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Karp's theorem in elastodynamic inverse scattering

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Abstract. Time-harmonic elastic waves are incident upon a bounded obstacle in three dimensions. The resulting scattered waves are characterized by their far-field patterns. We consider some simple questions concerning the determination of the shape of the obstacle from information on the far-field patterns. Specifically, we refine and extend some results of Wall, allowing mixed boundary conditions on piecewise-smooth surfaces. We also prove two elastodynamic analogues of a theorem due to Karp, giving sufficient conditions on the far-field patterns for the obstacle to be spherical. The proofs are indirect and are based on symmetry arguments, as used for scalar problems by Ramm. The possibility of a direct proof is also explored.

1. Introduction

When a time-harmonic elastic wave encounters an obstacle in an unbounded solid, it is scattered to infinity in all directions. The scattered waves separate into a compressional wave (P-wave) and a shear wave (S-wave); each wave is characterized by an amplitude, called a *far-field pattern*. The *direct problem*, described in section 2, is concerned with the calculation of the two far-field patterns, given the incident field, the shape of the obstacle's surface, S, and the boundary condition on S. There is an extensive literature on direct problems; see, for example, [13, 20, 21].

In the corresponding *inverse problems*, the goal is to determine the shape of S and, perhaps, the boundary condition on S, given some information on the far-field patterns for at least one known incident field. There is a considerable literature on these inverse problems in acoustics and in electromagnetism; see [7, 26, 28] and the recent extensive review by Bates *et al* [5]. However, the elastodynamic problem has received much less attention.

Most work in elastodynamics has been concerned with inclusions or cracks, where an inclusion is an obstacle composed of an elastic material differing from the surrounding solid and a crack is defined to be an open smooth surface across which the elastic displacement vector is discontinuous. The motivation behind these studies comes mainly from ultrasonic non-destructive evaluation; for a general review in this area see [33].

For inverse scattering by inclusions, one is interested in determining the inclusion's location and composition. Methods using long waves have been devised; see, e.g. [15, 32, 35]. Born approximations have also been used; see, e.g. [4, 17, 24]. For cavities, several approximate methods have been devised [12] in which only P-waves are used.

For inverse scattering by cracks in a solid, we cite the papers of Gubernatis [14, 15], Achenbach [1, 2, 31] and their coworkers. For example, Gubernatis and Domany [14] note that, in the long-wavelength limit, an obstacle must be a crack if the far-field patterns have equal magnitudes at all pairs of diametrically opposite directions.

All the elastodynamic work cited above involves some form of approximation. In the rest of the current paper, we are concerned with exact results. One question to ask is: how much scattering information is sufficient to uniquely determine, in principle, the obstacle? This question has been addressed by Wall [34]; his work is extended and refined in section 4.

In section 5, we consider elastodynamic analogues of Karp's theorem. Karp [19] proved the following result in two-dimensional acoustics for the scattering of plane waves by a sound-soft (Dirichlet condition) obstacle, S: suppose that the far-field pattern $F(\theta;\alpha)$ is a function of $\theta-\alpha$

$$F(\theta;\alpha)=f(\theta-\alpha)$$

say, for all θ and for all α, where θ is the angle of observation and α is the angle of incidence; then S is a circle. This provides the explicit solution to an inverse-scattering problem. New proofs were given later by Colton and Kirsch [6], in three dimensions and for sound-hard (Neumann condition) obstacles. Colton and Kress [6] have proved an analogous result for electromagnetic scattering by a perfectly conducting obstacle. We give proofs of two elastodynamic analogues of Karp's theorem in three dimensions, giving sufficient conditions on the far-field patterns for S to be a sphere. The proofs are based on some symmetry arguments, as used recently by Ramm [27] for several scalar problems. Thus, the proofs are indirect, whereas the proofs in [6, 8, 19] are direct. In section 6, we sketch how a direct proof for elastodynamics might proceed, following the ideas in [8], and highlight the remaining difficulties; the resolution of these difficulties may lead to new insights into elastodynamic inverse scattering.

2. The direct problem

Let $B_{\rm i}$ denote a bounded, three-dimensional domain, with boundary S, and simply-connected unbounded exterior, $B_{\rm e}$. We suppose that the surface S is *properly regular*, in the terminology of Gurtin [16, section 5]; this means, roughly, that S is closed, connected and piecewise smooth (so that edges and corners are allowed).

The exterior domain $B_{\rm e}$ is filled with homogeneous isotropic elastic material, with Lamé moduli λ and μ, Poisson's ratio ν, and mass density ρ. A stress wave is incident upon the obstacle $B_{\rm i}$; this leads to the following scattering problem.

Direct problem. Find a displacement vector $\boldsymbol{v}(P)$ which satisfies

$$k^{-2}\operatorname{grad}\operatorname{div}\boldsymbol{v}-K^{-2}\operatorname{curl}\operatorname{curl}\boldsymbol{v}+\boldsymbol{v}=\boldsymbol{0} \tag{2.1}$$

for $P\in B_{\rm e}$, radiation conditions at infinity (these are specified below in section 3) and boundary conditions on S. The latter are

$$\boldsymbol{u}(p)=\boldsymbol{0} \qquad \text{for } p\in S_{\rm u} \tag{2.2}$$

and

$$Tu(p) = \mathbf{0} \qquad \text{for } p \in S_t \tag{2.3}$$

where S is partitioned according to

$$S = S_u \cup S_t \qquad S_u \cap S_t = \emptyset$$

and the total displacement in B_e is

$$u(P) = v(P) + u_{inc}(P) \qquad P \in B_e. \tag{2.4}$$

The given incident wave, u_{inc}, is assumed to satisfy (2.1) everywhere. The wavenumbers k and K are defined by

$$\rho\omega^2 = (\lambda + 2\mu)k^2 = \mu K^2$$

and the time-dependence $e^{-i\omega t}$ is suppressed throughout. The traction operator T is defined on smooth parts of S by

$$(Tu)_m(p) = \lambda n_m \frac{\partial u_j}{\partial x_j} + \mu n_j \left(\frac{\partial u_m}{\partial x_j} + \frac{\partial u_j}{\partial x_m} \right) \tag{2.5}$$

where $n(p)$ is the unit normal at $p \in S$, pointing into B_e.

