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FLUID–SOLID INTERACTION: ACOUSTIC SCATTERING BY A SMOOTH ELASTIC OBSTACLE*

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Abstract. A bounded three-dimensional elastic obstacle is surrounded by an unbounded inviscid compressible fluid. Acoustic waves are scattered by the obstacle; the problem is to find the scattered waves and the response of the obstacle. This problem is formulated mathematically; existence and uniqueness theorems are proved. Various systems of boundary integral equations over the interface between the fluid and the solid are derived and analysed. These systems include a system obtained by a straightforward direct method and a smaller system for a single vector field.

Key words. fluid–solid interaction, boundary integral equations, transmission problem

AMS subject classifications. 45F15, 73K70, 76Q05

1. Introduction. Let us consider the interaction between an elastic body (the *target*) and a compressible, inviscid fluid in three dimensions. The target is coupled to the fluid, which occupies the unbounded exterior region, via the smooth interface S. In fact, the two media are coupled in two distinct ways. The first coupling is through the *kinematic boundary condition*: to ensure that a well-defined boundary between the fluid and the solid persists, the normal velocity of the fluid on one side of the boundary must match the normal velocity of the solid on the other side. (There is no such restriction on the tangential component of velocity, because the fluid has zero viscosity and thus can slip freely over the surface of the solid.) The second coupling is through the *dynamic boundary condition*, resulting from the balance of forces on all parts of S: each boundary element is massless, so a nonzero resultant force acting on it is prohibited. We suppose that a time-harmonic acoustic wave is incident upon the target and are required to determine its response and the scattered wave.

The fluid–solid interaction problem, described above, has received much attention in the engineering literature; see, for example, [10], [17]. Many authors have developed coupled schemes, using various boundary integral formulations for the unbounded fluid domain together with a finite-element method to model the elastic target [1], [2], [11], [14], [26], [34]. Boström has used T-matrix methods [3], [4], whereas Dallas [7] has used the limiting-amplitude principle. However, in many cases, there is a small parameter in the problem that can be used in an asymptotic analysis. For example, often the ratio of the fluid density to the solid density is small; this has been exploited [29], [30]. Long-wave approximations have also been given [28].

In this paper, we start by formulating the problem (§ 2) and then address the question of uniqueness (§ 3); for some frequencies (*Jones frequencies*) and some geometries, the ideal (inviscid) fluid–solid problem is not uniquely solvable for the elastic field, although the acoustic field in the exterior fluid is always unique. Uniqueness results for solid-solid transmission problems are given in [24].

Next, we develop several coupled boundary integral methods in which integral representations are used for both the fluid and the solid. This leads to integral equations over the fluid–solid interface. We are mainly interested in questions of existence and uniqueness. In particular, we identify frequencies (if any) at which the integral equations are not uniquely solvable (*irregular frequencies*).

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The simplest pair of direct boundary integral equations, discussed in § 5.1, has been used previously [12], [32], [33]. In [12], the boundary integral equations were solved both directly and in an iterative fashion; excellent results were obtained: irregular frequencies were observed, but Jones frequencies did not appear to pollute the numerical solution (perhaps because only acoustic fields were reported [12, Fig. 19], and these are unaffected by the presence of a Jones mode). We identify the irregular frequencies (analogous results for the fluid–fluid problem are given in [23]). We prove that the system of integral equations is solvable, using the theory of systems of multidimensional singular integral equations (outlined in the Appendix). We then go on to prove the existence of a solution to the fluid–solid problem at all frequencies for both elastic and viscoelastic targets. An analogous analysis for the simplest pair of indirect boundary integral equations is sketched in § 5.2.

The systems of integral equations discussed in §§ 5.1 and 5.2 consist of four equations in four unknowns (one scalar and one vector). In § 5.3, we derive systems for a single unknown vector field; this extends previous work on the acoustic transmission (fluid–fluid) problem [19] and on electromagnetic scattering by a dielectric obstacle [25]. In particular, we obtain such a system that is shown to be uniquely solvable at all frequencies.

2. Formulation of the problem. Let Ω_{i} denote a bounded, three-dimensional domain with smooth boundary S and simply connected unbounded exterior Ω_{e}. (Later, we shall state more precisely the smoothness conditions required of S.) Choose an origin of coordinates, O, in Ω_{i}. The exterior domain Ω_{e} is filled with homogeneous compressible inviscid fluid with density ρ_{f}. The target Ω_{i} is composed of a homogeneous isotropic elastic material with Lamé moduli λ and μ, Poisson's ratio ν, and mass density ρ_{s}. A time-harmonic sound wave of small amplitude and frequency ω is incident upon the target Ω_{i}; the problem is to determine the scattered wave in the fluid and the transmitted elastic wave in the solid target. Henceforth, the time dependence $\mathrm{e}^{-\mathrm{i}\omega t}$ is suppressed throughout.

The motion of the fluid is irrotational, whence a velocity potential, ϕ, exists. Thus, the fluid velocity, $\boldsymbol{v}$, can be expressed as $\boldsymbol{v}(\boldsymbol{x}) = \operatorname{grad}\phi(\boldsymbol{x})$ for $\boldsymbol{x} \in \Omega_{\mathrm{e}}$. The corresponding dynamic component of the fluid pressure is given by

$$p(\boldsymbol{x}) = \rho_{\mathrm{f}}\mathrm{i}\omega\phi(\boldsymbol{x}). \tag{2.1}$$

The governing partial differential equation is the wave equation; for time-harmonic motions, this reduces to the Helmholtz equation,

$$\nabla^2 p + k^2 p = 0, \tag{2.2}$$

where $k^2 = \omega^2/c^2$ and c is the speed of sound in the fluid.

For the solid, the elastodynamic displacement field, $\boldsymbol{u}(\boldsymbol{x})$, satisfies

$$\nabla \cdot \sigma(\boldsymbol{u}) + \rho_{\mathrm{s}}\omega^2 \boldsymbol{u}(\boldsymbol{x}) = \mathbf{0}, \tag{2.3}$$

where $\sigma(\boldsymbol{u})$ is the stress tensor. For homogeneous isotropic elastic solids, we have

$$\sigma_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij}, \tag{2.4}$$

where

$$e_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)$$

is the strain tensor, δ_{ij} is the Kronecker delta, and the usual summation convention is employed; the Lamé moduli are real constants and satisfy

$$\lambda + \tfrac{2}{3}\mu > 0 \qquad \text{and} \qquad \mu > 0. \tag{2.5}$$

Substitution of (2.4) into (2.3) gives

$$k_{\mathrm{p}}^{-2}\operatorname{grad}\operatorname{div}\boldsymbol{u} - k_{\mathrm{s}}^{-2}\operatorname{curl}\operatorname{curl}\boldsymbol{u} + \boldsymbol{u} = \boldsymbol{0}, \tag{2.6}$$

where the wavenumbers k_{p} and k_{s} are defined by $\rho_{\mathrm{s}}\omega^2 = (\lambda + 2\mu)k_{\mathrm{p}}^2 = \mu k_{\mathrm{s}}^2$.

We are also interested in viscoelastic targets. Such materials possess a "memory" of their strain history. If they are homogeneous and isotropic, (2.4) still applies, except that λ and μ are now functions of time. Furthermore, if the motion is time harmonic, it can be shown that the Fourier transforms of λ and μ—$\tilde{\lambda}(\omega)$ and $\tilde{\mu}(\omega)$, respectively—must satisfy (see [15, Chap. 9])

$$\operatorname{Re}(\tilde{\lambda} + \tfrac{2}{3}\tilde{\mu}) > 0, \quad \operatorname{Re}\tilde{\mu} > 0, \quad \operatorname{Im}(\tilde{\lambda} + \tfrac{2}{3}\tilde{\mu}) \le 0, \quad \text{and} \quad \operatorname{Im}\tilde{\mu} \le 0. \tag{2.7}$$

Thus, these materials are effectively modelled like elastic materials, but with the real Lamé moduli, λ and μ, replaced by the complex quantities $\tilde{\lambda}(\omega)$ and $\tilde{\mu}(\omega)$, respectively. We note that complex moduli have also been used to model damping in the solid [21].

The kinematic interface condition is

$$\frac{\partial \phi}{\partial n} = -\mathrm{i}\omega\, \boldsymbol{n}\cdot\boldsymbol{u} \qquad \text{on } S,$$

where $\boldsymbol{n}(\boldsymbol{x})$ is the unit normal at $\boldsymbol{x} \in S$, pointing into Ω_{e}. This and (2.1) imply that

$$\frac{\partial p}{\partial n} = \rho_{\mathrm{f}}\omega^2\, \boldsymbol{n}\cdot\boldsymbol{u} \qquad \text{on } S. \tag{2.8}$$

The dynamic interface condition is

$$-p\boldsymbol{n} = \boldsymbol{n}\cdot\sigma(\boldsymbol{u}) = T\boldsymbol{u} \qquad \text{on } S, \tag{2.9}$$

where the traction operator T is defined on S by

$$(T\boldsymbol{u})_m(\boldsymbol{x}) = \lambda n_m \frac{\partial u_j}{\partial x_j} + \mu n_j \left(\frac{\partial u_m}{\partial x_j} + \frac{\partial u_j}{\partial x_m} \right).$$

We can now formulate the ideal problem of fluid–solid interaction.

