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A Normal Crack in an Elastic Half-space with Stress-free Surface

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The problem of an elastic half-space with stress-free surface and a crack of arbitrary shape with prescribed displacements or tractions is reduced to an equivalent system of integral equations on the crack. For a pressurized crack in a plane perpendicular to the free surface, a scalar integral equation is derived. In properly chosen function spaces, unique solvability of the integral equation and regularity of solutions for regular data are proven.

1. Introduction

The problem of a crack terminating at a boundary has been attacked by several authors and different methods. The general method of [3, 5, 18, 21] and others lead typically to systems of integral equations involving layer potentials on the crack and on the boundary. In special cases one may expect simplifications due to a particular simple geometry and/or simplified boundary conditions.

The simplest case of a pressurized plane crack in an unbounded elastic solid leads to a scalar integral equation, which is investigated e.g. in [4, 16, 21]. Special geometries of plane cracks are treated in [6, 12, 13].

Now, consider a body containing a crack, where the crack does not intersect the body's boundary. The interaction between the crack and the boundary may be resolved by a version of Schwartz's alternating process, in which one alternates between (i) solving a problem in the uncracked body and (ii) solving a problem for the crack in an unbounded solid. The convergence of the process is proved in [5] for

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a plane crack, provided that the distance between the crack and the boundary is sufficiently large.

Here we use another method. We derive the integral equations starting from Green's function for the uncracked body. Then a system of integral equations on the crack only arises. Although the method can be applied for general domains explicit formulae can be expected only in cases when the Green's function is known analytically. For the half-space one can use Mindlin's fundamental solution. This is carried out in section 2.

In section 3 we make the additional assumption that the crack-plane is orthogonal to the free surface. Then the system of integral equations decouples and a scalar integral equation remains. Its kernel is given by (3.5).

Section 4 is devoted to the scalar integral equation. It is shown that for each given pressure a uniquely determined jump of the normal displacement component exists, provided data and solution are in appropriate spaces. In the proof, we construct a homotopy through Fredholm operators connecting the given operator with a simpler one which has index zero. In contrast to the works [3] and [21] here no Gårding inequality is known up to now. A regularity result is given in Theorem 4.3. It states that in a certain interval of the smoothness parameter, better data give rise to better solutions. This result cannot be improved in the sense that it definitely fails for each larger interval.

2. The half-space with stress-free surface

Consider a homogeneous isotropic elastic half-space $y > 0$ (x, y, z are Cartesian co-ordinates), whose surface $y = 0$ is stress-free. For given body forces $\mathbf{f}$, one has to solve the following boundary-value problem.

Problem $\mathscr{B}$

Find $\mathbf{u}$ where

$$\Delta^*\mathbf{u} = \mu\,\Delta\mathbf{u} + (\lambda + \mu)\,\mathrm{grad\,div}\,\mathbf{u} = \mathbf{f} \quad \text{in } \mathbb{R}^3_+, \text{ i.e. } y > 0, \tag{2.1}$$

$$T(\mathbf{u}) = \lambda(\mathrm{div}\,\mathbf{u})\mathrm{n} + 2\mu\frac{\partial\mathbf{u}}{\partial\mathbf{n}} + \mu\mathbf{n}\times\mathrm{curl}\,\mathbf{u} = \mathbf{0} \quad \text{on } \partial\mathbb{R}^3_+, \text{ i.e. } y = 0 \tag{2.2}$$

and

$$\mathbf{u} = o(1), \quad \nabla\mathbf{u} = o(|\mathbf{x}|^{-1}) \quad \text{as } |\mathbf{x}| \to \infty. \tag{2.3}$$

Here, Δ^* denotes the Lamé operator, λ and μ are the Lamé moduli, T is the traction operator and $\mathbf{n}$ is the exterior unit normal. In terms of Cartesian co-ordinates, the traction vector has components

$$(T(\mathbf{u}))_i = n_j\tau_{ij}(\mathbf{u}),$$

where

$$\tau_{ij}(\mathbf{u}) = c_{ijkl}e_{kl}(\mathbf{u}),$$

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

$$e_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.4}$$

and, as usual, repeated subscripts are summed from 1 to 3. Explicitly, we have

$$(T(\mathbf{u}))_i = \lambda n_i \frac{\partial u_k}{\partial x_k} + \mu n_j \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

For all given $\mathbf{f} \in H^{-1}_{\text{comp}}(\mathbb{R}^3_+)$ the existence of a unique solution $\mathbf{u} \in H^1_{\text{loc}}(\mathbb{R}^3_+)$ is well-known [9]. For simplicity, we take the notation $H^s(\mathbb{R}^3_+)$, etc. also for spaces of vector-valued functions instead of the notation $[H^s(\mathbb{R}^3_+)]^3$, indicating that each component belongs to the given function space. Further details on the function spaces are given in the appendix.

When (2.1) is replaced by

$$\Delta^* \mathbf{u} = \mathbf{0}, \tag{2.5}$$

we denote the corresponding homogeneous boundary-value problem by $\mathscr{B}_0$.

The solution of Problem $\mathscr{B}$ can be expressed explicitly by means of a special fundamental solution G with matrix-components G_{ij}. $G_{ij}(\mathbf{x}, \mathbf{x}')$ is the ith component of displacement at $\mathbf{x}$ due to a point force acting at $\mathbf{x}'$ in the jth direction, where $y = 0$ remains stress-free. This fundamental solution was first calculated by Mindlin in 1936 [15].

Now we give the formulae. G_{ij} can be written as

$$G_{ij} = G^{\mathrm{F}}_{ij} + G^{\mathrm{C}}_{ij}, \tag{2.6}$$

where G^{F}_{ij} denotes the corresponding solution for the full space (Kelvin's fundamental solution for the Lamé operator)

$$\mathscr{A}\, G^{\mathrm{F}}_{ij} = -\frac{\kappa}{R}\delta_{ij} + \frac{1}{R^3}(x_i - x_i')(x_j - x_j'),$$

$$\mathscr{A} = 16\pi\mu(1 - \nu), \quad \kappa = 3 - 4\nu, \; \nu = \tfrac{1}{2}\lambda/(\lambda + \mu)$$

is Poisson's ratio, $R = |\mathbf{x} - \mathbf{x}'|$, $\mathbf{x} = (x, y, z) = (x_1, x_2, x_3)$ and $\mathbf{x}' = (x', y', z') = (x_1', x_2', x_3')$. Note that G^{F}_{ij} is a convolution kernel since it depends only on the difference between co-ordinates.