We shall use the following notation: capital letters P, Q denote points of $B_e \cup B_i$; lower-case letters p, q denote points of S; r is the position vector of P with respect to the origin O, which is chosen at some point in B_i; $r = |r|$ and $\hat{r} = r/r$.

If $S_u = S$, the boundary condition (2.3) is absent; this is appropriate if the obstacle B_i is a rigid, immovable body. On the other hand, B_i is a cavity if $S_t = S$.

We shall also require some properties of the following related interior problem.

Vibration problem. Find a non-trivial displacement vector $u(P)$ which satisfies (2.1) in the bounded domain B_i, together with the boundary conditions (2.2) and (2.3) on the properly-regular surface S.

This eigenvalue problem only has non-trivial solutions for certain values of the frequency ω. It is known that these eigenfrequencies form an infinite, discrete set, and that each eigenfrequency has a finite multiplicity. For proofs of these results, see Gurtin [16, sections 75–78], Roseau [29, ch 6, subsection 3.3] and Sanchez Hubert and Sanchez Palencia [30, ch 2, section 7].

3. Radiation conditions and far-field patterns

The formulation of radiation conditions is given in [21, pp 124–130]. One formulation is the following: decompose the scattered field as

$$v(P) = v^P + v^S$$

where

$$v^P = -k^{-2}\text{grad div } v \qquad v^S = v - v^P$$

then, we require that

$$r\left(\frac{\partial v^P}{\partial r} - \mathrm{i}kv^P\right) \to \mathbf{0} \qquad r\left(\frac{\partial v^S}{\partial r} - \mathrm{i}Kv^S\right) \to \mathbf{0} \qquad \text{as} \quad r \to \infty \tag{3.1}$$

uniformly with respect to all directions $\hat{r}$. These are the radiation conditions. It is common to require also that both $v^P \to \mathbf{0}$ and $v^S \to \mathbf{0}$ as $r \to \infty$. However, these conditions are implied by (3.1).

The fields v^P and v^S are the longitudinal and transverse parts, respectively, of the scattered field v; they satisfy

$$(\nabla^2 + k^2)v^P = \mathbf{0} \qquad \text{and} \qquad (\nabla^2 + K^2)v^S = \mathbf{0}$$

and correspond to radiated P-waves and S-waves, respectively.

We can specify the behaviour of $v(P)$ for large r more precisely. We have

$$v(r\hat{r}) = F^P(\hat{r})\frac{\mathrm{e}^{\mathrm{i}kr}}{r} + F^S(\hat{r})\frac{\mathrm{e}^{\mathrm{i}Kr}}{r} + \mathrm{O}(r^{-2}) \tag{3.2}$$

as $r \to \infty$, uniformly with respect to all directions $\hat{r}$. The vectors F^P and F^S are called the *far-field patterns* (or scattering amplitudes). It turns out that

$$F^P(\hat{r}) = F^P(\hat{r})\hat{r} \qquad \text{and} \qquad \hat{r}\cdot F^S(\hat{r}) = 0. \tag{3.3}$$

Thus, the radiated P-wave propagates in the outward radial direction, whereas the radiated S-wave is polarized in a plane perpendicular to the radial direction.

The far-field patterns can be calculated in terms of the displacements and tractions on the surface of the obstacle. The starting point is the familiar representation

$$v(P) = \frac{1}{2}\int_S \{(Tv)\cdot \mathsf{G}(q;P) - v\cdot T^q \mathsf{G}(q;P)\}\,\mathrm{d}s_q \tag{3.4}$$

for $P \in B_\mathrm{e}$, where T^q means T applied at $q \in S$ and $\mathsf{G}(P;Q)$ is the fundamental the Green tensor (Kupradze matrix), defined by

$$(\mathsf{G}(P;Q))_{ij} = \frac{1}{\mu}\left\{\Psi\delta_{ij} + \frac{1}{K^2}\frac{\partial^2}{\partial x_i \partial x_j}(\Psi - \Phi)\right\}$$

where

$$\Phi = -\mathrm{e}^{\mathrm{i}kR}/(2\pi R) \qquad \Psi = -\mathrm{e}^{\mathrm{i}KR}/(2\pi R)$$

and R is the distance between P and Q. Asymptotic approximation of (3.4) for large r yields (3.2) and (3.3), where, for example (see [10])

$$F^P(\hat{r}) = \frac{1}{4\pi(\lambda + 2\mu)}\,\{v(q), \hat{r}\exp(-\mathrm{i}k\hat{r}\cdot q)\}_S\,. \tag{3.5}$$

Here, we have used the notation

$$\{v(q), w(q)\}_S = \int_S (v\cdot Tw - w\cdot Tv)\,\mathrm{d}s_q\,.$$

We can also obtain a formula for F^P in terms of the total field u; this is more convenient when using the boundary conditions on S. Thus, let $\phi(Q) = \exp(-\mathrm{i}k\hat{r}\cdot q)$. Since the displacement field

$$U = (\mathrm{i}/k)\,\mathrm{grad}_Q\,\phi = \phi\hat{r}$$

is regular everywhere, an application of the Betti reciprocal theorem in B_i to U and u_inc gives

$$\{u_\mathrm{inc}(q), \phi(q)\hat{r}\}_S = 0.$$

Adding this result to (3.5), using (2.4), gives

$$F^P(\hat{r}) = \frac{1}{4\pi(\lambda+2\mu)}\{u(q), \phi(q)\hat{r}\}_S.$$

Now, for any constant vector b, we have

$$u(q)\cdot T\{\phi b\} = -\mathrm{i}k\{\lambda(u\cdot n)(b\cdot\hat{r}) + \mu(n\cdot\hat{r})(u\cdot b) + \mu(n\cdot b)(u\cdot\hat{r})\}\phi(q).$$