Fluid–Solid Problem. Find a pair $(p, \boldsymbol{u})$, with $p_{\mathrm{sc}} \in C^2(\Omega_{\mathrm{e}}) \cap C(\overline{\Omega_{\mathrm{e}}})$ and $\boldsymbol{u} \in C^2(\Omega_{\mathrm{i}}) \cap C(\overline{\Omega_{\mathrm{i}}})$, so that

$$p = p_{\mathrm{sc}} + p_{\mathrm{inc}}, \tag{2.10}$$

p_{sc} satisfies (2.2) in Ω_{e}, and $\boldsymbol{u}$ satisfies (2.6) in Ω_{i}. In addition, $(p, \boldsymbol{u})$ must satisfy the interface conditions (2.8) and (2.9), and p_{sc} must satisfy the Sommerfeld radiation condition at infinity,

$$(\boldsymbol{x}/|\boldsymbol{x}|)\cdot\operatorname{grad} p_{\mathrm{sc}} - \mathrm{i}kp_{\mathrm{sc}} = o\left(|\boldsymbol{x}|^{-1}\right)$$

as $|\boldsymbol{x}| \to \infty$, uniformly with respect to all directions $\boldsymbol{x}/|\boldsymbol{x}|$.

Here, p_{inc} is the given incident wave and p_{sc} is the unknown scattered wave. We assume that p_{inc} satisfies (2.2) everywhere in Ω_{e}, except possibly at some isolated points. (This allows for the possibility that p_{inc} is generated by, for example, a point source situated somewhere in Ω_{e}.)

3. Uniqueness. Suppose that there were two solutions to the fluid–solid problem; call them $(p_1, \boldsymbol{u}_1)$ and $(p_2, \boldsymbol{u}_2)$, and then let $p = p_1 - p_2$ and $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$. Clearly, p satisfies (2.2) in Ω_e and the radiation condition, $\boldsymbol{u}$ satisfies (2.6) in Ω_i, and $(p, \boldsymbol{u})$ satisfies the interface conditions (2.8) and (2.9).

By an application of the divergence theorem

$$(3.1)\qquad \begin{aligned}\int_{S_a} p\frac{\partial \overline{p}}{\partial n}\, ds &= \int_S p\frac{\partial \overline{p}}{\partial n}\, ds + \int_{\Omega_a} \{\nabla p \cdot \nabla \overline{p} + p\nabla^2 \overline{p}\}\; dV \\ &= -\rho_\mathrm{f}\overline{\omega}^2 \int_S \overline{\boldsymbol{u}} \cdot (T\boldsymbol{u})\, ds + \int_{\Omega_a} \left\{\nabla p \cdot \nabla \overline{p} - \overline{k}^2 |p|^2\right\}\; dV,\end{aligned}$$

where the overbar denotes the complex conjugate; S_a is the surface of the sphere of radius a and centre O, which encloses Ω_i; Ω_a is the region between S_a and S; and we have used (2.2) and the interface conditions. We now consider several cases.

(i) *ω and k are real.* Take the imaginary part of (3.1):

$$(3.2)\qquad \mathrm{Im}\left(\int_{S_a} p\frac{\partial \overline{p}}{\partial n}\, ds\right) = -\rho_\mathrm{f}\omega^2 \mathrm{Im}\left(\int_S \overline{\boldsymbol{u}} \cdot (T\boldsymbol{u})\, ds\right).$$

As p satisfies the radiation condition, we have

$$(3.3)\qquad \mathrm{Im}\left(\int_{S_a} p\frac{\partial \overline{p}}{\partial n}\, ds\right) \to -k \lim_{a\to\infty} \int_{S_a} |p|^2\, ds.$$

Furthermore, by the divergence theorem in Ω_i and (2.3),

$$(3.4)\qquad \int_S \overline{\boldsymbol{u}} \cdot (T\boldsymbol{u})\, ds = \int_{\Omega_\mathrm{i}} \{\sigma(\boldsymbol{u}) : \nabla \overline{\boldsymbol{u}} - \rho_\mathrm{s}\omega^2 \boldsymbol{u}.\overline{\boldsymbol{u}}\}\; dV.$$

For purely elastic bodies, $\sigma(\boldsymbol{u}) : \nabla\overline{\boldsymbol{u}} = \lambda|e_{kk}|^2 + 2\mu e_{ij}\overline{e_{ij}}$, which is real. Therefore, (3.2), (3.3), and (3.4) imply that

$$(3.5)\qquad \lim_{a\to\infty} \int_{S_a} |p|^2\, ds = 0.$$

Rellich's lemma [6, p. 77] then implies that $p \equiv 0$ in Ω_e. Thus, $\partial p/\partial n = 0$ on S. Therefore, from (2.8) and (2.9),

$$(3.6)\qquad \boldsymbol{u} \cdot \boldsymbol{n} = 0 \qquad \text{and} \qquad T\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } S.$$

Equation (2.3), along with the boundary conditions (3.6), does not necessarily imply that $\boldsymbol{u}$ vanishes in Ω_i. It is known that, for certain geometries and for certain frequencies, there are nontrivial solutions to this problem. We call these *Jones modes* and the associated frequencies *Jones frequencies*, as they were first discussed by D. S. Jones [16] in a related context (a thin layer of ideal fluid between an elastic body and a surrounding elastic exterior); Dallas calls them "complex amplitudes of nonradiating modes" [7, p. 7]. Thus, it is known that Jones frequencies exist for spheres: Lamb and Chree found that an elastic sphere could sustain "torsional oscillations," in which the radial component of the displacement is identically zero (see, for example, [9, § 8.14]). Jones frequencies also exist for any axisymmetric body; such bodies can sustain torsional oscillations in which only the azimuthal component of displacement is nonzero. However, intuitively, we do not expect Jones frequencies to exist for an "arbitrary" body; this has been proved recently by Hargé [13].

(ii) *ω and k have positive imaginary parts.* In this case, it is readily shown that p decays exponentially at infinity. Hence, we have (3.5); when this is combined with (3.1), we deduce that

$$0 = -\rho_f \overline{\omega}^2 \int_S \overline{\boldsymbol{u}} \cdot (T\boldsymbol{u})\, ds + \int_{\Omega_e} \left\{ \nabla p \cdot \nabla \overline{p} - \overline{k}^2 |p|^2 \right\} dV. \tag{3.7}$$

Equations (3.4) and (3.7) imply that

$$0 = -\rho_f \overline{\omega}^2 \int_{\Omega_i} \{\sigma(\boldsymbol{u}) : \nabla \overline{\boldsymbol{u}} - \rho_s \omega^2 \boldsymbol{u} \cdot \overline{\boldsymbol{u}}\}\, dV + \int_{\Omega_e} \left\{ \nabla p \cdot \nabla \overline{p} - \overline{k}^2 |p|^2 \right\} dV. \tag{3.8}$$

Take the imaginary part of (3.8) to give

$$\operatorname{Im}(\omega^2) \left(\rho_f \int_{\Omega_i} \sigma(\boldsymbol{u}) : \nabla \overline{\boldsymbol{u}}\, dV + \frac{1}{c^2} \int_{\Omega_e} |p|^2\, dV \right) = 0.$$

Thus, $\operatorname{Im}(\omega^2) = 0$, or p vanishes in Ω_e and $\sigma(\boldsymbol{u}) : \nabla \overline{\boldsymbol{u}}$ vanishes in Ω_i. In the latter case, $\boldsymbol{u}$ is constant in Ω_i; the interface condition (2.8), together with the fact that p vanishes in Ω_e, implies that $\boldsymbol{u}$ vanishes on S, whence $\boldsymbol{u}$ vanishes in Ω_i. If $\operatorname{Im}(\omega^2) = 0$, ω^2 must be negative (as ω is not real); thus, each term in (3.8) is positive and so $p = 0$ and $\boldsymbol{u} = \boldsymbol{0}$.

(iii) *Viscoelastic materials.* For real ω^2, we proceed as before, to (3.4). We have

$$\sigma(\boldsymbol{u}) : \nabla \overline{\boldsymbol{u}} = (\tilde{\lambda} + \tfrac{2}{3}\tilde{\mu})|e_{kk}|^2 + 2\tilde{\mu}(e_{ij} - \tfrac{1}{3}e_{kk}\delta_{ij})(\overline{e_{ij}} - \tfrac{1}{3}\overline{e_{kk}}\delta_{ij}). \tag{3.9}$$

The third and fourth conditions of (2.7) imply that

$$\operatorname{Im} \left(\int_{\Omega_i} \sigma(\boldsymbol{u}) : \nabla \overline{\boldsymbol{u}}\, dV \right) \leq 0. \tag{3.10}$$

Equations (3.2) and (3.3) still apply; they and (3.10) imply that (3.5) holds, whence $p = 0$ in Ω_e, as before. Equations (3.3) and (3.4) then imply that the inequality (3.10) is actually an equality. Assuming that the material is genuinely *visco*elastic, (3.9) implies that $\boldsymbol{u}$ is constant in Ω_i. Since p vanishes, the interface conditions make it clear that $\boldsymbol{u}$ vanishes in Ω_i. So, for real frequencies, the solution to the problem of the interaction of a viscoelastic material and an acoustic medium, if it exists, is unique.