The additional, 'correction' term G^{C}_{ij} is more complicated. It is a function of

$$R_+ = \sqrt{[(x - x')^2 + (y + y')^2 + (z - z')^2]},$$

the distance between $\mathbf{x}$ and the mirror image of $\mathbf{x}'$ with respect to the plane $y = 0$. It also depends on the combinations $y + y'$, $y - y'$ and yy', as well as $x - x'$ and $z - z'$. Here are the explicit formulae:

$$\mathscr{A}\, G^{\mathrm{C}}_{11} = \frac{1}{R_+} + \kappa\frac{(x - x')^2}{R_+^3} + 2\frac{yy'}{R_+^3}\left[1 - \frac{3(x - x')^2}{R_+^2}\right] + \frac{D}{(R_+ + y + y')}\left[1 - \frac{(x - x')^2}{R_+(R_+ + y + y')}\right],$$

$$\mathscr{A}\, G^{\mathrm{C}}_{33} = \frac{1}{R_+} + \kappa\frac{(z - z')^2}{R_+^3} + 2\frac{yy'}{R_+^3}\left[1 - \frac{3(z - z')^2}{R_+^2}\right] + \frac{D}{(R_+ + y + y')}\left[1 - \frac{(z - z')^2}{R_+(R_+ + y + y')}\right],$$

$$\mathscr{A}\, G^{\mathrm{C}}_{13} = \mathscr{A}\, G^{\mathrm{C}}_{31} = \frac{(x-x')(z-z')}{R_+}\left[\frac{\kappa}{R_+^2} - \frac{6yy'}{R_+^4} - \frac{D}{(R_+ + y + y')^2}\right],$$

$$\mathscr{A}(G^{\mathrm{C}}_{12} + G^{\mathrm{C}}_{21}) = 2\kappa\frac{(x-x')(y-y')}{R_+^3},$$

$$\mathscr{A}(G^{\mathrm{C}}_{12} - G^{\mathrm{C}}_{21}) = 2\frac{(x-x')}{R_+}\left[\frac{6yy'(y+y')}{R_+^4} - \frac{D}{R_+ + y + y'}\right],$$

$$\mathscr{A}(G^{\mathrm{C}}_{32} + G^{\mathrm{C}}_{23}) = 2\kappa\frac{(z-z')(y-y')}{R_+^3},$$

$$\mathscr{A}(G^{\mathrm{C}}_{32} - G^{\mathrm{C}}_{23}) = 2\frac{(z-z')}{R_+}\left[\frac{6yy'(y+y')}{R_+^4} - \frac{D}{R_+ + y + y'}\right],$$

$$\mathscr{A}\, G^{\mathrm{C}}_{22} = \frac{8(1-\nu)^2 - \kappa}{R_+} + \frac{\kappa(y+y')^2 - 2yy'}{R_+^3} + \frac{6yy'(y+y')^2}{R_+^5},$$

where $D = 4(1-\nu)(1-2\nu)$. Note the behaviour of G at infinity,

$$G(\mathbf{x};\mathbf{x}') = O(|\mathbf{x}'|^{-1}), \qquad \nabla G(\mathbf{x};\mathbf{x}') = O(|\mathbf{x}'|^{-2}) \quad \text{as } |\mathbf{x}'| \to \infty, \tag{2.7}$$

uniformly for $\mathbf{x}$ in a compact subset of $\mathbb{R}^3_+$.

The solution $\mathbf{u}$ to Problem $\mathscr{B}$ is then given by the formula

$$\mathbf{u}(\mathbf{x}') = \int_{\mathbb{R}^3_+} G(\mathbf{x};\mathbf{x}')\mathbf{f}(\mathbf{x})\,d\mathbf{x} = \mathbf{G}\mathbf{f},$$

say. Obviously, the integral operator $\mathbf{G}$ is linear and continuous,

$$\mathbf{G}: C_0^\infty(\overline{\mathbb{R}^3_+}) \to C^\infty(\mathbb{R}^3_+).$$

It extends continuously to the scale of Sobolev spaces.

Theorem 2.1. $\mathbf{G}$ *defines a continuous mapping*

$\mathbf{G}: H^s_{\mathrm{comp}}(\overline{\mathbb{R}^3_+}) \to H^{s+2}_{\mathrm{loc}}(\mathbb{R}^3_+)$ *for each* $s > -\frac{3}{2}$.

Proof. Problem $\mathscr{B}$ is an elliptic boundary-value problem in the half-space. The homogeneous problem $\mathscr{B}_0$ only has the trivial solution. Therefore, according to the general theory of elliptic boundary-value problem ([1, 19]), we have the following estimate: for each $\varphi \in C_0^\infty(\overline{\mathbb{R}^3_+})$ and each $s > -\frac{3}{2}$ there is a constant c such that

$$\|\varphi\mathbf{G}\mathbf{f}|H^{s+2}(\mathbb{R}^3_+)\| \leqslant c\|\mathbf{f}|H^s(\mathbb{R}^3_+)\|$$

for all $\mathbf{f} \in H^s_{\mathrm{comp}}(\overline{\mathbb{R}^3_+})$. This is the stated continuity property.

Let Ω be an open subset of $\mathbb{R}^3$ with piecewise smooth boundary satisfying a two-sided cone condition. Then the symmetric bilinear form associated with Δ^* ([9]) is given by

$$\Phi_\Omega(\mathbf{u},\mathbf{v}) = \int_\Omega \lambda \operatorname{div}\mathbf{u}\operatorname{div}\mathbf{v} + 2\mu e_{ij}(\mathbf{u})e_{ij}(\mathbf{v}),$$

where $\mathbf{u}, \mathbf{v} \in H^1(\Omega)$ and e_{ij} is defined by (2.4).

Green's formulae have the form

$$\Phi_\Omega(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u}\,\Delta^*\mathbf{v} + \langle T(\mathbf{v}), \gamma(\mathbf{u})\rangle, \tag{2.8}$$

$$\int_\Omega (\mathbf{u}\,\Delta^*\mathbf{v} - \mathbf{v}\,\Delta^*\mathbf{u}) = \langle T(\mathbf{v}), \gamma(\mathbf{u})\rangle - \langle T(\mathbf{u}), \gamma(\mathbf{v})\rangle, \tag{2.9}$$

where $\mathbf{u}, \mathbf{v} \in H^1(\Omega)$, $\Delta^*\mathbf{u}$, $\Delta^*\mathbf{v} \in L^2(\Omega)$ and γ denotes the ordinary trace operator at $\partial\Omega$. Moreover, $\langle\cdot,\cdot\rangle$ denotes the extension of the standard bilinear pairing of $L^2(\partial\Omega)$ to the space $H^{-s}(\partial\Omega) \times H^s(\partial\Omega)$. Note that at first the formulae (2.8) and (2.9) make sense only for twice continuously differentiable functions, but they extend from this dense subspace by continuity.

We recall the following well-known result.

Lemma 2.1. *The trace operator γ gives a linear and continuous mapping*

$\gamma: H^s_{\mathrm{loc}}(\Omega) \to H^{s-1/2}_{\mathrm{loc}}(\partial\Omega)$ *for each* $s > \frac{1}{2}$.

If we know in addition that $\Delta^*\mathbf{u} \in L^2(\Omega)$ then the restriction on s can be weakened, i.e. Lemma 2.1 is then true for all $s > -\frac{1}{2}$.

Denote by $D(\Delta^*)$ the subspace of $L^2(\Omega)$ of all $\mathbf{u}$ such that $\Delta^*\mathbf{u} \in L^2(\Omega)$ and consider $H^s(\Omega) \cap D(\Delta^*)$ for $s > \frac{1}{2}$ with its natural topology. Then we have the following lemma.

Lemma 2.2. *The traction operator T gives by* (2.8) *a linear and continuous mapping*

$T: H^s_{\mathrm{loc}}(\Omega) \cap D(\Delta^*) \to H^{s-3/2}_{\mathrm{loc}}(\partial\Omega)$ *for each* $s > \frac{1}{2}$.

Proof. For each $s > \frac{3}{2}$ the assertion is clear immediately since T is a composition of a first-order differentiation and restriction. For each s, with $\frac{1}{2} < s < \frac{3}{2}$ and $\mathbf{w} \in H^{-s+3/2}_{\mathrm{comp}}(\partial\Omega)$ let $\mathbf{v} \in H^{-s+2}_{\mathrm{comp}}(\Omega)$ be such that $\mathbf{w} = \gamma\mathbf{v}$. Then (2.8) gives

$$\langle T(\mathbf{u}), \gamma(\mathbf{v})\rangle = \Phi_\Omega(\mathbf{u}, \mathbf{v}) - \int_\Omega \mathbf{v}\,\Delta^*\mathbf{u}$$

which shows that $T(u)$ defines a continuous linear functional on the dual of $H^{s-3/2}_{\mathrm{loc}}(\partial\Omega)$. The case $s = \frac{3}{2}$ follows by interpolation.