Hence

$$F^P(\hat{r}) = \frac{-\hat{r}}{4\pi(\lambda+2\mu)} \times \int_S [\mathrm{i}k\{\lambda(u\cdot n) + 2\mu(n\cdot\hat{r})(u\cdot\hat{r})\} + t\cdot\hat{r}]\exp(-\mathrm{i}k\hat{r}\cdot q)\,\mathrm{d}s_q \quad (3.6)$$

where $t = Tu$, u and n are all evaluated at q, the integration point on S with position vector q. Similar calculations for F^S give

$$F^S(\hat{r}) = \frac{-1}{4\pi\mu}\hat{r}\int_S [\mathrm{i}K\mu\{(u\times\hat{r})(n\cdot\hat{r}) + (n\times\hat{r})(u\cdot\hat{r})\} + t\times\hat{r}] \times \exp(-\mathrm{i}K\hat{r}\cdot q)\,\mathrm{d}s_q. \quad (3.7)$$

Wall [34] has given similar formulae for the far-field patterns, but his involve the scattered field v rather than the total field u.

4. The inverse problem

We are interested in the following *inverse problem*: given some information on the far-field patterns, determine the shape of S and/or the boundary condition on S. To begin with, suppose that we know F^P and F^S for all $\hat{r} \in \Omega$, the unit sphere (since F^P and F^S are analytic functions of $\hat{r}$, it is enough to know them on an open patch of Ω). We can then reconstruct the scattered field everywhere outside the smallest ball containing S, B_S [9]. This field can then be continued analytically into a portion of B_S (this portion certainly includes $B_S \setminus \overline{B_\mathrm{i}}$).

In the rest of this section, we consider the general question of uniqueness: is it possible that two different obstacles can give rise to the same far-field patterns?

Firstly, let us consider the determination of the boundary condition on a known surface S.

Theorem 4.1. Suppose that the obstacle B_i with boundary S has known non-zero far-field patterns, $\boldsymbol{F}^P(\hat{\boldsymbol{r}})$ and $\boldsymbol{F}^S(\hat{\boldsymbol{r}})$, for all $\hat{\boldsymbol{r}} \in \Omega$. Then the boundary condition on S, that is, the partition $S = S_u \cup S_t$ with $S_u \cap S_t = \emptyset$, is uniquely determined.

Proof. Let $S = S_u^j \cup S_t^j$ $(j = 1, 2)$ denote two distinct partitions of S, with boundary conditions given by (2.2) and (2.3). Since the far-field patterns are the same for each partition, the corresponding total fields are also identical throughout B_e. Choose a non-empty piece $S_{ut} \subset S_u^1 \cap S_t^2$, whence

$$\boldsymbol{u}(p) = \boldsymbol{0} \qquad \text{and} \qquad T\boldsymbol{u}(p) = \boldsymbol{0} \qquad \text{for } p \in S_{ut}.$$

These imply that $\boldsymbol{u} \equiv \boldsymbol{0}$ in B_e, which contradicts the known far-field behaviour. Hence, $S_{ut} = \emptyset$ and the result follows. □

Next, we consider two obstacles with different shapes. Let B_i^j denote the interior of the obstacle with boundary S_j and exterior B_e^j $(j = 1, 2)$. Denote the corresponding scattered fields by $\boldsymbol{v}_j$. Wall [34] gives the following result.

Theorem. Suppose that B_i^1 and B_i^2 have the same non-zero far-field patterns, $\boldsymbol{F}^P(\hat{\boldsymbol{r}})$ and $\boldsymbol{F}^S(\hat{\boldsymbol{r}})$, for all $\hat{\boldsymbol{r}} \in \Omega$. Then B_i^1 and B_i^2 are not disjoint, that is $B_i^1 \cap B_i^2 \neq \emptyset$.

Proof. Following Jones [18], suppose that B_i^1 and B_i^2 are disjoint. We have $\boldsymbol{v}_1 \equiv \boldsymbol{v}_2$ everywhere outside $B_i^1 \cup B_i^2$. From (3.4), we have, for $P \in B_e^1$

$$\begin{aligned} \boldsymbol{v}_1(P) &= \frac{1}{2} \int_{S_1} \{(T\boldsymbol{v}_1) \cdot \mathsf{G}(q; P) - \boldsymbol{v}_1 \cdot T^q \mathsf{G}(q; P)\} \, \mathrm{d}s_q \\ &= \frac{1}{2} \int_{S_1} \{(T\boldsymbol{v}_2) \cdot \mathsf{G}(q; P) - \boldsymbol{v}_2 \cdot T^q \mathsf{G}(q; P)\} \, \mathrm{d}s_q \\ &= \boldsymbol{0} \end{aligned}$$

since $\boldsymbol{v}_2$ is a regular elastodynamic field in B_i^1. This is a contradiction. □

Wall [34] proves a more general result, allowing B_i^1 and B_i^2 to be inhomogeneous inclusions.

Theorems 4.1 and 4.2 do not require the incident field or the frequency of oscillation to be specified. Moreover, theorem 4.2 does not require a specification of the boundary conditions on S_1 and S_2 (which can be different). However, they both require a knowledge of both $\boldsymbol{F}^P$ and $\boldsymbol{F}^S$.

