq 0 \text{ (no sum)} \quad (27)$$

$$D_{jk} = \overline{D}_{kj} \text{ for } j \neq k \text{ and } \mathscr{I}m(D_{kk}) \leq 0 \text{ (no sum)} \quad (28)$$

and $E = cI$ where c is a real constant. These conditions are sufficient to ensure that $\mathbf{u} \equiv \mathbf{0}$ in B_e, whence the interface conditions (19) give

$$\left.\begin{aligned}(I + B)\cdot\mathbf{t}_i + G\cdot\mathbf{u}_i &= \mathbf{0} \\ F\cdot\mathbf{t}_i + (I + A)\cdot\mathbf{u}_i &= \mathbf{0}\end{aligned}\right\}$$

on S. If these equations imply that $\mathbf{u}_i = \mathbf{0}$ and $\mathbf{t}_i = \mathbf{0}$, we can deduce that $\mathbf{u}_i \equiv \mathbf{0}$ in B_i, as required.

As an example, suppose that A, B, F, and G have the properties described at the end of Section 2: F and G are real diagonal matrices, whence (23) and (24) are satisfied; $A = B^T$ has negative imaginary elements apart from zeros on the diagonal, whence E is real diagonal, but not a multiple of I; moreover, C and D are purely imaginary and skew-diagonal, so the first parts of (27) and (28) are not satisfied. Thus, we are unable to establish uniqueness with these choices for A, B, F, and G. However, it is not known whether non-uniqueness can actually occur (as in Section 5.2).

6. BOUNDARY INTEGRAL EQUATIONS

In this section, we derive (direct) boundary integral equations over S for inclusions with various imperfect interfaces, in the plane-strain case (a similar analysis can be made for three-dimensional problems). In fact, our aim is to derive (quasi-) *Fredholm* systems of singular integral equations, for all the usual Fredholm theorems hold for such systems.[41] In particular, we can analyse solvability by showing that the corresponding homogeneous system has only the trival solution.

First, we introduce two fundamental Green's tensors, $\mathbf{G}_a(P; Q)$ $(a = e, i)$:

$$(\mathbf{G}_a(P; Q))_{ij} = \frac{1}{\mu_a}\left\{\psi_a\delta_{ij} + \frac{1}{K_a^2}\frac{\partial^2}{\partial x_i \partial x_j}(\psi_a - \phi_a)\right\}$$

where $\phi_a = -(i/2)H_0^{(1)}(k_aR)$, $\psi_a = -(i/2)H_0^{(1)}(K_aR)$ and $R = |P - Q|$. Next, we define elastic single-layer and double-layer potentials by

$$(S_a\mathbf{u})(P) = \int_S \mathbf{u}(q)\cdot\mathbf{G}_a(q; P)\, ds_q$$

and

$$(D_a\mathbf{u})(P) = \int_S \mathbf{u}(q)\cdot T_a^q\, \mathbf{G}_a(q; P)\, ds_q$$

respectively, where T_a^q means T_a applied at $q \in$ S. Then, three applications of Betti's theorem (one in B_e to $\mathbf{u}_e$ and $\mathbf{G}_e$, one in B_i to u_{inc} and $\mathbf{G}_e$, and one in B_i to $\mathbf{u}_i$ and $\mathbf{G}_i$) yield the familiar representations

$$2\mathbf{u}_e(P) = (S_e\mathbf{t})(P) - (D_e\mathbf{u})(P), \quad P \in B_e \qquad (29)$$

and

$$-2\mathbf{u}_i(P) = (S_i\mathbf{t}_i)(P) - (D_i\mathbf{u}_i)(P), \quad P \in B_i \qquad (30)$$

Letting $P \to p \in S$, (29) and (30) give

$$(I + \overline{K}_e^*)\mathbf{u} - S_e\mathbf{t} = 2\mathbf{u}_{\text{inc}} \qquad (31)$$

and

$$(I - \overline{K}_i^*)\mathbf{u}_i + S_i\mathbf{t}_i = \mathbf{0} \qquad (32)$$

respectively, where

$$\overline{K}_a^*\mathbf{u} = \int_S \mathbf{u}(q)\cdot T_a^q\, \mathbf{G}_a(q; p)\, ds_q$$

is a singular integral operator. We can obtain two further relations by calculating the tractions on S corresponding to (29) and (30). However, we shall forego this possibility here. This self-imposed restriction prevents us from obtaining Fredholm systems for some choices of F and G, including the case of a perfect interface ($F = G = 0$); Martin[23] has given such a system, involving a regularization of the operator TD.

6.1 Flexibly-Bonded Interfaces

Consider interfaces modeled by (15). If we use (15) to eliminate $\mathbf{u}_i$ and $\mathbf{t}_i$ from (32), and combine with (31), we obtain the system

$$\begin{pmatrix} I + \overline{K}_e^* & -S_e \\ I - \overline{K}_i^* & S_i - 2(I - \overline{K}_i^*)F \end{pmatrix}\begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix} = \begin{pmatrix} 2\mathbf{u}_{\text{inc}} \\ \mathbf{0} \end{pmatrix} \qquad (33)$$

More precisely, this is a system of four coupled singular integral equations for the four components of the two vectors $\mathbf{u}(p)$ and $\mathbf{t}(p)$, $p \in S$. This system was derived by Martin.[1] To analyze it, let

$$M = \sigma(I + \overline{K}_e^*) \quad \text{and} \quad N = \sigma(I - \overline{K}_i^*)$$

where $\sigma(L)$ is the symbol matrix of the singular integral operator L (see Muskhelishvili[41] or Martin[23]). Then, we have to examine the determinant

$$\det\begin{pmatrix} M & 0 \\ N & -2NF \end{pmatrix} = 4\det(M)\det(N)\det(F).$$

It is well known that det M and det N are non-zero, whence (33) is a Fredholm system provided that det F does not vanish. Thus, we require that F be a non-singular matrix (for all $p \in S$ if F varies with p).