4. Representation theorems and applications.

In this section, we use both acoustic and elastodynamic fundamental solutions to obtain integral representations in the fluid and in the solid.

4.1. Acoustic potential theory.

A suitable acoustic fundamental solution is $G(\boldsymbol{x}, \boldsymbol{y}) = -e^{ikR}/(2\pi R)$, where $R = |\boldsymbol{x} - \boldsymbol{y}|$. Using it, we define single-layer and double-layer potentials by

$$(Sf)(\boldsymbol{x}) = \int_S f(\boldsymbol{y}) G(\boldsymbol{x}, \boldsymbol{y})\, ds(\boldsymbol{y}) \qquad \text{and} \qquad (Df)(\boldsymbol{x}) = \int_S f(\boldsymbol{y}) \frac{\partial G(\boldsymbol{x}, \boldsymbol{y})}{\partial n(\boldsymbol{y})}\, ds(\boldsymbol{y}),$$

respectively, where f is a function defined on S. (We shall say more below on the required smoothness of f.) Then, two applications of Green's theorem (one in Ω_e to p_{sc} and G and one in Ω_i to p_{inc} and G) yield the familiar representation

$$2p_{sc}(\boldsymbol{x}) = (S(\partial p/\partial n))(\boldsymbol{x}) - (Dp)(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x} \in \Omega_e. \tag{4.1}$$

The single-layer and double-layer potentials have well-known properties. If $f(\boldsymbol{y})$ is continuous for $\boldsymbol{y} \in S$, $(Sf)(\boldsymbol{x})$ is defined up to and including S and is continuous as $\boldsymbol{x}$ crosses S. However, both $(Df)(\boldsymbol{x})$ and $(\partial/\partial n(\boldsymbol{x}))(Sf)(\boldsymbol{x})$ have jumps, given by

$$Df = (\mp I + \overline{K^*})f \qquad \text{and} \qquad (\partial/\partial n(\boldsymbol{x}))(Sf) = (\pm I + K)f, \tag{4.2}$$

where, in each case, the upper (lower) sign corresponds to $\boldsymbol{x} \to \boldsymbol{x}_0 \in S$ from Ω_{e} (Ω_{i}). Here, K and $\overline{K^*}$ are integral operators, defined, for $\boldsymbol{x} \in S$ and $f \in C(S)$, by

$$(\overline{K^*}f)(\boldsymbol{x}) = \int_S f(\boldsymbol{y}) \frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{y})}\, ds(\boldsymbol{y}) \qquad \text{and}$$
$$(Kf)(\boldsymbol{x}) = \int_S f(\boldsymbol{y}) \frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{x})}\, ds(\boldsymbol{y}).$$

We can also take the normal derivative of Df, but for existence up to the boundary we require f to be smoother: it is sufficient to have $f \in C^{1,\alpha}(S)$, with $\alpha > 0$; see [6] for information on classes of Hölder-continuous functions. Call this operator N:

$$(Nf)(\boldsymbol{x}) = \frac{\partial}{\partial n(\boldsymbol{x})} \int_S f(\boldsymbol{y}) \frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{y})}\, ds(\boldsymbol{y}). \tag{4.3}$$

It is known that Nf is continuous across S [6, p. 62].

It is not difficult to show that S, K, and $\overline{K^*}$ have weakly singular kernels. As operators, they are compact on $C(S)$, and also on $C^{0,\beta}(S)$, for any β with $0 < \beta < 1$. Furthermore, if f is in $C(S)$, then Sf, Kf, and $\overline{K^*}f$ belong to $C^{0,\beta}(S)$, for any β with $0 < \beta < 1$. Similarly, all these operators map $C^{0,\beta}(S)$ into $C^{1,\beta}(S)$. S and N are self-adjoint when the inner product is taken as

$$\langle f, g\rangle = \int_S f\,\overline{g}\, ds;$$

K and $\overline{K^*}$ are mutually adjoint with this inner product. N is a hypersingular operator; it maps $C^{1,\beta}(S)$ into $C^{0,\beta}(S)$. See [6] and [18] for proofs of these results.

4.2. Elastic potential theory. In the elastic target, we use the fundamental Green's tensor (Kupradze matrix) defined by

$$(\mathbf{G}(\boldsymbol{x};\boldsymbol{y}))_{ij} = \frac{1}{\mu}\left\{\Psi\delta_{ij} + \frac{1}{k_{\mathrm{s}}^2}\frac{\partial^2}{\partial x_i \partial x_j}(\Psi - \Phi)\right\},$$

where $\Phi = -\exp\{\mathrm{i}k_{\mathrm{p}}R\}/(2\pi R)$ and $\Psi = -\exp\{\mathrm{i}k_{\mathrm{s}}R\}/(2\pi R)$. Next, we define elastic single-layer and double-layer potentials by

$$(\mathbf{S}\boldsymbol{f})(\boldsymbol{x}) = \int_S \boldsymbol{f}(\boldsymbol{y}) \cdot \mathbf{G}(\boldsymbol{y};\boldsymbol{x})\, ds(\boldsymbol{y}) \qquad \text{and} \qquad (\mathbf{D}\boldsymbol{f})(\boldsymbol{x}) = \int_S \boldsymbol{f}(\boldsymbol{y}) \cdot T_y\mathbf{G}(\boldsymbol{y};\boldsymbol{x})\, ds(\boldsymbol{y}),$$

respectively, where T_y means T applied at $\boldsymbol{y} \in S$. Then, in the elastic target, we can apply the Betti reciprocal theorem to $\boldsymbol{u}$ and $\mathbf{G}$, giving the representation

$$-2\boldsymbol{u}(\boldsymbol{x}) = (\mathbf{S}(T\boldsymbol{u}))(\boldsymbol{x}) - (\mathbf{D}\boldsymbol{u})(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x} \in \Omega_{\mathrm{i}}. \tag{4.4}$$

The elastic single-layer and double-layer potentials also have well-known properties [20]. $(\mathbf{S}\boldsymbol{f})(\boldsymbol{x})$ is continuous as $\boldsymbol{x}$ crosses S, whereas both $\mathbf{D}$ and $T_x\mathbf{S}$ exhibit jumps given by

$$\mathbf{D}\boldsymbol{f} = (\mp\mathbf{I} + \overline{\mathbf{K}^*})\boldsymbol{f} \qquad \text{and} \qquad T_x\mathbf{S}\boldsymbol{f} = (\pm\mathbf{I} + \mathbf{K})\boldsymbol{f} \tag{4.5}$$

(cf. (4.2)), where $\mathbf{I}$ is the 3×3 identity matrix. Here, $\mathbf{K}$ and $\overline{\mathbf{K}^*}$ are *singular* integral operators, defined, for $\boldsymbol{x} \in S$, by

$$(\overline{\mathbf{K}^*}\boldsymbol{f})(\boldsymbol{x}) = \int_S \boldsymbol{f}(\boldsymbol{y}) \cdot T_y \mathbf{G}(\boldsymbol{y};\boldsymbol{x})\, ds(\boldsymbol{y}) \qquad \text{and}$$
$$(\mathbf{K}\boldsymbol{f})(\boldsymbol{x}) = \int_S \boldsymbol{f}(\boldsymbol{y}) \cdot T_x \mathbf{G}(\boldsymbol{y};\boldsymbol{x})\, ds(\boldsymbol{y}).$$

In all of the above formulae, it is sufficient that the density $\boldsymbol{f}$ be Hölder continuous on S [20, Chap. 5]. However, we shall also require the tractions corresponding to the elastic double-layer potential, defined by

$$\mathbf{N}\boldsymbol{f} = T_x \mathbf{D}\boldsymbol{f}. \tag{4.6}$$

The existence of $\mathbf{N}\boldsymbol{f}$ requires that $\boldsymbol{f}$ be smoother: a sufficient condition is that the tangential derivative of $\boldsymbol{f}(\boldsymbol{y})$ be Hölder continuous for $\boldsymbol{y} \in S$ ($\boldsymbol{f} \in C^{1,\beta}(S)$), and then the right-hand side of (4.6) is continuous across S [8], [20, p. 320].

$\mathbf{S}$ has a weakly singular kernel, but $\mathbf{K}$ and $\overline{\mathbf{K}^*}$ have singular kernels. Thus, for the existence of $\mathbf{S}\boldsymbol{f}$, it is sufficient that $\boldsymbol{f}$ be continuous on S. For the existence of $\mathbf{K}\boldsymbol{f}$ and $\overline{\mathbf{K}^*}\boldsymbol{f}$, $\boldsymbol{f}$ must belong to $C^{0,\beta}(S)$, with $0 < \beta < 1$; however, $\mathbf{K}$ and $\overline{\mathbf{K}^*}$ are not compact on this space. $\mathbf{S}$ and $\mathbf{N}$ are self-adjoint when the inner product is taken as

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle = \int_S \boldsymbol{f} \cdot \overline{\boldsymbol{g}}\, ds;$$

$\mathbf{K}$ and $\overline{\mathbf{K}^*}$ are mutually adjoint with this inner product. $\mathbf{N}$ is a hypersingular operator. For further information on these operators, see [20].