One way of making further progress is to suppose that we have information for a finite range of frequencies. This leads to an elastodynamic analogue of Schiffer's theorem. Before stating this theorem, we specify the allowable incident fields. Thus, we suppose that the incident field is a plane wave of unit amplitude, propagating in the direction of the unit vector $\hat{\boldsymbol{\alpha}}$. In particular, for an incident P-wave, we have

$$\boldsymbol{u}_{\text{inc}}(P) = \hat{\boldsymbol{\alpha}} \exp(\mathrm{i}k\boldsymbol{r} \cdot \hat{\boldsymbol{\alpha}}) \tag{4.1}$$

whereas for an incident S-wave, we have

$$\boldsymbol{u}_{\text{inc}}(P) = \hat{\boldsymbol{\beta}} \exp(\mathrm{i}K\boldsymbol{r} \cdot \hat{\boldsymbol{\alpha}})$$

where $\hat{\boldsymbol{\beta}}$ is any unit vector satisfying

$$\hat{\boldsymbol{\alpha}} \cdot \hat{\boldsymbol{\beta}} = 0.$$

For any of these incident fields, there will be, in general, a scattered P-wave and a scattered S-wave.

Theorem 4.3. Suppose that B_i^1 and B_i^2 have the same non-zero far-field patterns, $\boldsymbol{F}^P(\hat{\boldsymbol{r}})$ and $\boldsymbol{F}^S(\hat{\boldsymbol{r}})$, for all $\hat{\boldsymbol{r}} \in \Omega$ and for all frequencies in the interval $\omega_1 \leqslant \omega \leqslant \omega_2$, with $\omega_1 < \omega_2$. Then $B_i^1 = B_i^2$.

Proof. Theorem 4.2 implies that $B_i^1 \cap B_i^2 \neq \emptyset$ and that $\boldsymbol{v}_1 \equiv \boldsymbol{v}_2$ in the exterior of $B_i^1 \cup B_i^2$. Let B_0 be any connected component of $B_i^1 \setminus B_i^2$, with boundary S_0. It is clear that S_0 is properly regular, composed of a piece of S_1 and a piece of S_2. Since $B_0 \subset B_e^2$, it follows that $\boldsymbol{u}_2(P) \equiv \boldsymbol{v}_2 + \boldsymbol{u}_{\text{inc}}$ solves the vibration problem in B_0, for all ω with $\omega_1 \leqslant \omega \leqslant \omega_2$. There are now three possibilities: $\boldsymbol{u}_2 \not\equiv \boldsymbol{0}$ or $\boldsymbol{u}_2 \equiv \boldsymbol{0}$ or $B_0 = \emptyset$. The first possibility is excluded since the eigenfrequencies of the vibration problem are discrete, whereas the second is excluded since

$$|\boldsymbol{u}_2(P)| = |\boldsymbol{u}_{\text{inc}} + \boldsymbol{v}_2| \geqslant |\boldsymbol{u}_{\text{inc}}| - |\boldsymbol{v}_2| = 1 + \mathrm{O}(r^{-1})$$

as $r \to \infty$. Thus, $B_0 = \emptyset$ and $B_i^1 = B_i^2$. □

The above proof is basically Schiffer's proof [22, p 173]; see also [26, subsection 2.1.2.]. Note that it is essential that B_0 is independent of ω. Thus, we cannot rephrase the result to assert that the given far-field information determines the shape of S uniquely. For, it may be possible that two different obstacles generate the same far-field patterns, but that these obstacles vary with ω; see also [18, p 187]. Note also that S_0 always has corners and edges, even if S_1 and S_2 are smooth; see also [34, p 236].

The remarks in the previous paragraph are also applicable to the next theorem, in which the frequency ω is *fixed* but different incident waves are used. Here, 'different' means different angles of incidence (vary $\hat{\boldsymbol{\alpha}}$), different types ($P$-waves, S-waves or both) or different polarizations (vary $\hat{\boldsymbol{\beta}}$ for incident S-waves). We have the following result.

Theorem 4.4. Suppose that B_i^1 and B_i^2 have the same non-zero far-field patterns, $\boldsymbol{F}^P(\hat{\boldsymbol{r}})$ and $\boldsymbol{F}^S(\hat{\boldsymbol{r}})$, for all $\hat{\boldsymbol{r}} \in \Omega$ and for an infinite number of different incident waves. Then $B_i^1 = B_i^2$.

Proof. Let $\boldsymbol{u}_{\text{inc}}^m(P)$ denote the mth incident wave, and let $\boldsymbol{u}_j^m$ denote the corresponding total field exterior to B_i^j for $j = 1, 2$. Proceeding as in the proof of theorem 4.3, we see that $\boldsymbol{u}_2^m$ solves the vibration problem in B_0. It is proved in the appendix that the eigenfunctions $\boldsymbol{u}_2^m$ are linearly independent, whence ω is an eigenfrequency of infinite multiplicity. However, the vibration problem only has eigenfrequencies of finite multiplicity; this contradiction implies that $B_i^1 = B_i^2$. □

So far, we have placed only mild restrictions on S and on the boundary conditions on S. If we tighten these restrictions, we can give a result that only requires information from a single incident wave at a single fixed frequency. Its proof relies on analyticity with respect to frequency.

Theorem 4.5. Suppose that the obstacle B_i with smooth boundary S has known non-zero far-field patterns, $\boldsymbol{F}^P(\hat{\boldsymbol{r}})$ and $\boldsymbol{F}^S(\hat{\boldsymbol{r}})$, for all $\hat{\boldsymbol{r}} \in \Omega$. Suppose that B_i is either a rigid body ($S_t = \emptyset$) or a cavity ($S_u = \emptyset$). Then, the shape of S and the boundary condition on S are uniquely determined.