Now, let Ω be an open subset of $\mathbb{R}^3_+$ with smooth boundary. We seek solutions of (2.5) in Ω, subject to (2.2) and (2.3). We derive in the standard manner a version of Betti's representation formula by applying Green's formula to $\mathbf{u}$ and to Mindlin's fundamental solution G with respect to a domain $\Omega\backslash B_\varepsilon$, where B_ε is a ball of radius ε with centre at a distinguished point $\mathbf{x}'$. With $\varepsilon \to 0$ we get for any solution $\mathbf{u} \in H^1_{\mathrm{loc}}(\Omega)$

$$p\mathbf{u}(\mathbf{x}') = \int_{\partial_+\Omega} \{(T_\mathbf{x} G(\mathbf{x}; \mathbf{x}'))^t \mathbf{u}(\mathbf{x}) - G(\mathbf{x}; \mathbf{x}')\,T(\mathbf{u}(\mathbf{x}))\}\, ds_\mathbf{x}, \tag{2.10}$$

where we have set $\partial_+\Omega = \partial\Omega \cap \mathbb{R}^3_+$ in order to distinguish the boundary in $\mathbb{R}^3_+$ from the boundary in $\mathbb{R}^3$, and

$$p = \begin{cases} 1 & \text{for } \mathbf{x}' \in \Omega, \\ \frac{1}{2} & \text{for } \mathbf{x}' \in \partial_+\Omega \\ 0 & \text{for } \mathbf{x}' \notin \Omega. \end{cases}$$

At first (2.10) makes sense for sufficiently smooth functions $\mathbf{u}$. Due to Lemmas 2.1, 2.2 and Theorem 2.1, the representation remains valid for $\mathbf{u} \in H^s_{\mathrm{loc}}(\Omega) \cap D(\Delta^*)$ for $s > \frac{1}{2}$.

The right-hand side of (2.10) involves operators of potential type, which map functions or distributions on $\partial_+\Omega$ into those on Ω. Denote by $P(\mathbf{G})$ the operator

$$P(\mathbf{G})(\mathbf{w})(\mathbf{x}') = \int_{\partial_+\Omega} G(\mathbf{x}; \mathbf{x}')\mathbf{w}(\mathbf{x})\, ds_{\mathbf{x}}$$

and by $P(T\mathbf{G})$ the operator

$$P(T\mathbf{G})(\mathbf{w})(\mathbf{x}') = \int_{\partial_+\Omega} (T_{\mathbf{x}} G(\mathbf{x}; \mathbf{x}'))^t \mathbf{w}(\mathbf{x})\, ds_{\mathbf{x}},$$

where $\mathbf{w}$ is given on $\partial_+\Omega$.

Lemma 2.3. *The operator $P(\mathbf{G})$ gives a linear and continuous mapping*

$$P(\mathbf{G}): H^s_{\mathrm{comp}}(\partial_+\Omega) \to H^{s+3/2}_{\mathrm{loc}}(\Omega) \quad \textit{for each } s > -1.$$

The operator $P(T\mathbf{G})$ gives a linear and continuous mapping

$$P(T\mathbf{G}): H^s_{\mathrm{comp}}(\partial_+\Omega) \to H^{s+1/2}_{\mathrm{loc}}(\Omega) \quad \textit{for each } s > 0.$$

According to Lemmas 2.1–2.3 the operators occurring on the right-hand side of (2.10) define continuous operators

$$P(\mathbf{G}) \circ T: H^s_{\mathrm{comp}}(\Omega) \to H^s_{\mathrm{loc}}(\Omega)$$
$$P(T\mathbf{G}) \circ \gamma: H^s_{\mathrm{comp}}(\Omega) \to H^s_{\mathrm{loc}}(\Omega)$$

for all $s > \frac{1}{2}$. Moreover, we have the following result.

Lemma 2.4. *The following compositions of operators are continuous:*

$$\gamma \circ P(\mathbf{G}): H^s(\partial_+\Omega) \to H^{s+1}(\partial_+\Omega) \quad \textit{for } s > -1,$$
$$T \circ P(\mathbf{G}): H^s(\partial_+\Omega) \to H^s(\partial_+\Omega) \quad \textit{for } s > -1,$$
$$\gamma \circ P(T\mathbf{G}): H^s(\partial_+\Omega) \to H^s(\partial_+\Omega) \quad \textit{for } s > 0, \textit{ and}$$
$$T \circ P(T\mathbf{G}): H^s(\partial_+\Omega) \to H^{s-1}(\partial_+\Omega) \quad \textit{for } s > 0.$$

Remark 2.1. When the intersection $\partial\Omega \cap \partial\mathbb{R}^3_+$ is empty, all the operators in Lemma 2.4 are pseudo-differential operators of order $-1, 0, 0$ and 1, respectively. This is no longer true near intersection points, as Mellin-type operators also occur. These will be analysed in the proofs of Theorems 4.1 and 4.2.

Now it is easy to derive from (2.10) representation formulae for domains with cracks. Let $\Gamma \subset \mathbb{R}^3_+$ be a bounded submanifold with boundary of a smooth two-dimensional surface S, which intersects $\partial\mathbb{R}^3_+$ transversally. We may assume that each ball B_R of radius R and centre at the origin is divided by S into the two domains Ω^+_R and Ω^-_R which satisfy the two-sided cone condition. The orientation of the normal $\mathbf{n}$ on S is taken with respect to Ω^+_R, and so points into Ω^-_R. Adding up the equations

(2.10) for the domains Ω_R^+ and Ω_R^- we obtain for each $\mathbf{x}' \in B_R \backslash S$

$$\mathbf{u}(\mathbf{x}') = \int_S \{(T_\mathbf{x} G(\mathbf{x}; \mathbf{x}'))^t [\mathbf{u}(\mathbf{x})] - G(\mathbf{x}; \mathbf{x}') \langle T(\mathbf{u})(\mathbf{x}) \rangle\} \, ds_\mathbf{x}$$
$$+ \int_{\partial_+ B_R} \{(T_\mathbf{x} G(\mathbf{x}; \mathbf{x}'))^t \mathbf{u}(\mathbf{x}) - G(\mathbf{x}; \mathbf{x}') T(\mathbf{u})(\mathbf{x})\} \, ds_\mathbf{x},$$

where

$$[\mathbf{u}(\mathbf{x})] = \gamma^+ \mathbf{u}(\mathbf{x}) - \gamma^- \mathbf{u}(\mathbf{x}),$$
$$\langle T(\mathbf{u})(\mathbf{x}) \rangle_i = n_j(\mathbf{x}) \{(\tau_{ij}(\mathbf{u}))^+ + (\tau_{ij}(\mathbf{u}))^-\}$$

and the superscripts + and − denote limits taken from Ω_R^+ and Ω_R^-, respectively. The integrands in the first integral vanish outside Γ, so that we can replace S by Γ. When $R \to \infty$, the integrals over $\partial_+ B_R$ vanish according to (2.3) and (2.7) and, so, we obtain

$$\mathbf{u}(\mathbf{x}') = \int_\Gamma \{(T_\mathbf{x} G(\mathbf{x}; \mathbf{x}'))^t [\mathbf{u}(\mathbf{x})] - G(\mathbf{x}; \mathbf{x}') \langle T(\mathbf{u})(\mathbf{x}) \rangle\} \, ds_\mathbf{x}, \tag{2.11}$$

for $x' \in B_R \backslash \Gamma$. For $\mathbf{x}' \in \Gamma$ we get

$$\frac{1}{2} \langle \mathbf{u}(\mathbf{x}') \rangle = \int_\Gamma \{(T_\mathbf{x} G(\mathbf{x}; \mathbf{x}'))^t [\mathbf{u}(\mathbf{x})] - G(\mathbf{x}; \mathbf{x}') \langle T(\mathbf{u})(\mathbf{x}) \rangle\} \, ds_\mathbf{x}, \tag{2.12}$$

where

$$\langle \mathbf{u}(\mathbf{x}) \rangle = \gamma^+ \mathbf{u}(\mathbf{x}) + \gamma^- \mathbf{u}(\mathbf{x}).$$