6.2. Inertial Interfaces

Consider interfaces modeled by (17). It is easily seen that the use of (17) and (32), as before, does not lead to a Fredholm system for any G (including $G = 0$). For an exceptional special case, see Section 7.

6.3. The Baik–Thompson Model

Consider the Baik–Thompson model (18). When combined with (17) and (18), we obtain the system

$$\begin{pmatrix} I + \overline{K}_e^* & 0 & -S_e & 0 \\ 0 & I - \overline{K}_i^* & 0 & S_i \\ -G & -G & I & -I \\ -I & I & F & F \end{pmatrix}\begin{pmatrix} \mathbf{u} \\ \mathbf{u}_i \\ \mathbf{t} \\ \mathbf{t}_i \end{pmatrix} = \begin{pmatrix} 2\mathbf{u}_{\text{inc}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

the corresponding symbol matrix has determinant

$$4 \det(M) \det(N) \det(F)$$

which does not depend on G. Thus, the 8×8 system is Fredholm if, and only if, F is non-singular; with this assumption, the interface conditions give

$$\mathbf{t} = \mathcal{A}\mathbf{u} - \mathcal{B}\mathbf{u}_i \quad \text{and} \quad \mathbf{t}_i = \mathcal{B}\mathbf{u} - \mathcal{A}\mathbf{u}_i$$

on S, where the matrices $\mathcal{A}$ and $\mathcal{B}$ are given by

$$\mathcal{A} = \frac{1}{2}(F^{-1} + G) \quad \text{and} \quad \mathcal{B} = \frac{1}{2}(F^{-1} - G)$$

Substituting into (31) and (32) gives

$$\begin{pmatrix} I + \overline{K_e^*} - S_e\mathcal{A} & S_e\mathcal{B} \\ S_i\mathcal{B} & I - \overline{K_i^*} - S_i\mathcal{A} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{u}_i \end{pmatrix} = \begin{pmatrix} 2\mathbf{u}_{\text{inc}} \\ \mathbf{0} \end{pmatrix} \tag{34}$$

This is a Fredholm system for $\mathbf{u}(p)$ and $\mathbf{u}_i(p)$. In particular, if $G = 0$, we have $\mathcal{A} = \mathcal{B} = 1/2F^{-1}$, whence (34) reduces to (33), since $\mathbf{t} = 1/2F^{-1}[\mathbf{u}]$.

6.4. The Rokhlin–Wang Model

Consider the Rokhlin–Wang model (19). When combined with (17) and (18), we obtain the system

$$\begin{pmatrix} I + \overline{K_e^*} & 0 & -S_e & 0 \\ 0 & I - \overline{K_i^*} & 0 & S_i \\ 0 & -G & I & -I - B \\ -I & I + A & 0 & F \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{u}_i \\ \mathbf{t} \\ \mathbf{t}_i \end{pmatrix} = \begin{pmatrix} 2\mathbf{u}_{\text{inc}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

This system is Fredholm if, and only if, F is non-singular; with this assumption, we can eliminate the tractions to give the Fredholm system

$$\begin{pmatrix} I + \overline{K_e^*} - S_e(I+B)F^{-1} & S_e((I+B)F^{-1}(I+A) - G) \\ S_iF^{-1} & I - \overline{K_i^*} - S_iF^{-1}(I+A) \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{u}_i \end{pmatrix} = \begin{pmatrix} 2\mathbf{u}_{\text{inc}} \\ \mathbf{0} \end{pmatrix} \tag{35}$$

7. DISCUSSION

We have described several ways of reducing the inclusion problem to systems of singular integral equations over S. Many other reductions are possible. For example, we have not used the hypersingular operator, TD. Rather than consider more complicated situations, let us conclude by mentioning a simpler situation. This arises when the matrix and the inclusion are composed of the *same* material. Then, for inertial interfaces, defined by (17), we simply add (31) and (32) to give

$$(I - S_eG)\mathbf{u} = \mathbf{u}_{\text{inc}} \tag{36}$$

which is a Fredholm integral equation of the second kind for $\mathbf{u}(p)$; here, we have assumed that the common material has Lamé moduli λ_e and μ_e, and density ρ_e. Having found $\mathbf{u}(p)$, $\mathbf{t}(p)$ is given by (31) as the solution of a Fredholm integral equation of the first kind. Alternatively, we can look for a solution in the form

$$\mathbf{u}(P) = \mathbf{u}_{\text{inc}}(P) + (S_e\mathbf{v})(P), \quad P \in B_e$$
$$\mathbf{u}_i(P) = \mathbf{u}_{\text{inc}}(P) + (S_e\mathbf{v})(P), \quad P \in B_i$$

where $\mathbf{v}(p)$ is an unknown function. These representations satisfy $[\mathbf{u}] = \mathbf{0}$ automatically. Equation $(17)_1$ then gives

$$(I - GS_e)\mathbf{v} = G\mathbf{u}_{\text{inc}} \tag{37}$$

which is adjoint to (36). One can make similar reductions for flexibly-bonded interfaces, defined by (15), but the result involves hypersingular equations. For an analysis of the analogous problems in acoustics and further references, see Angell *et al.*[42]

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