Proof. We proceed as in the proof of theorem 4.3. Since S_1 and S_2 are assumed to be smooth, with a single boundary condition on each (which may be different), we can solve the direct problem using boundary integral equations [21]. It follows that the solutions $\boldsymbol{u}_j(P)$ are analytic functions of ω in B_e^j. In particular, $\boldsymbol{u}_2(P)$ is an analytic function of ω in $B_0 \subset B_e^2$ and so

$$\boldsymbol{w}(P) = \frac{\partial \boldsymbol{u}_2}{\partial \omega}$$

exists. The result on the shape of S now follows exactly as in Wall [34, theorem 4.1] (which is modelled on Jones [18 theorem 3]), by an application of the elastodynamic version of Green's second theorem in B_0 to $\boldsymbol{w}$ and the complex conjugate of $\boldsymbol{u}_2$. The uniqueness of the boundary condition then follows from theorem 4.1. □

We remark that the above proof will work for non-smooth S and mixed boundary conditions once the direct solution $\boldsymbol{u}$ is known to be an analytic function of ω (in a neighbourhood of the positive real axis in the complex ω-plane).

5. Karp's theorem

The far-field patterns depend on the shape of the obstacle S, as well as on the incident wave. We make this dependence explicit with the notation

$$\boldsymbol{F}^P(\hat{\boldsymbol{r}}; \hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}; S) \qquad \text{and} \qquad \boldsymbol{F}^S(\hat{\boldsymbol{r}}; \hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}; S).$$

Let R be a rotation matrix. Thus, R is a real, orthogonal matrix, which satisfies $R^{-1} = R^T$. Since the elastic material in B_e is isotropic, we have

$$R\boldsymbol{F}^Q(\hat{\boldsymbol{r}}; \hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}; S) = \boldsymbol{F}^Q(R\hat{\boldsymbol{r}}; R\hat{\boldsymbol{\alpha}}, R\hat{\boldsymbol{\beta}}; RS) \tag{5.1}$$

where $Q = P, S$. This identity holds for all unit vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$, and for all rotations R.

If the surface S is spherical, we have

$$RS = S \qquad \text{for all rotations } R$$

and (5.1) reduces to

$$R\boldsymbol{F}^Q(\hat{\boldsymbol{r}}; \hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}; S) = \boldsymbol{F}^Q(R\hat{\boldsymbol{r}}; R\hat{\boldsymbol{\alpha}}, R\hat{\boldsymbol{\beta}}; S) \qquad \text{for all rotations } R \tag{5.2}$$

where $Q = P, S$. Under certain conditions, the converse is true, as we shall now show.

Theorem 5.1. Suppose that $\boldsymbol{F}^P(\hat{\boldsymbol{r}})$ and $\boldsymbol{F}^S(\hat{\boldsymbol{r}})$ are both known for all $\hat{\boldsymbol{r}} \in \Omega$ and both satisfy the symmetry relation (5.2). Suppose further that they are both known for:

(i) one incident wave and all frequencies in the interval $\omega_1 \leqslant \omega \leqslant \omega_2$, with $\omega_1 < \omega_2$ or

(ii) one frequency and an infinite number of different incident waves.

Then S is a sphere.

Proof. Replace S by $R^T S$ in (5.1) and subtract the result from (5.2), giving

$$\boldsymbol{F}^Q(\hat{\boldsymbol{r}};\hat{\boldsymbol{\alpha}},\hat{\boldsymbol{\beta}};S) = \boldsymbol{F}^Q(\hat{\boldsymbol{r}};\hat{\boldsymbol{\alpha}},\hat{\boldsymbol{\beta}};R^T S)$$

for $Q = P, S$. By theorem 4.3 or 4.4, the given information implies that

$$S = R^T S \qquad \text{for all rotations } R$$

and the result follows. □

The basic idea of the above proof is due to Ramm [27]. We remark that case (ii) is the closest analogue of Karp's theorem in acoustic scattering.

Note that it was assumed in theorem 5.1 that S was in the class of properly-regular surfaces and that the boundary conditions on S could be mixed. If we make stronger assumptions, we can get the same result with weaker assumptions on the far-field patterns.

Theorem 5.2. Suppose that B_i is either a rigid body ($S_t = \emptyset$) or a cavity ($S_u = \emptyset$), with smooth boundary S. Suppose that $\boldsymbol{F}^P(\hat{\boldsymbol{r}})$ and $\boldsymbol{F}^S(\hat{\boldsymbol{r}})$ are both known for all $\hat{\boldsymbol{r}} \in \Omega$, for one frequency and for one incident wave. Suppose further that $\boldsymbol{F}^P$ or $\boldsymbol{F}^S$ satisfies the symmetry relation (5.2). Then S is a sphere. □

Proof. By theorem 4.5, the given information on both $\boldsymbol{F}^P$ and $\boldsymbol{F}^S$ is sufficient to determine the shape of S, uniquely. However, we already know that both symmetry relations are satisfied if S is a sphere. Hence, the additional information on $\boldsymbol{F}^P$ or $\boldsymbol{F}^S$ implies that S must be a sphere.

6. Discussion: a direct proof?

Direct proofs of Karp's theorem have been given by Colton and Kirsch [6] for acoustics, and by Colton and Kress [8] for electromagnetism. We have attempted to extend these proofs to elastodynamics, but have failed. It is of interest to see where difficulties remain. Thus, let us consider the case where B_i is a cavity, whence (3.6) gives

$$\boldsymbol{F}^P(\hat{\boldsymbol{r}};\hat{\boldsymbol{\alpha}},\hat{\boldsymbol{\beta}};S) = \frac{-\mathrm{i}k}{4\pi(\lambda+2\mu)}\hat{\boldsymbol{r}} \times \int_S [\lambda(\boldsymbol{u}\cdot\boldsymbol{n}) + 2\mu(\boldsymbol{n}\cdot\hat{\boldsymbol{r}})(\boldsymbol{u}\cdot\hat{\boldsymbol{r}})]\exp(-\mathrm{i}k\hat{\boldsymbol{r}}\cdot\boldsymbol{q})\,\mathrm{d}s_q \tag{6.1}$$

where $\boldsymbol{u}(q)$ is the (unknown) displacement vector at $q \in S$. There is a similar formula for $\boldsymbol{F}^S$, but we shall only examine the scattered P-waves here.