5. Boundary integral equations. In this section, we derive three different systems of boundary integral equations and use them to prove the existence of a solution to the coupled fluid–solid problem. The first two systems consist of four equations in four (scalar) unknowns; each of these systems exhibits spurious irregular frequencies, at which the system of equations is not uniquely solvable.

It is sometimes important not to have these irregular frequencies. For example, if the ratio $\rho_{\mathrm{f}}/\rho_{\mathrm{s}}$ is small (which is often the case in practice), then there are complex scattering frequencies with a small negative imaginary part [29], [31, Chap. 9]. In this case, the response curve will have peaks near the scattering frequencies and they may be difficult to distinguish from the peaks due to the irregular frequencies. This situation could be further complicated if Jones modes are also possible. In view of this, the third system derived is designed so that irregular frequencies do not occur. Moreover, it consists of only three equations in three unknowns; for three-dimensional problems, this is likely to be optimal. However, the price to be paid here is in the increased complexity of the surface potentials utilised.

5.1. The simplest direct boundary integral equations. Let us begin with the representation (4.1) for the scattered pressure field. Taking the limit as we pass to a point on the surface S, using (2.10) and (4.2), we obtain

$$p + \overline{K^*}p - S(\partial p/\partial n) = 2p_{\mathrm{inc}}. \tag{5.1}$$

Similarly, if we use the representation (4.4) for the displacement in the solid and then evaluate it on S, using (4.5), we obtain

$$\boldsymbol{u} - \overline{\mathbf{K}^*}\boldsymbol{u} + \mathbf{S}(T\boldsymbol{u}) = \mathbf{0}.$$

Now use the interface conditions (2.8) and (2.9) to get

$$\left\{ \begin{array}{rcl} p + \overline{K^*}p - \rho_{\mathrm{f}}\omega^2 S(\boldsymbol{u}\cdot\boldsymbol{n}) & = & 2p_{\mathrm{inc}}, \\ \\ \boldsymbol{u} - \overline{\mathbf{K}^*}\boldsymbol{u} - \mathbf{S}(p\boldsymbol{n}) & = & \mathbf{0}. \end{array} \right. \tag{5.2}$$

This is our first system of boundary integral equations; it has also been given in [12], [32], and [33]. We look for a solution with p and $\boldsymbol{u}$ in $C^{1,\beta}(S)$, for some $\beta > 0$.

5.1.1. Solvability of the system (5.2). To analyse the system (5.2), using the theory outlined in the appendix, it is helpful to write it in the form

$$\begin{pmatrix} I+\overline{K^*} & -\rho_{\mathrm{f}}\omega^2 Sn_1 & -\rho_{\mathrm{f}}\omega^2 Sn_2 & -\rho_{\mathrm{f}}\omega^2 Sn_3 \\ -S_{1j}n_j & I-\overline{K^*_{11}} & -\overline{K^*_{12}} & -\overline{K^*_{13}} \\ -S_{2j}n_j & -\overline{K^*_{21}} & I-\overline{K^*_{22}} & -\overline{K^*_{23}} \\ -S_{3j}n_j & -\overline{K^*_{31}} & -\overline{K^*_{32}} & I-\overline{K^*_{33}} \end{pmatrix} \begin{pmatrix} p \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 2p_{\mathrm{inc}} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{5.3}$$

where

$$(\overline{K^*_{ij}}f)(\boldsymbol{x}) = \int_S f(\boldsymbol{y})(T_y\mathbf{G}(\boldsymbol{y},\boldsymbol{x}))_{ij}\,ds(\boldsymbol{y}) \qquad \text{and}$$
$$(S_{ij}f)(\boldsymbol{x}) = \int_S f(\boldsymbol{y})(\mathbf{G}(\boldsymbol{y},\boldsymbol{x}))_{ij}\,ds(\boldsymbol{y}).$$

The system (5.3) is not of the form "identity plus compact"; nevertheless, it is Fredholm, under certain conditions. To show this, we must show that the corresponding symbol matrix is invertible. To calculate the symbol matrix, we first identify the singular terms in (5.3), as weakly singular operators do not contribute. Thus, we consider the system matrix

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I-\overline{K^*_{11}} & -\overline{K^*_{12}} & -\overline{K^*_{13}} \\ 0 & -\overline{K^*_{21}} & I-\overline{K^*_{22}} & -\overline{K^*_{23}} \\ 0 & -\overline{K^*_{31}} & -\overline{K^*_{32}} & I-\overline{K^*_{33}} \end{pmatrix},$$

where all the remaining elastic terms can be evaluated at $\omega = 0$. Next, we choose a particular point $\boldsymbol{x} \in S$ and then define $\xi(\boldsymbol{x})$ to be the unitary matrix that rotates the coordinate system so that the new $\boldsymbol{e}_3$ axis is normal to S at $\boldsymbol{x}$. Explicit calculation in the new frame then shows that the operator on the left-hand side of (5.3) is in $G'(\alpha)$, the class of operators defined in the appendix, where S is assumed to be a $C^{2,\alpha}$ surface; the elastic part of this calculation is given in, for example, [27, p. 387]. The corresponding symbol matrix is

$$\Theta(\boldsymbol{x},\theta) = X^T(\boldsymbol{x}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\mathrm{i}\delta\cos\theta \\ 0 & 0 & 1 & -\mathrm{i}\delta\sin\theta \\ 0 & \mathrm{i}\delta\cos\theta & \mathrm{i}\delta\sin\theta & 1 \end{pmatrix} X(\boldsymbol{x}),$$

where θ is the angle a line in the $\boldsymbol{e}_1$-$\boldsymbol{e}_2$ plane makes with the $\boldsymbol{e}_1$-axis,

$$X(\boldsymbol{x}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \xi_{11} & \xi_{12} & \xi_{13} \\ 0 & \xi_{21} & \xi_{22} & \xi_{23} \\ 0 & \xi_{31} & \xi_{32} & \xi_{33} \end{pmatrix},$$

and $X^T(\boldsymbol{x})$ is the transpose of $X(\boldsymbol{x})$. The material parameter δ is defined by

$$\delta = \tfrac{1}{2}(1-2\nu)/(1-\nu) = \mu/(\lambda+2\mu), \tag{5.4}$$

where ν is Poisson's ratio. Since $\det\Theta(\boldsymbol{x},\theta) = 1-\delta^2$, the condition

$$\inf_{\substack{x\in S\\ 0\le\theta\le 2\pi}} |\det\Theta(\boldsymbol{x},\theta)| > 0, \tag{5.5}$$

required by Theorem A.3, is fulfilled if $\nu \neq \frac{3}{4}$; the conditions on the Lamé constants in (2.5) and (2.7) make a Poisson's ratio of this value impossible. We conclude that the system of integral equations (5.2) is Fredholm when the solid is elastic or viscoelastic. The symbol matrix is Hermitian and so the index of the system is zero. Consequently, a unique solution to (5.2) exists if the corresponding homogeneous system,

$$\left\{ \begin{array}{rcl} p + \overline{K^*}p - \rho_{\mathrm{f}}\omega^2 S(\boldsymbol{u}\cdot\boldsymbol{n}) & = & 0, \\ \\ \boldsymbol{u} - \overline{\mathbf{K}^*}\boldsymbol{u} - \mathbf{S}(p\boldsymbol{n}) & = & \mathbf{0}, \end{array} \right. \tag{5.6}$$

has just one solution. So, suppose that the system (5.6) has a nontrivial solution, $(p',\boldsymbol{u}')$ in $L^2(S)$. The fact that the operator is in $G'(\alpha)$ implies that p' and $\boldsymbol{u}'$ are actually in $C^{1,\alpha}(S)$. Define p_{e}, p_{i}, $\boldsymbol{u}_{\mathrm{e}}$ and $\boldsymbol{u}_{\mathrm{i}}$ by

$$(Dp')(\boldsymbol{x}) - \rho_{\mathrm{f}}\omega^2(S(\boldsymbol{u}'\cdot\boldsymbol{n}))(\boldsymbol{x}) = \left\{ \begin{array}{ll} p_{\mathrm{e}}(\boldsymbol{x}) & \text{if } \boldsymbol{x}\in\Omega_{\mathrm{e}}, \\ p_{\mathrm{i}}(\boldsymbol{x}) & \text{if } \boldsymbol{x}\in\Omega_{\mathrm{i}}, \end{array} \right. \tag{5.7}$$

and

$$(\mathbf{D}\boldsymbol{u}')(\boldsymbol{x}) + (\mathbf{S}(p'\boldsymbol{n}))(\boldsymbol{x}) = \left\{ \begin{array}{ll} \boldsymbol{u}_{\mathrm{e}}(\boldsymbol{x}) & \text{if } \boldsymbol{x}\in\Omega_{\mathrm{e}}, \\ \boldsymbol{u}_{\mathrm{i}}(\boldsymbol{x}) & \text{if } \boldsymbol{x}\in\Omega_{\mathrm{i}}. \end{array} \right.$$