The integral equation (2.12) can be used for solving the Dirichlet problem in a half-space with a thin rigid inclusion Γ:

Problem $\mathscr{D}$

Find $\mathbf{u}$ such that

$$\begin{aligned} \Delta^* \mathbf{u} &= \mathbf{0} && \text{in } \mathbb{R}^3_+ \backslash \Gamma, \\ T(\mathbf{u}) &= \mathbf{0} && \text{on } \partial \mathbb{R}^3_+, \\ \gamma^\pm \mathbf{u} &= \mathbf{g}^\pm && \text{on } \Gamma^\pm \end{aligned}$$

and

$$\mathbf{u} = o(1), \quad \nabla \mathbf{u} = o(|\mathbf{x}|^{-1}) \quad \text{as } |\mathbf{x}| \to \infty.$$

Here, Γ^+ and Γ^- are the two sides of the surface Γ. We have the following theorem.

Theorem 2.2. *Let* $\mathbf{g}^+$, $\mathbf{g}^- \in H^{1/2}(\Gamma)$ and $\mathbf{g}^+ - \mathbf{g}^- = [\mathbf{g}] \in H^{1/2}(\mathbb{R}^2_+, \Gamma)$. *Then* $\mathbf{u} \in H^1_{\text{loc}}(\mathbb{R}^3_+ \backslash \Gamma)$ *solves Problem* $\mathscr{D}$ *iff* $\mathbf{u}$ *is given by* (2.11), *where* $\langle T(\mathbf{u})(\mathbf{x}) \rangle \in H^{-1/2}(\mathbb{R}^2_+, \Gamma)$ *solves the integral equation*

$$\frac{1}{2} \langle (\mathbf{g}(\mathbf{x}') \rangle = \int_\Gamma \{(T_\mathbf{x} G(\mathbf{x}; \mathbf{x}'))^t [\mathbf{g}(\mathbf{x})] - G(\mathbf{x}; \mathbf{x}') \langle T(\mathbf{u})(\mathbf{x}) \rangle\} \, ds_\mathbf{x}, \quad \mathbf{x}' \in \Gamma. \tag{2.13}$$

In the special case $\mathbf{g}^+ = \mathbf{g}^- = \mathbf{g}$, say, (2.13) reduces to

$$\int_\Gamma G(\mathbf{x};\mathbf{x}')\langle T(\mathbf{u})(\mathbf{x})\rangle\,\mathrm{d}s_{\mathbf{x}} = -\mathbf{g}(\mathbf{x}')\quad \mathbf{x}'\in\Gamma.$$

Equations of this type are discussed in [1] and [21] for thin rigid inclusions in full spaces.

For a loaded crack Γ in an elastic half-space, we must solve the following Neumann problem:

Problem $\mathcal{N}$

Find $\mathbf{u}$ such that

$$\begin{aligned} \Delta^*\mathbf{u} &= \mathbf{0} && \text{in } \mathbb{R}^3_+\backslash\Gamma,\\ T(\mathbf{u}) &= \mathbf{0} && \text{on } \partial\mathbb{R}^3_+,\\ n_j(\mathbf{x})\,(\tau_{ij}(\mathbf{u}))^\pm &= h_i^\pm && \text{on } \Gamma^\pm \end{aligned}$$

and

$$\mathbf{u} = o(1),\qquad \nabla\mathbf{u} = o(|\mathbf{x}|^{-1})\qquad \text{as } |\mathbf{x}|\to\infty.$$

for this problem, we first have to calculate the tractions on the crack. From (2.11), we obtain

$$\frac{1}{2}[T(\mathbf{u})(\mathbf{x}')] = \int_\Gamma \{T_{\mathbf{x}'}(T_{\mathbf{x}}G(\mathbf{x};\mathbf{x}'))^t[\mathbf{u}(\mathbf{x})] - T_{\mathbf{x}'}G(\mathbf{x};\mathbf{x}')\langle T(\mathbf{u})(\mathbf{x})\rangle\}\,\mathrm{d}s_{\mathbf{x}}$$

for all $\mathbf{x}'\in\Gamma$, where

$$[T(\mathbf{u})(\mathbf{x})]_i = \mathbf{n}_j(\mathbf{x})\,\{(\tau_{ij}(\mathbf{u}))^+ - (\tau_{ij}(\mathbf{u}))^-\}.$$

This is the integral equation for Problem $\mathcal{N}$; it is to be solved for the jump in the displacement $[\mathbf{u}]$ across Γ. The equivalence of the integral equation and the original boundary-value problem is given by the next theorem.

Theorem 2.3. *Let* $\mathbf{h}^+,\ \mathbf{h}^-\in H^{-1/2}(\Gamma)$, $\mathbf{h}^+ + \mathbf{h}^- = \langle\mathbf{h}\rangle\in H^{-1/2}(\mathbb{R}^2_+,\ \Gamma)$. *Then* $\mathbf{u}\in H^1_{\mathrm{loc}}(\mathbb{R}^3_+\backslash\Gamma)$ *solves Problem $\mathcal{N}$ iff $\mathbf{u}$ is given by* (2.11), *where* $[\mathbf{u}(\mathbf{x})]\in H^{1/2}(\mathbb{R}^2_+,\Gamma)$ *solves the integral equation*

$$\frac{1}{2}[\mathbf{h}(\mathbf{x}')] = \int_\Gamma \{T_{\mathbf{x}'}(T_{\mathbf{x}}G(\mathbf{x};\mathbf{x}'))^t[\mathbf{u}(\mathbf{x})] - T_{\mathbf{x}'}G(\mathbf{x};\mathbf{x}')\langle\mathbf{h}(\mathbf{x})\rangle\}\,\mathrm{d}s_{\mathbf{x}},\quad \mathbf{x}'\in\Gamma. \tag{2.14}$$

Theorems 2.2 and 2.3 are valid for quite general cracks in the half-space. In fracture mechanics, the most important problem is Problem $\mathcal{N}$, in which the given tractions satisfy

$$\mathbf{h}^+ = -\mathbf{h}^- = \mathbf{h},$$

say, whence the integral equation (2.14) reduces to

$$\int_\Gamma T_{\mathbf{x}'}(T_{\mathbf{x}}G(\mathbf{x};\mathbf{x}'))^t[\mathbf{u}(\mathbf{x})]\,\mathrm{d}s_{\mathbf{x}} = \mathbf{h}(\mathbf{x}'). \tag{2.15}$$

In the special case of a crack perpendicular to the free surface it is possible to calculate the kernel of the integral equation (2.15) explicitly and, hence, to study its properties.

This will be done in the next section. We remark that equations similar to (2.15) are discussed in [21] and [22] for cracks in full spaces.