For any fixed incident wave, the vectors $\hat{\boldsymbol{\alpha}}$, $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\gamma}}$, where $\hat{\boldsymbol{\alpha}}\cdot\hat{\boldsymbol{\beta}} = 0$ and $\hat{\boldsymbol{\gamma}} = \hat{\boldsymbol{\alpha}}\times\hat{\boldsymbol{\beta}}$, form an orthonormal basis, whence

$$\boldsymbol{F}^P(\hat{\boldsymbol{r}};\hat{\boldsymbol{\alpha}},\hat{\boldsymbol{\beta}};S) = f_1(\hat{\boldsymbol{r}};\hat{\boldsymbol{\alpha}},\hat{\boldsymbol{\beta}})\hat{\boldsymbol{\alpha}} + f_2(\hat{\boldsymbol{r}};\hat{\boldsymbol{\alpha}},\hat{\boldsymbol{\beta}})\hat{\boldsymbol{\beta}} + f_3(\hat{\boldsymbol{r}};\hat{\boldsymbol{\alpha}},\hat{\boldsymbol{\beta}})\hat{\boldsymbol{\gamma}}. \tag{6.2}$$

If we assume that the symmetry condition (5.2) holds, we deduce that

$$f_j(\hat{\boldsymbol{r}};\hat{\boldsymbol{\alpha}},\hat{\boldsymbol{\beta}}) = f_j(R\hat{\boldsymbol{r}};R\hat{\boldsymbol{\alpha}},R\hat{\boldsymbol{\beta}})$$

and then a change of variables gives

$$\int_{\Omega} f_j(\hat{r};\hat{\alpha},\hat{\beta})\,\mathrm{d}s(\hat{r}) = \int_{\Omega} f_j(\hat{r};R^T\hat{\alpha};R^T\hat{\beta})\,\mathrm{d}s(\hat{r})$$

for $j = 1,2,3$. Integrating (5.6) over the unit sphere then gives

$$\int_{\Omega} \boldsymbol{F}^P(\hat{r};\hat{\alpha},\hat{\beta};S)\,\mathrm{d}s(\hat{r}) = c_1\hat{\alpha} + c_2\hat{\beta} + c_3\hat{\gamma} \tag{6.3}$$

where the coefficients c_j do not depend on $\hat{\alpha}$ and $\hat{\beta}$.

Let us now restrict attention to incident P-waves, propagating in the direction of the unit vector $\hat{\alpha}$. Since $\boldsymbol{F}^P$ does not depend on $\hat{\beta}$, we must have $c_2 = c_3 = 0$, and thus (6.3) reduces to

$$\int_{\Omega} \boldsymbol{F}^P(\hat{r};\hat{\alpha})\,\mathrm{d}s(\hat{r}) = c_1\hat{\alpha} \tag{6.4}$$

where the dependence on S is now implicit.

Next, we reduce both sides of (6.4) to integrals over S. Substituting from (6.1) gives

$$\frac{-k}{\lambda+2\mu}\int_S \boldsymbol{I}^P(q)\,\mathrm{d}s_q = c_1\hat{\alpha} \tag{6.5}$$

where

$$\boldsymbol{I}^P(q) = \frac{\mathrm{i}}{4\pi}\int_{\Omega} \hat{r}[\lambda(\boldsymbol{u}\cdot\boldsymbol{n}) + 2\mu(\boldsymbol{n}\cdot\hat{r})(\boldsymbol{u}\cdot\hat{r})]\exp(-\mathrm{i}k\hat{r}\cdot q)\,\mathrm{d}s(\hat{r}).$$

We can evaluate $\boldsymbol{I}^P$ by appropriate differentiations of the formula

$$\frac{1}{4\pi}\int_{\Omega}\exp(-\mathrm{i}k\hat{r}\cdot q)\,\mathrm{d}s(\hat{r}) = j_0(kr_q)$$

where $r_q = |q|$ and $j_n(x)$ is a spherical Bessel function. The result is

$$\begin{aligned}\boldsymbol{I}^P(q) = {} & (\lambda j_1 + 2\mu\psi)(\boldsymbol{u}\cdot\boldsymbol{n})\hat{q} + 2\mu(j_1 - 5\psi)(\boldsymbol{u}\cdot\hat{q})(\boldsymbol{n}\cdot\hat{q})\hat{q} \\ & + 2\mu\psi[(\boldsymbol{n}\cdot\hat{q})\boldsymbol{u} + (\boldsymbol{u}\cdot\hat{q})\boldsymbol{n}]\end{aligned} \tag{6.6}$$

where $\hat{q} = q/r_q$, $j_1 \equiv j_1(kr_q)$ and

$$\psi = (kr_q)^{-2}j_1(kr_q) - (kr_q)^{-1}j_1'(kr_q).$$

We can write $\hat{\alpha}$ (and hence the right-hand side of (6.4)) as an integral over S as follows. For $P \in B_\mathrm{i}$, the formula (3.4) becomes

$$\boldsymbol{0} = \int_S \{(T\boldsymbol{v})\cdot\mathsf{G}(q;P) - \boldsymbol{v}\cdot T^q\mathsf{G}(q;P)\}\,\mathrm{d}s_q.$$