The continuity of the single-layer and double-layer potentials up to the boundary implies the continuity of p_{e}, p_{i}, $\boldsymbol{u}_{\mathrm{e}}$, and $\boldsymbol{u}_{\mathrm{i}}$ up to the boundary. p_{e} and p_{i} are smooth solutions of the Helmholtz equation (2.2) in their respective domains. $\boldsymbol{u}_{\mathrm{e}}$ and $\boldsymbol{u}_{\mathrm{i}}$ are smooth solutions of Navier's equation (2.6) in their respective domains. Moreover, due to the far-field behaviour of G and $\mathbf{G}$, p_{e} satisfies the Sommerfeld radiation condition and $\boldsymbol{u}_{\mathrm{e}}$ satisfies the following elastic radiation condition:

$$|\boldsymbol{x}|\left(\frac{\partial\boldsymbol{u}_{\mathrm{e}}^p}{\partial|\boldsymbol{x}|} - \mathrm{i}k_{\mathrm{p}}\boldsymbol{u}_{\mathrm{e}}^p\right) \to \mathbf{0} \qquad \text{and} \qquad |\boldsymbol{x}|\left(\frac{\partial\boldsymbol{u}_{\mathrm{e}}^s}{\partial|\boldsymbol{x}|} - \mathrm{i}k_{\mathrm{s}}\boldsymbol{u}_{\mathrm{e}}^s\right) \to \mathbf{0} \tag{5.8}$$

as $|\boldsymbol{x}|\to\infty$, uniformly with respect to all directions $\boldsymbol{x}/|\boldsymbol{x}|$, where

$$\boldsymbol{u}_{\mathrm{e}}^p = -k_{\mathrm{p}}^{-2}\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u}_{\mathrm{e}} \qquad \text{and} \qquad \boldsymbol{u}_{\mathrm{e}}^s = \boldsymbol{u}_{\mathrm{e}} - \boldsymbol{u}_{\mathrm{e}}^p.$$

From the first part of (5.6) and (5.7), using (4.2), we have $p_{\mathrm{i}} = 0$ on S. It is well known that for each compact domain Ω_{i}, there is only a countably infinite number of wavenumbers at which a nontrivial, square-integrable function satisfying the Helmholtz equation in Ω_{i} and a homogeneous Dirichlet boundary condition on S exists [31, Chap. 2]. We shall call the squares of such wavenumbers *eigenvalues of the interior Dirichlet problem.*

Suppose that k^2 is not an eigenvalue of the interior Dirichlet problem. Then $p_{\mathrm{i}} \equiv 0$ in Ω_{i}. In particular, $\partial p_{\mathrm{i}}/\partial n = 0$ on S. From the continuity of the normal derivative of the double-layer potential across S and the jump relations (4.2), we obtain

$$\frac{\partial p_{\mathrm{e}}}{\partial n} - \frac{\partial p_{\mathrm{i}}}{\partial n} = \frac{\partial p_{\mathrm{e}}}{\partial n} = -2\rho_{\mathrm{f}}\omega^2 \boldsymbol{u}' \cdot \boldsymbol{n} \qquad \text{on } S. \tag{5.9}$$

Now, evaluate p_{e} on S:

$$p_{\mathrm{e}}(\boldsymbol{x}) = -p' + \overline{K^*}p' - \rho_{\mathrm{f}}\omega^2 S(\boldsymbol{u}' \cdot \boldsymbol{n}) = -2p' \qquad \text{on } S, \tag{5.10}$$

from the first part of (5.6). From the second part of (5.6) and the jump relations (4.5), we have $\boldsymbol{u}_{\mathrm{e}} = \boldsymbol{0}$ on S. In addition, since $\boldsymbol{u}_{\mathrm{e}}$ satisfies (2.6) and the radiation condition (5.8), it follows that $\boldsymbol{u}_{\mathrm{e}} \equiv \boldsymbol{0}$ in Ω_{e}. (For an elastic material, this is proved in [20, pp. 132–136]; for a viscoelastic material, it is proved in [22, Appendix D].) In particular, $\boldsymbol{T}\boldsymbol{u}_{\mathrm{e}} = \boldsymbol{0}$ on S. Evaluating the jump in the surface tractions across S, we have

$$\boldsymbol{T}\boldsymbol{u}_{\mathrm{e}} - \boldsymbol{T}\boldsymbol{u}_{\mathrm{i}} = -\boldsymbol{T}\boldsymbol{u}_{\mathrm{i}} = 2p'\boldsymbol{n} \qquad \text{on } S. \tag{5.11}$$

After using the second part of (5.6) to evaluate $\boldsymbol{u}_{\mathrm{i}}$ on S, we have

$$\boldsymbol{u}_{\mathrm{i}}(\boldsymbol{x}) = 2\boldsymbol{u}'(\boldsymbol{x}) \qquad \text{on } S. \tag{5.12}$$

Equations (5.9), (5.10), (5.11), and (5.12) imply that $(-p_{\mathrm{e}}, \boldsymbol{u}_{\mathrm{i}})$ solves the homogeneous fluid–solid problem. Therefore (from § 3), if the solid is either elastic or viscoelastic, $(-p_{\mathrm{e}}, \boldsymbol{u}_{\mathrm{i}}) = (0, \boldsymbol{0})$, unless a Jones mode is possible. If a Jones mode is ruled out, then (5.10) and (5.12) imply that

$$(p', \boldsymbol{u}') = (0, \boldsymbol{0}).$$

Thus, we have shown that the homogeneous system (5.6) has only a trivial solution, provided that k^2 is not an eigenvalue of the interior Dirichlet problem and that ω is not a Jones frequency. Subject to these conditions, Theorem A.6 shows that a solution $(p, \boldsymbol{u})$ of the system (5.2) exists and that this solution is in $C^{1,\alpha}(S)$ (because the right-hand side of (5.2) is in this space).

5.1.2. Solvability of the fluid–solid problem. We now show that we can use the solution of the system (5.2) to construct the solution of the fluid–solid problem, provided that k^2 is not an eigenvalue of the interior Dirichlet problem and that ω is not a Jones frequency. Define

$$P(\boldsymbol{x}) = p_{\mathrm{inc}} - \tfrac{1}{2}(Dp) + \tfrac{1}{2}\rho_{\mathrm{f}}\omega^2(S(\boldsymbol{u} \cdot \boldsymbol{n})) \qquad \text{for } \boldsymbol{x} \in \Omega_{\mathrm{e}} \text{ and} \tag{5.13}$$

$$\boldsymbol{U}(\boldsymbol{x}) = \tfrac{1}{2}(\mathbf{D}\boldsymbol{u}) + \tfrac{1}{2}(\mathbf{S}(p\boldsymbol{n})) \qquad \text{for } \boldsymbol{x} \in \Omega_{\mathrm{i}}. \tag{5.14}$$

It is clear that P satisfies the Helmholtz equation in Ω_{e}, that $P - p_{\mathrm{inc}}$ satisfies the Sommerfeld radiation condition, and that $\boldsymbol{U}$ satisfies Navier's equation (2.6) in Ω_{i}. It remains to check that the interface conditions (2.8) and (2.9) are satisfied.

We can also define P in Ω_{i} and $\boldsymbol{U}$ in Ω_{e}, using (5.13) and (5.14), respectively. Because of regularity, the normal derivative of P and the surface traction of $\boldsymbol{U}$ on S exist. Denote by P_- (P_+) the limiting value of P as S is approached from Ω_{i} (Ω_{e}), with similar definitions for other quantities. From the first part of (5.2), $P_- = 0$. Since we have assumed that k^2 is not an eigenvalue of the interior Dirichlet problem,

$P \equiv 0$ in Ω_{i}. In particular, $\partial P_{-}/\partial n = 0$ on S. So, as before, the jump in the normal derivative of P across S implies that

$$\frac{\partial P_{+}}{\partial n} = \rho_{\mathrm{f}}\omega^2 \boldsymbol{u}\cdot\boldsymbol{n} \qquad \text{on } S. \tag{5.15}$$

Similarly,

$$P_{+} = p, \quad T\boldsymbol{U}_{-} = -p\boldsymbol{n}, \quad \text{and} \quad \boldsymbol{U}_{-} = \boldsymbol{u} \quad \text{on } S. \tag{5.16}$$

Equations (5.15) and (5.16) imply that the interface conditions are satisfied.