3. The flat perpendicular crack

Let Γ be a flat crack in the xy-plane, so that the crack is in a plane perpendicular to the free surface $y = 0$. Assume that the crack is pressurized, so that

$$\mathbf{h}(\mathbf{x}) = \frac{\mu}{4\pi(1-\nu)} p(\mathbf{x})\mathbf{k}, \tag{3.1}$$

where $\mathbf{k}$ is a unit vector in the z-direction, $p(\mathbf{x})$ is a given (scalar) function of $\mathbf{x} \in \Gamma$ and the factor $\mu/(4\pi(1-\nu))$ is inserted for later convenience.

According to Theorem 2.3, the displacement field $\mathbf{u}$ can be represented as an elastic double-layer potential,

$$u_m(\mathbf{x}') = c_{ijkl} \int_\Gamma [u_i(\mathbf{x})] n_j(\mathbf{x}) \frac{\partial}{\partial x_k} G_{lm}(\mathbf{x}; \mathbf{x}')\, \mathrm{d}s_{\mathbf{x}}.$$

Computing the corresponding tractions on Γ gives the integral equation (2.15). It turns out that for the present geometry the problem decouples. Moreover, for a pressurized crack only $[u_3]$ is non-zero. It will be determined from a scaler integral equation.

In the solid, i.e. for $y' > 0$ and $z' \neq 0$, or $z' = 0$ and $(x', y') \in C\Gamma$, $C\Gamma$ being the complement of Γ in the plane, we get

$$u_m(\mathbf{x}') = \int_\Gamma [u_3(\mathbf{x})]\, \Sigma_m(\mathbf{x}; \mathbf{x}')\, \mathrm{d}s_{\mathbf{x}},$$

where

$$\Sigma_m(\mathbf{x}; \mathbf{x}') = \lambda \frac{\partial G_{\alpha m}}{\partial x_\alpha} + (\lambda + 2\mu) \frac{\partial G_{3m}}{\partial z}$$

is the τ_{33} component of the stress tensor at $\mathbf{x} \in \Gamma$ due to a point force acting at $\mathbf{x}'$ in the mth direction, and we have used the convention that repeated Greek subscripts are to be summed over 1 and 2. Σ_m can be calculated from Mindlin's solution; we find that it has the form

$$8\pi(1-\nu)\Sigma_1 = (x - x')[A_1 + B_1(z - z')^2],$$

$$8\pi(1-\nu)\Sigma_2 = A_2 + B_2(z - z')^2,$$

$$8\pi(1-\nu)\Sigma_3 = (z - z')[A_3 + B_3(z - z')^2],$$

with $z = 0$ on Γ. Just as in (2.6), we can write

$$\Sigma_m = \Sigma_m^{\mathrm{F}} + \Sigma_m^{\mathrm{C}},$$

with corresponding decompositions for A_m and B_m. The full-space solution is given by

$$A_1^{\mathrm{F}} = (1 - 2\nu)R^{-3}, \qquad A_2^{\mathrm{F}} = (y - y')A_1^{\mathrm{F}}, \qquad A_3^{\mathrm{F}} = -A_1^{\mathrm{F}}; \tag{3.2}$$

the expressions for B_m^{F} are not required here. The half-space correction is given by

$$A_1^{\mathrm{C}} = \frac{\kappa(1-2\nu)}{R_+^3} + \frac{6y'}{R_+^5}[2\nu y' - (1-2\nu)y] - \frac{D}{R_+(R_+ + y + y')^2},$$

$$A_2^{\mathrm{C}} = \frac{(1-2\nu)}{R_+^3}[\kappa y - (3+4\nu)y'] - \frac{6(y+y')}{R_+^5}[2\nu y'^2 - (1-2\nu)yy']$$

$$- \frac{D}{R_+(R_+ + y + y')^2},$$

$$A_3^{\mathrm{C}} = \frac{(5-4\nu)(1-2\nu)}{R_+^3} + \frac{6y'}{R_+^5}[2\nu y' - (3-2\nu)y] - \frac{3D}{R_+(R_+ + y + y')^2}.$$

Next, we compute $(T(\mathbf{u}))_3$ at $\mathbf{x}' \in \Gamma$. We need to evaluate

$$\lambda \frac{\partial \Sigma_\alpha}{\partial x'_\alpha} + (\lambda + 2\mu)\frac{\partial \Sigma_3}{\partial z'}$$

on $z' = 0$; denote this quantity by (cf. (3.1))

$$\frac{\mu}{4\pi(1-\nu)} K_A(\mathbf{x}, \mathbf{x}').$$

Then, from the structure of Σ_m, we have

$$2\mu K_A(\mathbf{x}, \mathbf{x}') = \lambda\left[(x - x')\frac{\partial A_1}{\partial x'} - A_1 + \frac{\partial A_2}{\partial y'}\right] - (\lambda + 2\mu)A_3. \tag{3.3}$$

In particular, using (3.2), we obtain

$$2\mu K_A^{\mathrm{F}} = (2\mu - \lambda)A_1^{\mathrm{F}} + \left[(x - x')\frac{\partial A_1^{\mathrm{F}}}{\partial x'} + (y - y')\frac{\partial A_1^{\mathrm{F}}}{\partial y'}\right],$$

which gives

$$K_A^{\mathrm{F}}(\mathbf{x}, \mathbf{x}') = R_1^{-3},$$

where $R_1 = |\mathbf{x} - \mathbf{x}'| = \sqrt{[(x - x')^2 + (y + y')^2]}$. The calculation of K_A^{C} is straightforward but tedious; one merely substitutes the expressions for A_m^{C} into (3.3). The final result is

$$K_A(\mathbf{x}, \mathbf{x}') = \frac{1}{R_1^3} + \frac{(-5 + 20\nu - 24\nu^2)}{R_2^3}$$

$$+ \frac{6}{R_2^5}[3yy' - 2\nu(1-2\nu)(y + y')^2] + \frac{12(1-\nu)(1-2\nu)}{R_2(R_2 + y + y')^2}, \tag{3.5}$$

where $R_2 = \sqrt{[(x - x')^2 + (y - y')^2]}$; this formula is in agreement with results in [6] and [10].

The integral equation to be solved is

$$\int_\Gamma v(\mathbf{x}) K_A(\mathbf{x}, \mathbf{x}')\,\mathrm{d}\mathbf{x} = p(\mathbf{x}'), \quad \mathbf{x}' \in \Gamma, \tag{3.6}$$

where we have set $v(\mathbf{x}) = [u_3(\mathbf{x})]$. The kernel K_A is seen to be function of $x - x'$, and no other x, x' dependence, whereas the dependence on y, y' is more complicated: it depends on $y - y'$ but also on yy' and $y + y'$. Let A denote the operator with kernel K_A. Clearly, A defines a linear and continuous mapping

$$A: C_0^\infty(\mathbb{R}^2_+) \to C^\infty(\mathbb{R}^2_+).$$

The mapping properties of the operator A in Sobolev spaces are direct consequences of Lemma 2.4. It gives a continuous operator

$$A: H^s_{\text{comp}}(\overline{\mathbb{R}^2_+}) \to H^{s-1}_{\text{loc}}(\mathbb{R}^2_+) \tag{3.7}$$

for $s > 0$.