Similarly, since u_{inc} satisfies (2.1), we have

$$-2u_{\mathrm{inc}}(P)=\int_S\{(Tu_{\mathrm{inc}})\cdot\mathbf{G}(q;P)-u_{\mathrm{inc}}\cdot T^q\mathbf{G}(q;P)\}\,\mathrm{d}s_q.$$

Adding these two formulae, using (2.4) and the boundary condition on S, gives

$$2u_{\mathrm{inc}}(P)=\int_S u\cdot T^q\mathbf{G}(q;P)\,\mathrm{d}s_q$$

whence (4.1) gives

$$\hat{\alpha}=\frac{1}{2}\int_S u\cdot T^q\mathbf{G}(q;O)\,\mathrm{d}s_q. \tag{6.7}$$

If we substitute (6.6) and (6.7) into (6.5), and then take the scalar product with an arbitrary constant vector b, we obtain

$$\int_S u(q;\hat{\alpha})\cdot h_b^P(q)\,\mathrm{d}s_q=0 \tag{6.8}$$

where the displacement u depends on $\hat{\alpha}$, and

$$\begin{aligned}h_b^P(q)=\tfrac{1}{2}(\lambda+2\mu)(c_1/k)(T^q\mathbf{G}(q;O))\cdot b+(\lambda j_1+2\mu\psi)(b\cdot\hat{q})n\\ +2\mu(j_1-5\psi)(n\cdot\hat{q})(b\cdot\hat{q})\hat{q}+2\mu\psi[(n\cdot\hat{q})b+(n\cdot b)\hat{q}]\end{aligned} \tag{6.9}$$

is independent of $\hat{\alpha}$.

If one considers the scattered S-waves, one obtains (6.8), but with h_b^P replaced by a different vector, h_b^S say. Moreover, if one considers incident S-waves, one also obtains similar equations, except that h_b^P and h_b^S are replaced by different vectors that depend on c_2 and c_3 (with $c_1=0$).

Equation (6.8) holds for all $\hat{\alpha}\in\Omega$ and for any constant vector b. What does it say about the vector $h_b^P(q)$? Certainly, the set of vectors $\{u(q;\hat{\alpha})$ for all $\hat{\alpha}\in\Omega\}$, comprising all displacement vectors on the cavity S generated by incident P-waves in all directions $\hat{\alpha}$, is not complete, so we cannot deduce that $h_b^P=\mathbf{0}$. (It should be possible to prove that $\{u(q)\}$ *is* complete if we use more incident fields, including P-waves in all directions *and* S-waves in all directions and with two distinct polarizations; Dassios and Rigou [11] have proved the corresponding result for tractions $\{t(q)\}$ on a rigid body.) However, we may expect that

$$a(q)\cdot h_b^P(q)=0 \tag{6.10}$$

for every $q\in S$ and for some vector $a(q)$. If we now fix q and choose b so that $b\cdot\hat{q}=0$, we can calculate the left-hand side of (6.10), using (6.9), giving

$$[(a\cdot b)(n\cdot\hat{q})+(n\cdot b)(a\cdot\hat{q})]\chi(r_q)=0 \tag{6.11}$$

where $\chi(r_q)$ is a certain analytic scalar function of r_q. If it follows from (6.11) that $\chi=0$, analyticity implies that $r_q=|q|$ is a constant, and so S is a sphere.

It remains to determine $a(q)$. In order to give some idea of the difficulties, let us consider the simpler problem in which the unknown u is replaced by u_{inc}, defined by (4.1). Thus, suppose that

$$\int_S u_{\text{inc}}(q;\hat{\alpha})\cdot h(q)\,\mathrm{d}s_q = 0 \qquad \text{for all } \hat{\alpha}\in\Omega. \tag{6.12}$$

We can expand u_{inc} using regular spherical vector wave-functions; following Morse and Feshbach [23, p 1865], these are defined by

$$L_l^1(P) = (1/k)\operatorname{grad}\phi_l \qquad M_l^1(P) = \operatorname{curl}(\phi_l r)$$
$$N_l^1(P) = (1/k)\operatorname{curl} M_l^1$$

where $\phi_l \equiv \phi_{\sigma mn}(r) = j_n(kr)Y_l$, $Y_l \equiv Y_{\sigma mn}(\hat{r})$ is a spherical harmonic, and $l \equiv \sigma mn$ is a multi-index. Since

$$u_{\text{inc}}(P;\hat{\alpha}) = (-\mathrm{i}/k)\operatorname{grad}\{\exp(\mathrm{i}k r\cdot\hat{\alpha})\} = -4\pi\mathrm{i}\sum_l \mathrm{i}^n Y_l(\hat{\alpha})L_l^1(P)$$

the orthogonality of $\{Y_l(\hat{\alpha})\}$ over Ω implies that (6.12) is equivalent to

$$\int_S L_l^1(q)\cdot h(q)\,\mathrm{d}s_q = 0 \qquad \text{for all } l. \tag{6.13}$$

The following result shows that, in general, (6.13) does not imply that $h = \mathbf{0}$.

Theorem 6.1 (Aydin and Hizal [3]). Suppose that the (square-integrable) vector field $h(q)$ satisfies (6.13) and

$$\int_S M_l^1(q)\cdot h(q)\,\mathrm{d}s_q = 0 \qquad \text{for all } l$$

and

$$\int_S N_l^1(q)\cdot h(q)\,\mathrm{d}s_q = 0 \qquad \text{for all } l.$$

Suppose further that k^2 is not an eigenvalue of the interior Dirichlet problem. Then $h(q) \equiv \mathbf{0}$.