5.1.3. Discussion on excluded frequencies. Let us examine here what happens when k^2 is an eigenvalue of the interior Dirichlet problem and/or ω is a Jones frequency. Let $P_D(\boldsymbol{x})$ be a nontrivial solution of the interior Dirichlet problem. By applying Green's theorem in Ω_{i} to P_D and G, we obtain

$$P_D(\boldsymbol{x}) = -\tfrac{1}{2}(S(\partial P_D/\partial n))(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x}\in\Omega_{\mathrm{i}}.$$

It follows that

$$(I+K)(\partial P_D/\partial n) = 0 \qquad \text{and} \qquad S(\partial P_D/\partial n) = 0 \qquad \text{on } S.$$

Let $\{\boldsymbol{U}_{\mathrm{J}}^{(i)}; i = 1,\dots,n\}$ be a base of the space of Jones modes. It is clear that

$$(\mathbf{I} - \overline{\mathbf{K}^*})\boldsymbol{U}_{\mathrm{J}}^{(i)} = \mathbf{0}.$$

We know that $(\mathbf{I} - \overline{\mathbf{K}^*})$ is Fredholm, whence $(\mathbf{I} - \mathbf{K})\boldsymbol{b} = \mathbf{0}$ has at least n independent solutions. Let $\{\boldsymbol{b}^{(i)}; i = 1,\dots,m\}$ be a base of the null-space of $(\mathbf{I}-\mathbf{K})$. It is easy to see that the space spanned by $\{\mathbf{S}\boldsymbol{b}^{(i)}; i = 1,\dots,m\}$ is the space of interior displacement fields with zero surface tractions. Thus, each Jones mode can be expressed as $\mathbf{S}\boldsymbol{b}$, for some $\boldsymbol{b}$. Clearly,

$$(\mathbf{I} - \mathbf{K})\boldsymbol{b} = \mathbf{0} \qquad \text{and} \qquad \boldsymbol{n}\cdot(\mathbf{S}\boldsymbol{b}) = 0.$$

Hence, the adjoint, homogeneous version of the system (5.2),

$$\begin{cases} (I+K)a - \boldsymbol{n}\cdot(\mathbf{S}\boldsymbol{b}) &= 0, \\ (\mathbf{I}-\mathbf{K})\boldsymbol{b} - \rho_{\mathrm{f}}\omega^2\boldsymbol{n}\,Sa &= \mathbf{0}, \end{cases} \tag{5.17}$$

has the solution $(a, \boldsymbol{b})$, where $a = \partial P_D/\partial n$ and $\boldsymbol{b}$ is as above.

Conversely, it can easily be shown that the only solutions of (5.17) have

$$\boldsymbol{n}\cdot(\mathbf{S}\boldsymbol{b}) = 0, \quad (\mathbf{I}-\mathbf{K})\boldsymbol{b} = \mathbf{0}, \quad Sa = 0, \quad \text{and} \quad (I+K)a = 0.$$

Apply Green's theorem in Ω_{i} to p_{inc} and G. Evaluating the result on S gives

$$2p_{\mathrm{inc}} = (I + \overline{K^*})p_{\mathrm{inc}} - S(\partial p_{\mathrm{inc}}/\partial n) \qquad \text{on } S.$$

Therefore, the inner product of $(2p_{\mathrm{inc}}, \mathbf{0})$ with $(a, \boldsymbol{b})$, with a and $\boldsymbol{b}$ as above, equals

$$\langle (I+\overline{K^*})p_{\mathrm{inc}} - S(\partial p_{\mathrm{inc}}/\partial n), a\rangle = \langle p_{\mathrm{inc}}, (I+K)a\rangle - \langle(\partial p_{\mathrm{inc}}/\partial n), Sa\rangle = 0.$$

Thus, all solutions of the adjoint homogeneous system (5.17) are orthogonal to the right-hand side of (5.2), whence the system (5.2) is solvable at all frequencies. Thus, the fluid–solid problem is solvable at all frequencies. The system is not uniquely solvable when k^2 coincides with an eigenvalue of the interior Dirichlet problem or when ω is a Jones frequency. The nonuniqueness at each eigenvalue of the interior Dirichlet problem is spurious, because we know that the fluid–solid problem is uniquely solvable at these frequencies, unless, of course, it happens to coincide with a Jones frequency.

We have identified the irregular frequencies as certain values of k^2; they are unaltered by changes in the composition of the target. They can be removed by using a different representation for p in Ω_e or by combining (5.1) with another equation. For example, if we evaluate the normal derivative of (4.1) on S, we obtain

$$(I - K)(\partial p/\partial n) + Np = 2(\partial p_{\mathrm{inc}}/\partial n), \tag{5.18}$$

where N is defined by (4.3). One can then replace (5.1) with (5.1)$+\mathrm{i}\eta$(5.18) and then proceed as before; here, η is a constant. For acoustic scattering by a rigid body, this method is from Burton and Miller [5]; for the fluid–solid problem, see [1], [26].

5.2. The simplest indirect boundary integral equations. The system derived in this section is indirect. We start by looking for a solution in the form

$$p(\boldsymbol{x}) = (S\mu)(\boldsymbol{x}) + p_{\mathrm{inc}}(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x} \in \Omega_\mathrm{e} \text{ and} \tag{5.19}$$

$$\boldsymbol{u}(\boldsymbol{x}) = (\mathbf{S}\boldsymbol{g})(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x} \in \Omega_\mathrm{i}. \tag{5.20}$$

These representations satisfy the appropriate field equations and the radiation condition, for any choice of $\boldsymbol{g}$ and μ, which we assume belong to $C^{0,\beta}(S)$ for some $\beta > 0$. Applying the interface conditions (2.8) and (2.9), we obtain

$$\left\{ \begin{array}{rcl} \mu + K\mu - \rho_\mathrm{f}\omega^2 \boldsymbol{n}\cdot(\mathbf{S}\boldsymbol{g}) & = & -\partial p_{\mathrm{inc}}/\partial n, \\ \boldsymbol{g} - \mathbf{K}\boldsymbol{g} - \boldsymbol{n}(S\mu) & = & p_{\mathrm{inc}}\boldsymbol{n}. \end{array} \right. \tag{5.21}$$

In § 5.1.1, we saw that the system

$$\begin{pmatrix} I + \overline{K^*} & -\rho_\mathrm{f}\omega^2 S\boldsymbol{n} \\ -\mathbf{S}\boldsymbol{n} & \mathbf{I} - \overline{\mathbf{K}^*} \end{pmatrix} \tag{5.22}$$

is Fredholm. This implies that the system

$$\begin{pmatrix} I + \overline{K^*} & -S\boldsymbol{n} \\ -\rho_\mathrm{f}\omega^2\mathbf{S}\boldsymbol{n} & \mathbf{I} - \overline{\mathbf{K}^*} \end{pmatrix} \tag{5.23}$$

is Fredholm too, because its singular part is identical to that of (5.22). Since the index is zero, the relationship between this system and its adjoint is symmetric. Therefore, the system (5.21) is Fredholm. In addition, (5.21) is solvable if and only if k^2 is not an eigenvalue of the interior Dirichlet problem and ω is not a Jones frequency. To see this, note that, due to the vanishing index, the null-space of the system matrix in (5.21) has the same dimension as the null-space of (5.23). Since

$$\begin{pmatrix} I & 0 \\ 0 & \rho_\mathrm{f}\omega^2\mathbf{I} \end{pmatrix} \begin{pmatrix} I + \overline{K^*} & -\rho_\mathrm{f}\omega^2 S\boldsymbol{n} \\ -\mathbf{S}\boldsymbol{n} & \mathbf{I} - \overline{\mathbf{K}^*} \end{pmatrix} = \begin{pmatrix} I + \overline{K^*} & -S\boldsymbol{n} \\ -\rho_\mathrm{f}\omega^2\mathbf{S}\boldsymbol{n} & \mathbf{I} - \overline{\mathbf{K}^*} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \rho_\mathrm{f}\omega^2\mathbf{I} \end{pmatrix},$$

the null-space of (5.22) has the same dimension as does the null-space of (5.23), and the claim follows. In this case, the system (5.21) has the unique solution $(\mu, \boldsymbol{g})$. The pressure and displacement fields are then given by (5.19) and (5.20); by construction, these fields solve the fluid–solid problem.

Again, the irregular frequencies can be removed by modifying the representation in Ω_e. Thus, replace (5.19) by

$$p(\boldsymbol{x}) = (S\mu)(\boldsymbol{x}) + \mathrm{i}(D\mu)(\boldsymbol{x}) + p_{\mathrm{inc}}(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x} \in \Omega_e,$$

and then proceed as before. The resulting system of integral equations can be analysed in a manner similar to that described in § 5.3; see [22] for details.

5.3. Single integral equations. In this section, we shall derive a system of three equations in three unknowns that is free from irregular frequencies.