4. Solvability and regularity

In this section we study the integral equation (3.6). We start with a special case, in which $\Gamma = E$, where

$$E = \{(x, y, z):\ -\infty < x < \infty, 0 < y < a, z = 0\}. \tag{4.1}$$

E is called an *edge crack*; it corresponds to a saw-cut of depth a in the surface of the elastic half-space. A Fourier transform with respect to x leads to a two-dimensional boundary-value problem in the y–z plane, with $y > 0$; references to the mechanics literature on this problem are given in [14]. Our analysis of this problem is given in section 4.1. Then, we consider the general situation in which Γ is bounded. There are two cases, depending on whether the crack edge $\partial\Gamma$ meets the free surface $\partial\mathbb{R}^3_+$, or not. In the latter case, we have the same situation as in [16], since the integral operator in (3.6) and the pseudo-differential operator with symbol $|\xi|$ have a compact (in fact, smoothing) difference, provided that either y or y' vary in a closed subset of $\mathbb{R}^2_+$ not intersecting $\partial\mathbb{R}^2_+$. The former case, in which $\partial\Gamma \cap \partial\mathbb{R}^2_+ \neq \emptyset$, demands some further efforts, due to the corner points at the free surface. This case is analysed in section 4.2.

4.1. *An edge crack*

Consider the integral equation (3.6), in which $\Gamma = E$.

Theorem 4.1. *Let* $\Gamma = E$, *defined by* (4.1). *The integral equation* (3.6) *has a unique solution* v *in* $H^s(\mathbb{R}^2_+, E)$ *for each given right-hand side* p *belonging to* $H^{s-1}(E)$, *for all* s, $0 < s < 1$.

Proof.

Step 1. Injectivity. The equation has at most one solution. Assuming the existence of a non-trivial solution v to the homogeneous equation then $P(TG)v$ solves homogeneous form of Problem $\mathcal{N}$, contradicting the (assumed) uniqueness.

Step 2. A is Fredholm. In order to prove that the operator

$$A: H^s(\mathbb{R}^2_+, E) \to H^{s-1}(E)$$

is Fredholm for all s, with $0 < s < 1$, we appeal to the general theory in [17, 19] and [20]. Thus, we have to show the bijectiveness of the interior symbol and of the boundary symbol. First, according to (3.7), A is a continuous linear operator that is elliptic in the interior, as its symbol is equal to $|\xi|$.

$$(\sigma_{\partial}(A, \xi)w)(y') = \int_0^\infty \hat{k}_A(\xi, y, y')w(y)\,dy,$$

where we have set $K_A(x, x', y, y') = k_A(x - x', y, y')$ and the caret denotes Fourier transform with respect to x:

$$\hat{k}_A(\xi, y, y') = \int_{-\infty}^{\infty} k_A(x, y, y')\,e^{i\xi x}\,dx.$$

We have to show the family of operators $\sigma_\partial(A, \xi)$ is invertible for every $\xi \in \mathbb{R}$. We reduce this to the case $\xi = 0$ (i.e. to the Mellin symbol) by the following observations. From (3.5), we have

$$k_A(\lambda x, \lambda y, \lambda y') = \lambda^{-3} k_A(x, y, y')$$

for any non-zero scalar λ, whence

$$\hat{k}_A(\xi, y, y') = \lambda^{-2}\hat{k}_A(\lambda\xi, y/\lambda, y'/\lambda).$$

Also, if we define

$$w_\lambda(y) = w(\lambda y)/\lambda,$$

we have

$$(\sigma_{\partial}(A, \xi)w)(y') = (\sigma_\partial(A, \lambda\xi)w_\lambda)(y'/\lambda).$$

Hence, by continuity in ξ and the above homogeneity, we can conclude that if $\sigma_\partial(A, \xi)$ is invertible at $\xi = 0$ then it is invertible for all ξ.

Next, we calculate the Mellin symbol $M(A) := \sigma_\partial(A, 0)$. From the kernel K_A of A we obtain the kernel $K_{M(A)}$ by

$$K_{M(A)}(y, y') = \frac{1}{2}\hat{k}_A(0, y, y') = \int_0^\infty k_A(x, y, y')\,dx$$

Inserting (3.5), we obtain

$$K_{M(A)}(y, y') = \frac{1}{(y - y')^2} - \frac{1}{(y + y')^2} + \frac{12yy'}{(y + y')^4},$$

where we have used the following elementary integrals:

$$\int_{-\infty}^{\infty} \frac{dx}{R_1^3} = \frac{2}{(y - y')^2}, \quad \int_{-\infty}^{\infty} \frac{dx}{R_2^3} = \frac{2}{(y + y')^2},$$

$$\int_{-\infty}^{\infty} \frac{dx}{R_2^5} = \frac{4}{3(y + y')^4}, \quad \int_{-\infty}^{\infty} \frac{dx}{R_2(R_2 + y + y')^2} = \frac{2}{3(y + y')^2}.$$

Note that $K_{M(A)}$ does not depend on v. The integral operator $M(A)$ with kernel $K_{M(A)}(y, y')$ arises in the two-dimensional (plain strain) problem of a line crack perpendicular to the free surface of an elastic half-plane, $y > 0$ ([8]). $M(A)$ admits a Mellin representation as follows. Set $k_{M(A)}(y'/y) = y^2 K_{M(A)}(y, y')$. Then

$$M(A)v(y') = \int_0^\infty K_{M(A)}(y, y')v(y)\,dy = \int_0^\infty k_{M(A)}\left(\frac{y'}{y}\right)\frac{v(y)}{y}\frac{dy}{y},$$

which is a Mellin convolution. Thus, with the Mellin transform taken as

$$\mathcal{M}u \equiv \tilde{u}(z) = \int_0^\infty t^{z-1} u(t)\,\mathrm{d}t,$$

we obtain

$$\mathcal{M}\{y' M(A) v(y')\} = m_A(z)\tilde{v}(z),$$

where the Mellin symbol $m_A(z)$ is given by

$$m_A(z) = \hat{k}_{M(A)}(z+1) = \int_0^\infty t^z k_{M(A)}(t)\,\mathrm{d}t.$$

Explicit evaluation gives [14]

$$m_A(z) = \frac{2\pi z}{\sin \pi z}\left[\sin^2\left(\frac{\pi}{2}z\right) - z^2\right].$$

The Mellin symbol $m_A(z)$ is analytic in the strip $|\Re z| < 2$; within this strip, $m_A(z)$ has a single (double) zero at $z = 0$. Hence, the Mellin symbol $M(A) = \sigma_\partial(A, 0)$ is invertible and, thus, the Fredholm property follows.

Step 3. The index vanishes. This property is shown by means of a homotopy through Fredholm operators. At the end, we arrive at an operator with index zero. Since the index is unchanged under homotopies, the same is true for the original operator A.

The kernel of the operator A is exactly the value at $t = 1$ of

$$k_{k,\nu}(x - x', y, y') = \frac{1}{R_1^3} + \frac{(-5 + 20\nu - 24\nu^2)}{R_2^3} + \frac{6}{R_2^5}\left[3tyy' - 2\nu(1 - 2\nu)(y + y')^2\right] + \frac{12(1-\nu)(1-2\nu)}{R_2(R_2 + y + y')^2}.$$

Consider a homotopy giving a connection between $k_{1,\nu}$ and $k_{0,1/2}$, taking, for instance, t as the path parameter and $\nu(t) = \frac{1}{2} + (\nu - \frac{1}{2})t$. Note the particularly simple form of the kernel at the end point $t = 0$,

$$k_{0,1/2}(x - x', y, y') = R_1^{-3} - R_2^{-3}.$$

Along the homotopy the ellipticity conditions remain valid. In fact, the interior symbol, which arises from the term R_1^{-3} only, is unaffected. It remains to consider the Mellin symbol. Similar calculations as in Step 2 show that $k_{t,\nu}$ has the associated Mellin operator with kernel

$$\frac{1}{(y - y')^2} - \frac{1}{(y + y')^2} + \frac{12tyy'}{(y + y')^4}.$$

The corresponding Mellin symbol is $m_t(z)$, where

$$m_t(z) = \frac{2\pi z}{\sin \pi z}\left[\sin^2\left(\frac{\pi}{2}z\right) - 1 + t(1 - z^2)\right],$$

so that $m_1 = m_A$. This symbol is also analytic within the strip $|\Re z| < 2$. Moreover, it does not vanish for $0 < |\Re z| \leqslant 1$ and for all $0 \leqslant t \leqslant 1$. Therefore, the homotopy runs through Fredholm operators only.