In order to make some progress, let us restrict the vector h. We can decompose h into its normal and tangential components, using

$$h = (h\cdot n)n - n\times(n\times h).$$

Similarly

$$kL_l^1 = \operatorname{grad}\phi_l = \frac{\partial\phi_l}{\partial n}n + \operatorname{Grad}\phi_l$$

where Grad is the surface gradient [7, p 33]. Thus, (6.13) reduces to

$$\int_S \left\{(h\cdot n)\frac{\partial\phi_l}{\partial n} + (n\times h)\cdot(n\times\operatorname{Grad}\phi_l)\right\}\mathrm{d}s = 0 \qquad \text{for all } l \tag{6.14}$$

leading to the following result.

Theorem 6.2. Suppose that $\boldsymbol{h}$ satisfies (6.12), and that $\boldsymbol{h}$ is a normal vector field. Suppose further that k^2 is not an eigenvalue of the interior Neumann problem. Then $\boldsymbol{h} \equiv \mathbf{0}$.

Proof. Since (6.12) is equivalent to (6.14), and $\boldsymbol{n} \times \boldsymbol{h} = \mathbf{0}$, the result follows by appealing to some known results from acoustics on the completeness of the set $\{\partial\phi_l/\partial n\}$ [25]. □

Note that we cannot *deduce* from (6.12) (or (6.13)) that $\boldsymbol{h} \cdot \boldsymbol{n} = 0$. This is the crux of the difficulty: the functions $\{\boldsymbol{u}_{\rm inc}(q;\hat{\boldsymbol{\alpha}})\}$ or $\{\boldsymbol{L}_l^1(q)\}$ do not pick out normal fields automatically. This is in contradistinction to the electromagnetic problem [8], where (6.12) is replaced by

$$\int_S (\boldsymbol{n} \times \boldsymbol{H}) \cdot \boldsymbol{h}(q)\, \mathrm{d}s_q = 0 \tag{6.15}$$

for a certain set of vector fields $\{\boldsymbol{H}\}$; thus, (6.15) immediately gives information on the tangential component of $\boldsymbol{h}(q)$.

Note that it does not seem to help to introduce more information by considering the scattered S-waves or incident S-waves or both. For, although we can prove a result analogous to theorem 6.1 for elastic waves (see the remarks above (6.10)), this would only be helpful here if we knew that a *single* vector $\boldsymbol{h}$ was orthogonal to every member of a complete set; as described below (6.9), we have several different vectors (such as $\boldsymbol{h}_b^P$ and $\boldsymbol{h}_b^S$), each of which is orthogonal to a subset of a complete set.

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Appendix

In this appendix, we use a modified form of an argument given in [26, pp 87–88] to prove that the eigenfunctions $\{\boldsymbol{u}_2^m(P)\}$ are linearly independent for $P \in B_0$, as required in the proof of theorem 4.4. Thus, we must show that

$$\sum_{m=1}^{M} c_m \boldsymbol{u}_2^m(P) = \mathbf{0} \tag{A1}$$

for $P \in B_0$ implies that $c_m = 0$ for $m = 1, 2, \ldots, M$.

Let $B_{\rm e}^R = B_R \cap B_{\rm e}^2$, where B_R is a large ball, of radius R and centre O, containing $B_{\rm i}^1 \cup B_{\rm i}^2$. Since $B_0 \subset B_{\rm e}^R$, we deduce that (A1) holds for all $P \in B_{\rm e}^R$, by analytic continuation. Hence, using (2.4) and (3.2), we obtain

$$\sum_{m=1}^{M} c_m \left\{\boldsymbol{u}_{\rm inc}^m(P) + \mathrm{O}(r^{-1})\right\} = \mathbf{0} \tag{A2}$$

for large r, with $P \in B_e^R$.

Now, we can write the mth incident field as

$$u_{\text{inc}}^m(P) = \hat{\delta}_m \exp(\mathrm{i}\kappa_m r \cdot \hat{\alpha}_m)$$

where the positive constant κ_m and the unit vector $\hat{\delta}_m$ are defined by

$$\kappa_m = k \qquad \hat{\delta}_m = \hat{\alpha}_m$$

for an incident P-wave, and by

$$\kappa_m = K \qquad \hat{\delta}_m = \hat{\beta}_m$$

for an incident S-wave.

Returning to (A2), we can isolate a particular coefficient, c_n say, by forming the scalar product of (A2) with the complex conjugate of u_{inc}^m to give

$$c_n + \sum_{\substack{m=1\\ m\neq n}}^{M} c_m \hat{\delta}_m \cdot \hat{\delta}_n \exp\{\mathrm{i}(\kappa_m \hat{\alpha}_m - \kappa_n \hat{\alpha}_n) \cdot r\} + \mathrm{O}(r^{-1}) = 0. \quad \text{(A3)}$$

Since the incident waves are different (as defined above theorem 4.4), $|\kappa_m \hat{\alpha}_m - \kappa_n \hat{\alpha}_n| > 0$ and we can set

$$\kappa_m \hat{\alpha}_m - \kappa_n \hat{\alpha}_n = \kappa_{mn} \hat{\alpha}_{mn}$$

where $\kappa_{mn} > 0$ and $\hat{\alpha}_{mn}$ is a unit vector. Let us now integrate (A3) over the spherical shell V, defined by $\frac{1}{2}R < r < R$, to give

$$\frac{7}{6}\pi R^3 c_n + \sum_{\substack{m=1\\ m\neq n}}^{M} c_m \hat{\delta}_m \cdot \hat{\delta}_n I_{mn} + \mathrm{O}(R^2) = 0 \qquad \text{(A4)}$$

for large R, where

$$I_{mn} = \int_V \exp(\mathrm{i}\kappa_{mn} \hat{\alpha}_{mn} \cdot r)\,\mathrm{d}V = \left[\frac{4\pi}{\kappa_{mn}} r^2 j_1(\kappa_{mn} r)\right]_{R/2}^{R}.$$

For large R, we have $I_{mn} = \mathrm{O}(R)$. Thus, (A4) can only be satisfied if $c_n = 0$.

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