We look for a solution in the target in the form

$$\boldsymbol{u}(\boldsymbol{x}) = (\mathbf{S}\boldsymbol{g}), \qquad \text{for } \boldsymbol{x} \in \Omega_i, \tag{5.24}$$

where $\boldsymbol{g} \in C^{0,\beta}(S)$, whence

$$\boldsymbol{u} = \mathbf{S}\boldsymbol{g} \qquad \text{and} \qquad \boldsymbol{t} = (-\mathbf{I} + \mathbf{K})\boldsymbol{g} \qquad \text{on } S, \tag{5.25}$$

where $\boldsymbol{t} = T\boldsymbol{u}$ is the traction vector. The dynamic interface condition (2.9) gives

$$\boldsymbol{t} = -p\boldsymbol{n} = (\boldsymbol{t} \cdot \boldsymbol{n})\boldsymbol{n},$$

whence (5.24) and the second part of (5.25) give

$$(\mathbf{I} - \mathbf{K})\boldsymbol{g} - [\boldsymbol{n} \cdot (\boldsymbol{g} - \mathbf{K}\boldsymbol{g})]\boldsymbol{n} = \mathbf{0}. \tag{5.26}$$

This equation does not give any information in the normal direction: we need an additional scalar equation before we can determine $\boldsymbol{g}$.

From (5.1) and the interface conditions, we have

$$(I + \overline{K^*})(\boldsymbol{t} \cdot \boldsymbol{n}) + \rho_f \omega^2 S(\boldsymbol{u} \cdot \boldsymbol{n}) = -2p_{\mathrm{inc}}.$$

Substituting from (5.25) gives

$$\boldsymbol{L} \cdot \boldsymbol{g} = 2p_{\mathrm{inc}}, \tag{5.27}$$

where

$$\boldsymbol{L} \cdot \boldsymbol{g} = (I + \overline{K^*})\{\boldsymbol{n} \cdot (\boldsymbol{g} - \mathbf{K}\boldsymbol{g})\} - \rho_f \omega^2 S\{\boldsymbol{n} \cdot (\mathbf{S}\boldsymbol{g})\}.$$

Alternatively, from (5.18) and the interface conditions, we have

$$\rho_f \omega^2 (I - K)(\boldsymbol{u} \cdot \boldsymbol{n}) - N(\boldsymbol{t} \cdot \boldsymbol{n}) = 2(\partial p_{\mathrm{inc}}/\partial n),$$

whence substituting from (5.25) gives

$$\boldsymbol{M} \cdot \boldsymbol{g} = 2(\partial p_{\mathrm{inc}}/\partial n), \tag{5.28}$$

where we assume now that $\boldsymbol{g} \in C^{1,\beta}(S)$ and

$$\boldsymbol{M} \cdot \boldsymbol{g} = N\{\boldsymbol{n} \cdot (\boldsymbol{g} - \mathbf{K}\boldsymbol{g})\} + \rho_f \omega^2 (I - K)\{\boldsymbol{n} \cdot (\mathbf{S}\boldsymbol{g})\}.$$

The sets (5.26) and (5.27), and (5.26) and (5.28), both give three scalar equations from which $\boldsymbol{g}$ is to be determined. (We can obtain further sets if we replace (5.24) by $\boldsymbol{u} = (\mathbf{D}\boldsymbol{g})$.) However, both sets suffer from irregular frequencies. Thus, in the spirit of Burton and Miller [5], we combine (5.27) and (5.28) and consider the system

$$(5.29) \qquad \begin{cases} \boldsymbol{L}\cdot\boldsymbol{g} + \mathrm{i}\eta\boldsymbol{M}\cdot\boldsymbol{g} &=& 2p_{\mathrm{inc}} + 2\mathrm{i}\eta(\partial p_{\mathrm{inc}}/\partial n), \\ (\mathbf{I}-\mathbf{K})\boldsymbol{g} - [\boldsymbol{n}\cdot(\boldsymbol{g}-\mathbf{K}\boldsymbol{g})]\boldsymbol{n} &=& \mathbf{0}, \end{cases}$$

where η is a constant to be chosen later. Having solved (5.29) for $\boldsymbol{g}$, we construct the solution of the fluid–solid problem, using

$$(5.30) \qquad p(\boldsymbol{x}) = \tfrac{1}{2}\rho_{\mathrm{f}}\omega^2\left(S\{\boldsymbol{n}\cdot(\mathbf{S}\boldsymbol{g})\}\right)(\boldsymbol{x}) - \tfrac{1}{2}(D\{\boldsymbol{n}\cdot(\boldsymbol{g}-\mathbf{K}\boldsymbol{g})\})(\boldsymbol{x}) + p_{\mathrm{inc}}(\boldsymbol{x})$$

for $\boldsymbol{x}\in\Omega_{\mathrm{e}}$ and (5.24) for $\boldsymbol{x}\in\Omega_{\mathrm{i}}$.

5.3.1. Uniqueness. Before we prove that the system (5.29) is Fredholm, let us first show that the corresponding homogeneous system has a nontrivial solution only if ω is a Jones frequency. So, suppose that $\boldsymbol{g}'$ is such a solution, belonging to $C^{1,\beta}(S)$. Construct fields $\boldsymbol{u}$ and p, using (5.24) and (5.30), respectively, but with $p_{\mathrm{inc}}=0$ and $\boldsymbol{g}$ replaced by $\boldsymbol{g}'$. Using subscripts $\pm$ as in § 5.1.2, we find that

$$p_- + \mathrm{i}\eta\frac{\partial p_-}{\partial n} = 0 \qquad \text{on } S;$$

p also solves the Helmholtz equation in Ω_{i}. A standard argument then shows that p vanishes identically in Ω_{i}, if we make a suitable choice for η: we choose

$$\eta = \overline{k}^2/|k|^2.$$

In particular, $\partial p_-/\partial n = 0$. If we now compute p_+, $\partial p_+/\partial n$, $\boldsymbol{u}_-$ and $\boldsymbol{t}_-$, we find that the interface conditions are satisfied. Clearly, the field equations are also satisfied, as is the radiation condition on p. Hence (§ 3), $p\equiv 0$ in Ω_{e}, and $\boldsymbol{u}\equiv\mathbf{0}$ in Ω_{i} unless a Jones mode is possible; if not, then $\boldsymbol{u}_-=\mathbf{0}$. The continuity of the elastic single-layer potential then implies that $\boldsymbol{u}_+=\mathbf{0}$, whence $\boldsymbol{u}\equiv\mathbf{0}$ in Ω_{e} [20]. Then, since $\boldsymbol{t}_+=\mathbf{0}$ and $\boldsymbol{t}_-=\mathbf{0}$, we deduce that $\boldsymbol{g}'=\mathbf{0}$, as required: we have uniqueness unless ω is a Jones frequency.

5.3.2. Existence. To prove the existence of solutions to the system (5.29), we adapt an argument given in [6, p. 93]. Choose a wavenumber k_0 so that k_0^2 is neither an eigenvalue of the interior Dirichlet problem nor an eigenvalue of the interior Neumann problem; denote operators evaluated with k_0 in place of k by a subscript zero. Let us write

$$N = N_0 + (N - N_0);$$

N_0 is invertible [6, p. 90] with

$$N_0^{-1} = S_0(I+K_0)^{-1}(-I+K_0)^{-1},$$

which shows that N_0^{-1} is compact on $C^{0,\beta}(S)$ and that N_0^{-1} maps $C^{0,\beta}(S)$ into $C^{1,\beta}(S)$. $(N-N_0)$ is compact on $C^{0,\beta}(S)$, since its kernel is weakly singular; its kernel also satisfies the conditions (A.2), (A.3), and (A.4).

We now operate on the left on the first part of (5.29) with $-\mathrm{i}\eta^{-1}N_0^{-1}$ to regularize the hypersingular operator, N, in $\boldsymbol{M}$; it becomes

$$\boldsymbol{n}\cdot(\boldsymbol{g}-\mathbf{K}\boldsymbol{g})+\text{compact terms}=-(2\mathrm{i}/\eta)N_0^{-1}\{p_{\mathrm{inc}}+\mathrm{i}\eta(\partial p_{\mathrm{inc}}/\partial n)\}.$$

When this is combined with the second part of (5.29), we find that the symbol matrix for the complete, regularized 3×3 system is

$$\Theta(\boldsymbol{x},\theta)=\xi^T(\boldsymbol{x})\begin{pmatrix}1 & -\mathrm{i}\delta\cos\theta & -\mathrm{i}\delta\sin\theta\\ \mathrm{i}\delta\cos\theta & 1 & 0\\ \mathrm{i}\delta\sin\theta & 0 & 1\end{pmatrix}\xi(\boldsymbol{x}),$$

where $\xi(\boldsymbol{x})$ is the rotation matrix defined in § 5.1.1 and δ is defined by (5.4). Clearly, the condition (5.5) is satisfied for all allowable values of Poisson's ratio, whence the system (5.29) is Fredholm. Since the only solution of the homogeneous system is the trivial one, we deduce the existence of a solution in $C^{1,\alpha}(S)$ of the inhomogeneous system.

Appendix. Regularization, regularity and the symbol matrix. Consider an operator $A: X\to X$, where X is a Banach space. The bounded operator $B: X\to X$ is called a *left equivalent regularizer* if

$$BA=I+T,$$

where I denotes the identity and T is compact in X, and if the equations

$$Au=f \qquad \text{and} \qquad BAu=Bf$$

are equivalent. Similarly, $C: X\to X$ is called a *right equivalent regularizer* if

$$AC=I+T',$$

where T' is compact in X, and if any solution of $Au=f$ can be written as $u=Cv$, for some v in X, and vice versa.