The operator A_0 with kernel $k_{0,1/2}$ in the half-space is closely connected with an operator B with kernel K_A^F (defined by (3.4)) in the full space. In fact, let l denote the extension operator by symmetric reflection,

$$lu(x, y) = \begin{cases} u(x, y) & \text{if } y > 0, \\ u(x, -y) & \text{if } y < 0; \end{cases}$$

the continuity of l,

$$l: H^s(\mathbb{R}_+^2) \to H^s(\mathbb{R}^2) \quad \text{for } 0 \leqslant s < 1,$$

is well known [11]. Then

$$A_0 u(x, y) = \int_{-\infty}^{\infty} \int_0^{\infty} \left(\frac{1}{R_1^3} - \frac{1}{R_2^3} \right) u(x', y') \, dx' \, dy' = Blu(x, y)$$

for $y > 0$. Since Blu is symmetric we see that $lA_0 = Bl$, i.e. the diagram

$$\begin{array}{ccccc} & H^{1/2}(\mathbb{R}_+^2) & \xrightarrow{l} & H^{1/2}(\mathbb{R}^2) & \\ A_0 & \downarrow & & \downarrow & B \\ & H^{-1/2}(\mathbb{R}_+^2) & \xrightarrow{l} & H^{-1/2}(\mathbb{R}^2) & \end{array}$$

commutes. Moreover, the restriction to a subdomain $\mathscr{G} \subset \mathbb{R}_+^2$ makes sense,

$$\begin{array}{ccccc} & H^{1/2}(\mathbb{R}_+^2, \mathscr{G}) & \xrightarrow{l} & H^{1/2}(\mathscr{G}_1) & \\ A_0 & \downarrow & & \downarrow & B, \\ & H^{-1/2}(\mathscr{G}) & \xrightarrow{l} & H^{-1/2}(\mathscr{G}_1) & \end{array}$$

where $\mathscr{G}_1$ is the double of $\mathscr{G}$, i.e. $\mathscr{G}_1 = \mathscr{G} \cup \mathscr{G}_-$, where $\mathscr{G}_-$ is the mirror image of $\mathscr{G}$ in $\mathbb{R}_-^3$; in particular, we can take $\mathscr{G} = E$.

It is well-known that B is an isomorphism and has index zero, in particular. This remains true when restricted to symmetric functions. Therefore, A_0 and, hence, the original operator A both have index zero.

Remark 4.1. The homotopy in the proof of Theorem 4.1 could have been avoided if the Mellin symbol had not depended on t. Then the difference between the operators A and A_0 would have been compact.

4.2. *A bounded crack*

In this section, we study the integral equation (3.6) wherein Γ is bounded; we assume that the crack edge, $\partial\Gamma$, meets the free surface, $\partial\Gamma \cap \partial\mathbb{R}_+^2 \neq \emptyset$.

Theorem 4.2. *The integral equation* (3.6) *has a unique solution v in $H^s(\mathbb{R}_+^2, \Gamma)$ for each given right-hand side p belonging to $H^{s-1}(\Gamma)$, for all s, $0 < s < 1$.*

Proof.

Step 1. Injectivity. This follows, as before, from uniqueness for Problem $\mathscr{N}$.

Step 2. A is Fredholm. According to the general theory [20], we have to check bijectivity of the interior symbol, the boundary symbol, and the corner symbol at each corner point. To begin with, let us reconsider the homotopy of integral operators described in the proof of Theorem 4.1. There, we checked bijectivity of the boundary symbols at $\partial\Gamma \cap \partial\mathbb{R}^2_+$ along the homotopy. At the rest of the boundary, $\partial\Gamma\backslash\partial\mathbb{R}^2_+$, the boundary symbol is bijective according to [16]. It remains to treat the corner symbols.

To be specific, suppose that the crack has two corner points, at $C_1(-b, 0)$ and $C_2(b, 0)$ in the x–y plane. Focussing attention on C_1, define polar coordinates r, θ by

$$x = -b + r\cos\theta \quad \text{and} \quad y = r\sin\theta,$$

so that, close to C_1, Γ occupies the interval $0 \leqslant \theta \leqslant \alpha$, where α is the opening angle at C_1. Now, recall that the corner symbol at a given corner arises by taking a Mellin transform in a radial direction. It yields a family of operators on the base of a cone, which, in our case, is the interval $0 \leqslant \theta \leqslant \alpha$. We denote these operators by $M_{t,v}(z)$. They are continuous,

$$M_{t,v}(z)\colon H^s(0, \mathring{\alpha}) \to H^{s-1}(0, \mathring{\alpha}), \tag{4.2}$$

where $H^s(0, \mathring{\alpha}) \subseteq H^s(0, \pi)$ denotes the subspace of functions which is embedded by extension by zero across α. They are Fredholm operators since the interior and boundary symbols are bijective for $0 < s < 1$.

Step 3. The index vanishes. Bijectivity is guaranteed for large $|z|$, with $c < \Re z < C$, where c and C are arbitrary constants. It remains to show injectivity for all z, with $0 < \Re z < 1$ (cf. [17]). Note that the bijectivity of the boundary symbol shown in the proof of Theorem 4.1 implies the bijectivity of

$$M_{t,v}(z)\colon H^s(0, \pi) \to H^{s-1}(0, \pi),$$

because we can treat the half-space as a special case of a cone. Then $M_{t,v}$, acting according to (4.2), must be injective as a restriction to subspaces. Thus, we have proved that the homotopy runs through Fredholm operators only. The end point has index zero and so does every other point along the homotopy. □

Finally, we shall give a regularity result.

Theorem 4.3. *Let $v \in H^s(\mathbb{R}^2_+, \Gamma)$ be the unique solution to* (3.6), *for a given $p \in H^{s-1}(\Gamma)$. Then $p \in H^{t-1}(\Gamma)$ with $t > s$, $|t - \frac{1}{2}| < \frac{1}{2}$ implies $v \in H^t(\mathbb{R}^2_+, \Gamma)$.*

Proof. By Theorem 4.2, the solutions in $H^s(\mathbb{R}^2_+, \Gamma)$ and $H^t(\mathbb{R}^2_+, \Gamma)$ for all s and t satisfying $|s - \frac{1}{2}| < \frac{1}{2}$ and $|t - \frac{1}{2}| < \frac{1}{2}$ must coincide. □

Remark 4.2. The strength of the singularity term $r^{1/2}$ at the interior boundary of Γ shows that this result cannot be improved in terms of Sobolev spaces. For totally characteristic spaces with weights ([20]) the smoothness order can be increased indefinitely, but the weight is subject to the same limitations.

Special interest lies in the singularity at the corner points of Γ. A quantitative determination as in the case of interior corner points in [17] would be highly desirable.

Appendix: function spaces

We begin with the necessary notations of the different function spaces. The notation for the standard function spaces we use as in [11]; in particular, for the Sobolev spaces H^s with arbitrary real smoothness index s in the euclidean space $\mathbb{R}^n$. We need a careful distinction between the different possibilities of definition of Sobolev spaces for subsets of $\mathbb{R}^n$. We suppose that all open and closed subsets have piecewise smooth boundaries.