The *index* of an operator A, $\operatorname{Ind} A$, is defined by

$$\operatorname{Ind} A=\dim\{\mathcal{N}(A)\}-\dim\{\mathcal{N}(A^*)\},$$

where $\mathcal{N}(A)$ is the null-space of A and A^* is the adjoint of A. We have the following important results.

THEOREM A.1. *If A admits both left and right regularization, then* $\operatorname{Ind} A$ *is finite.*

THEOREM A.2. *If a closed operator A admits a left regularization, then, for the solvability of $Au=f$, it is necessary and sufficient that f be orthogonal to every element of $\mathcal{N}(A^*)$. We say A is* normally solvable *when it has this property.*

An immediate corollary of Theorem A.2 is the fact that if A admits a right regularization, then A^* is normally solvable.

The question that concerns us here is, Given a system of operators of the form

$$(\mathbf{A}\boldsymbol{u})(\boldsymbol{x})=\boldsymbol{u}(\boldsymbol{x})+\int_S \mathbf{k}(\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{u}(\boldsymbol{y})\,ds(\boldsymbol{y}),$$

where $\mathbf{k}(\boldsymbol{x},\boldsymbol{y})$ is singular and $\mathbf{A}$ is considered to be acting on $L^2(S)$, under what conditions does a regularizer having the form

$$(\mathbf{B}\boldsymbol{u})(\boldsymbol{x})=\boldsymbol{u}(\boldsymbol{x})+\int_S \mathbf{k}'(\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{u}(\boldsymbol{y})\,ds(\boldsymbol{y})$$

exist? To answer this question, some general results will be used without proof; detailed accounts of the theory of regularization of two-dimensional singular integral operators are given in [20, Chap. 4], [27, Chap. 14], and [35, Chap. 2].

Since S is sufficiently smooth, a normal can be defined at every point of S. The normal at $\boldsymbol{x}_0 \in S$, $\boldsymbol{n}(\boldsymbol{x}_0)$, is in $C^{1,\alpha}(S)$. Define the cylinder $C(\boldsymbol{x}_0)$ by

$$C(\boldsymbol{x}_0) = \{\boldsymbol{y} : |(\boldsymbol{y}-\boldsymbol{x}) \times \boldsymbol{n}(\boldsymbol{x}_0)| \leq d,\ -l \leq (\boldsymbol{y}-\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}_0) \leq l\},$$

where l and d are chosen to be small enough so that the orthogonal projection of the intersection of S and $C(\boldsymbol{x}_0)$, which we shall refer to as $S(\boldsymbol{x}_0, d)$, onto the base of $C(\boldsymbol{x}_0)$ is conformal. Let $\tau(\boldsymbol{x}_0, d)$ be the intersection of $C(\boldsymbol{x}_0)$ and the tangent-plane to S at $\boldsymbol{x}_0$. If ζ is the image of the orthogonal projection of a point $\boldsymbol{x} \in S(\boldsymbol{x}_0, d)$ onto $\tau(\boldsymbol{x}_0, d)$ and f is any function with domain $S(\boldsymbol{x}_0, d)$, then we shall denote by f' the function in $\tau(\boldsymbol{x}_0, d)$ with $f'(\zeta) = f(\boldsymbol{x})$; we suppose that the point $\boldsymbol{x}_0$ is mapped to the origin under the orthogonal projection.

Suppose that

$$k'_{ij}(\zeta, \eta) = l_{ij}(\zeta, \zeta - \eta) + m_{ij}(\zeta, \eta) \tag{A.1}$$

and $l_{ij}(\zeta, t(\zeta - \eta)) = t^{-2} l_{ij}(\zeta, \zeta - \eta)$, for all $t > 0$ and $\zeta \neq \eta$. Suppose, further, that $l_{ij}(\zeta, \kappa)$ and all its derivatives with respect to κ when considered as a function of ζ belong to $C^{1,\alpha}(\tau(\boldsymbol{x}_0, d))$, for all κ with $|\kappa| = 1$. Finally, suppose that $m_{ij}(\zeta, \eta)$ satisfies the following two conditions:

$$|m_{ij}(\zeta', \eta) - m_{ij}(\zeta'', \eta)| \leq M|\zeta' - \zeta''|^{\beta}(\upsilon(\zeta', \zeta'', \eta))^{-2} \tag{A.2}$$

and

$$|m_{ij}(\zeta, \eta') - m_{ij}(\zeta, \eta'')| \leq M|\eta' - \eta''|^{\beta}(\upsilon(\eta', \eta'', \zeta))^{-2}, \tag{A.3}$$

where M is a positive constant and, for example, $\upsilon(\zeta', \zeta'', \eta) = \min\{|\zeta' - \eta|, |\zeta'' - \eta|\}$. If all three of these conditions hold, then $\mathbf{A}$ is said to belong to the class $G(\beta)$.

Suppose that $\mathbf{A} \in G(\beta)$ and that $\boldsymbol{t}(\boldsymbol{x}_0)$ is a unit vector in $\tau(\boldsymbol{x}_0, d)$ that makes an angle θ with some fixed line in $\tau(\boldsymbol{x}_0, d)$. All the derivatives of $l_{ij}(\zeta, \kappa)$ with respect to κ are supposed to exist. $l_{ij}(\zeta, \boldsymbol{t}(\boldsymbol{x}_0))$ may thus be expanded as a Fourier series,

$$l_{ij}(\boldsymbol{x}_0, \boldsymbol{t}(\boldsymbol{x}_0)) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} a_{ij}^{(n)} \exp(in\theta);$$

the term for $n = 0$ is missing because we have to assume that

$$\int_{|\kappa|=1} l_{ij}(\zeta, \kappa)\, d\kappa = 0$$

for the existence of $\mathbf{A}\boldsymbol{u}$ as a principal-value integral. Define the *symbol matrix* $\Theta(\boldsymbol{x}_0, \theta)$ to be the matrix with entries

$$\Theta_{ij}(\boldsymbol{x}_0, \boldsymbol{t}(\boldsymbol{x}_0)) = \delta_{ij} + 2\pi \mathrm{i} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\mathrm{i}^{|n|}}{|n|} a_{ij}^{(n)} \exp(in\theta)$$

The main result of the general theory is the following.

THEOREM A.3. *Suppose that*

$$\inf_{\substack{x_0 \in S \\ 0 \leq \theta \leq 2\pi}} |\det \Theta(\boldsymbol{x}_0, \theta)| > 0.$$

Then a double-sided regularizer of $\mathbf{A}$ *of the correct form exists in* $L^2(S)$. *Moreover, this regularizer is in the class* $G(\beta)$.

We shall need the following theorems.

THEOREM A.4. *If* $\mathbf{B}$ *is the regularizer of* $\mathbf{A}$ *in Theorem A.3, then*

$$\operatorname{Ind} \mathbf{B} + \operatorname{Ind} \mathbf{A} = 0.$$

THEOREM A.5. *If the symbol matrix is Hermitian, then* $\operatorname{Ind} \mathbf{B} = 0$.

Thus, if $\mathbf{A}$ satisfies the conditions of Theorem A.3 and its symbol matrix is Hermitian, then its index is zero. Therefore, this and Theorem A.2 imply that $\mathbf{A}$ satisfies the Fredholm properties, and thus $\mathbf{A}$ is said to be a (quasi)*Fredholm operator.*

Suppose that $\mathbf{A}$ is in the class $G(\beta)$. Suppose that

$$k_{ij}(\boldsymbol{x}, \boldsymbol{y}) \in C^{1,\alpha}(S(\boldsymbol{x}_0, \delta)),$$

as a function of its first argument uniformly in $\boldsymbol{y} \in S(\boldsymbol{x}_0, d) \setminus S(\boldsymbol{x}_0, \delta)$, where δ is any positive number less than $d/2$, and that the function $m_{ij}(\zeta, \eta)$ in (A.1) satisfies the following property:

$$\int_{\tau(x_0,d)} m_{ij}(\zeta, \eta) u(\eta)\, ds(\eta) \in C^{1,\beta}(\tau(\boldsymbol{x}_0, d)), \tag{A.4}$$

whenever $u \in C^{0,\beta}(\tau(\boldsymbol{x}_0, d))$, for $0 < \beta \leq \alpha$. Then $\mathbf{A}$ is said to belong to the class $G'(\beta)$. For such operators, we have the following important regularity result.

THEOREM A.6. *Let* β *be any positive number with* $\beta \leq \alpha$. *If* $\mathbf{A}\boldsymbol{u} = \boldsymbol{f}$, *where* $\mathbf{A}$ *is a singular integral operator in the class* $G'(\beta)$, $\boldsymbol{f} \in C^{1,\beta}(S)$, *and* $\boldsymbol{u} \in L^2(S)$, *then* $\boldsymbol{u} \in C^{1,\beta}(S)$.

The effect of this theorem and the preceding results is that any operator in $G'(\beta)$ that satisfies the conditions in Theorems A.3 and A.5 is Fredholm on $C^{1,\beta}(S)$, for any positive number β with $\beta \leq \alpha$.

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