Let $\Gamma \subset \mathbb{R}^n$ be closed. Then

$$H^s(\mathbb{R}^n, \Gamma) = \{u \in H^s(\mathbb{R}^n): \operatorname{supp} u \subset \Gamma\}$$

is a closed subset of $H^s(\mathbb{R}^n)$ and, hence, complete with the induced topology.

Let $\Omega \subset \mathbb{R}^n$ be open. Then we set

$$H^s(\Omega) = r_\Omega H^s(\mathbb{R}^n),$$

where r_Ω denotes the restriction operator of distributions to the open subset Ω. The exact sequence

$$0 \to H^s(\mathbb{R}^n, \mathbb{R}^n \backslash \Omega) \to H^s(\mathbb{R}^n) \to H^s(\Omega) \to 0$$

implies the algebraical isomorphism

$$H^s(\Omega) \cong H^s(\mathbb{R}^n)/H^s(\mathbb{R}^n, \mathbb{R}^n \backslash \Omega),$$

which turns $H^s(\Omega)$ into a Hilbert space with the factor topology. Its elements can be represented as distributions on Ω which are extendible to $H^s(\mathbb{R}^n)$-distributions. In particular, we need the cases $n = 2$, $n = 3$, $\Omega = \mathbb{R}^2_+$ and $\Gamma = \overline{\mathbb{R}^2_+}$.

This construction also makes sense when Ω has interior boundary parts, like cracks. Then, for $s > \frac{1}{2}$, the spaces $H^s(\Omega)$ inherit certain compatibility conditions of the boundary values from both sides of the crack. For instance, for $\frac{1}{2} < s < \frac{3}{2}$, the traces from both sides must coincide, and for $\frac{1}{2} < s - k < \frac{3}{2}$ the same for traces of all normal derivatives of order $\leqslant k$. In order to avoid these compatibility conditions we introduce another notion of extendible distributions. The local model is $\mathbb{R}^n$ with a crack Γ which is a closed subset of $\mathbb{R}^{n-1}$. Then $\tilde{H}^s(\mathbb{R}^n \backslash \Gamma)$ is defined as the closed subspace of $H^s(\mathbb{R}^n_+) \bigoplus H^s(\mathbb{R}^n_-)$ with the compatibility conditions only on $\mathbb{R}^{n-1} \backslash \Gamma$. If $\Gamma \subset \mathbb{R}^n$ is the closure of an $(n-1)$-dimensional manifold $\Omega \subset \mathbb{R}^n$, then $H^s(\mathbb{R}^n \backslash \Gamma)$ is defined in the standard way with a partition of unity, local charts and the above local model for the crack.

Let now $\Gamma \subset \mathbb{R}^n_+$ be closed. As far as $\partial\Gamma \cap \partial\mathbb{R}^n_+ \neq \emptyset$, Γ is no longer closed in $\mathbb{R}^n$. Then there is a closed $\Gamma_1 \subset \mathbb{R}^n$ such that $\Gamma = \mathbb{R}^n_+ \cap \Gamma_1$. We define

$$H^s(\mathbb{R}^n_+, \Gamma) = H^s(\mathbb{R}^n, \Gamma_1)/H^s(\mathbb{R}^n, \overline{\Gamma_1 \backslash \Gamma}).$$

Elements of this factor space can be represented as distributions in the half-space $\mathbb{R}^n_+$ with support in Γ which can be extended to $H^s(\mathbb{R}^n)$-distributions.

We shalll need mainly the spaces $\tilde{H}^s(\mathbb{R}^3_+ \backslash \mathbb{R}^2_+)$, $\tilde{H}^s(\mathbb{R}^3_+ \backslash \Gamma)$ and $\tilde{H}^s(\mathbb{R}^2_+, \Gamma)$.

References

1. Agmon, S., Douglis, A. and Nirenberg, L., 'Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I and II', *Comm. Pure Appl. Math.*, **12**, 623–727 (1959); **17**, 35–62 (1964).
2. Benthem, J. P., 'The quarter-infinite crack in a half-space; alternative and additional solutions', *Int. J. Solids Struct.*, **16**, 119–130 (1980).
3. Costabel, M. and Wendland, W. L., 'Strong ellipticity of boundary integral operators', *J. Reine Angew. Math.*, **372**, 39–62 (1986).

4. Eskin, G. I., *Boundary Value Problems for Elliptic Pseudodifferential Equations*. Translations of Mathematical Monographs, Vol. 52, American Math. Soc., Providence, RI, 1981.
5. Gol'dshtein, R. V. and Shifrin, E. I., 'Plane crack of an arbitrary discontinuity in a bounded elastic body', *Appl. Math. Mech.*, **46**, 369–377 (1982).
6. Hayashi, K. and Abé, H., 'Stress intensity factors for a semi-elliptic crack in the surface of a semi-infinite solid', *Int. J. Fracture*, **16**, 275–285 (1980).
7. Hsiao, G. C., Stephan, E. P. and Wendland, W. L., 'On the Dirichlet problem in elasticity for a domain exterior to an arc', *J. Comp. Appl. Math.*, **34**, 1–19 (1991).
8. Kaya, A. C. and Erdogan, F., 'On the solution of integral equations with strongly singular kernels', *Q. Appl. Math.*, **45**, 105–122 (1987).
9. Kupradze, V. D., Gegelia, T. G., Basheleishvili, M. O. and Burchuladze, T. V., *Three-dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity*, North-Holland, Amsterdam, 1979.
10. Lee, J. C. and Keer, L. M., 'Study of a three-dimensional crack terminating at an interface', *J. Appl. Mech.*, **53**, 311–316 (1986).
11. Lions, J. L. and Magenes, E., *Non-homogeneous Boundary Value Problems and Applications I*, Springer, Berlin, 1972.
12. Martin, P. A., 'Orthogonal polynomial solutions for pressurized elliptical cracks', *Quart. J. Mech. Appl. Math.*, **39**, 269–287 (1986).
13. Martin, P. A., 'Orthogonal polynomial solutions for elliptical cracks under shear loadings', *Quart. J. Mech. Appl. Math.*, **39**, 520–534 (1986).
14. Martin, P. A., 'End-point behaviour of solutions to hypersingular integral equations', *Proc. Roy. Soc.*, **A432**, 301–320 (1991).
15. Mindlin, R. D., 'Force at a point in the interior of a semi-infinite solid', *Phys.* **7**, 195–202 (1936).
16. Päivärinta, L. and Rempel, S., 'A deconvolution problem with the kernel $1/|x|$ on the plane', *Applicable Analysis*, **26**, 105–128 (1987).
17. Päivärinta, L. and Rempel, S., 'Corner singularities of solutions to $\Delta^{\pm 1/2}u = f$ in two dimensions', *Asymptotic Analysis*, to appear.
18. von Petersdorff, T., Randwertprobleme der Elastizitätstheorie für Polyeder-Singularitäten und Approximation mit Randelementmethoden. *Dissertation*, TH Darmstadt, 1989.
19. Rempel, S. and Schulze, B.-W., *Index Theory of Elliptic Boundary Problems*, Akademie-Verlag, Berlin, 1982.
20. Rempel, S. and Schulze, B.-W., *Asymptotics for Elliptic Mixed Boundary Value Problems*, Math. Research, Vol. 50, Akademie-Verlag, Berlin, 1989.
21. Stephan, E. P., 'A boundary integral equation method for three-dimensional crack problems in elasticity', *Math. Meth. in the Appl. Sci.*, **8**, 609–623 (1986).
22. Wendland, W. L. and Stephan, E. P., 'A hypersingular boundary integral method for two-dimensional screen and crack problems', *Arch. Rat. Mech. Anal.*, **112**, 363–390 (1990).