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Asymptotic approximations for functions defined by series, with some applications to the theory of guided waves

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Mellin transforms are used here to find asymptotic approximations for functions defined by series. The simplest cases are those of the form $\sum_{n=1}^{\infty} u(nx)$. Such series are called *separable* here, because the given function u is sampled at points whose variation with n and x is separated. Nonseparable series are analysed by first approximating them by separable series. Both types of series arise in the theory of electromagnetic waveguides and in the theory of linear water waves; several examples are worked out in detail.

1. Introduction

Asymptotic approximations for functions defined by integrals in the form

$$f(x) = \int_0^\infty c(\mu)u(\mu x)\,\mathrm{d}\mu, \tag{1}$$

for small or large values of x, can be found using Mellin transforms [2, 18]. We are interested in obtaining analogous results for functions defined by series,

$$f(x) = \sum_{n=1}^{\infty} c_n u(\mu_n x). \tag{2}$$

Here, c_n are known constants and μ_n is an increasing sequence, with $n = 1, 2, \ldots$; the function $u(y)$ is defined for all $y > 0$. We assume that the series is convergent for all positive real x (it may diverge at $x = 0$), and seek the asymptotic behaviour of $f(x)$ as $x \to 0+$. (The behaviour as $x \to \infty$ can be found in a similar manner.) Some problems of this type were considered by Ramanujan [1: Chap. 15]: he took $\mu_n = n^p$ and $u(x) = \mathrm{e}^{-x}$ or $u(x) = (1+x)^{-l}$, for integer values of p and l, and is believed to have obtained his results using the Euler–Maclaurin formula.

In (2), the function $u(y)$ is sampled at points $y = \mu_n x$; we describe such points, and the series (2), as *separable*. Separable series can often be analysed directly using Mellin transforms (Section 3). For example, this method was used by Macfarlane [10] on one example; he showed that

$$\sum_{n=1}^{N} \frac{(1 - xn^{\frac{2}{3}})^{\frac{1}{2}}}{n^{\frac{2}{3}}} \sim \frac{3\pi}{4\sqrt{x}} + \zeta(\tfrac{2}{3}) + \tfrac{1}{4}x - \tfrac{1}{8}\zeta(-\tfrac{2}{3})x^2$$

as $x \to 0$, where N is the largest integer such that $xN^{\frac{2}{3}} < 1$ and ζ is the Riemann zeta function. Macfarlane's work is described in the books by Sneddon [16: § 4-7] and by Davies [3: § 13.1]; Davies also gives some other examples. Mellin transforms were also used by Berndt [1] to confirm some of Ramanujan's results. Estrada & Kanwal [5] and Estrada [4] have obtained similar results, using the theory of generalized functions.

A natural generalization of (1) is

$$f(x) = \int_0^\infty c(\mu) U(\mu, x)\, \mathrm{d}\mu, \tag{3}$$

where U is a function of two variables. We are interested in a similar generalization of (2). Thus, we consider series where $u(y)$ is sampled at nonseparable points, $y = \lambda_n(x)$, where $\lambda_n(x)$ is a known function of x for each n;

$$f(x) = \sum_{n=1}^{\infty} c_n u(\lambda_n(x)) \tag{4}$$

is called a *nonseparable* series.

There are no general methods for the asymptotic approximation of integrals of the form (3). Similarly, we do not expect to find a general method for the asymptotic approximation of nonseparable series; indeed, we are not aware of any previous results for such series. However, in certain applications (described below), the quantities $\lambda_n(x)$ occur as the solutions of a transcendental equation, and then progress can be made. Our method proceeds in two stages. First, we look for suitable separable approximations to $\lambda_n(x)$, and then we use Mellin-transform techniques (Sections 4 and 5).

Both separable and nonseparable series arise in waveguide problems. Such problems are often solved using various modal expansions (separation of variables, matched eigenfunction expansions, Wiener–Hopf techniques, etc.). In these expansions, the lateral variation is represented by a series of eigenfunctions, which depend on the width of the waveguide (related to x). Moreover, if the waveguide walls are hard (Neumann boundary condition) or soft (Dirichlet condition), the associated eigenvalues are separable (in the above sense), and so (2) is typical. However, if the walls are impedance boundaries (Robin condition), the lateral eigenvalues are usually determined as the roots of a transcendental equation, leading to the nonseparable series (4).

In the context of waveguide problems, the limit $1/h \to 0$ is of interest, where h is the width of the waveguide. This limit has been discussed by Mittra & Lee [13: § 3-11.(2)]. They consider the infinite bifurcated waveguide shown in Fig. 1, with

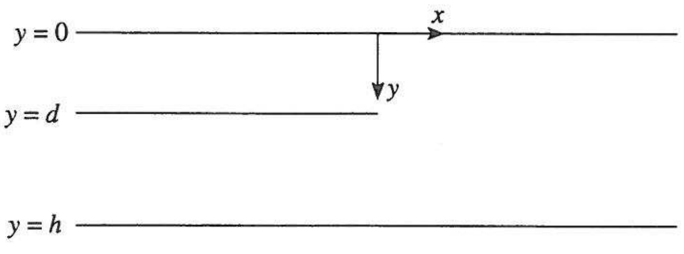

FIG. 1. Closed region—finite depth.

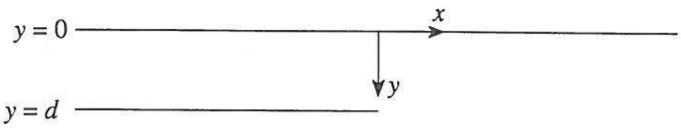

FIG. 2. Open region—infinite depth.

a semi-infinite plate (the septum) at a fixed distance d from the wall at $y = 0$; this geometry is referred to as a *closed region.* The governing partial differential equation is the Helmholtz equation,

$$(\nabla^2 + k^2)\phi = 0,$$

where k is the positive real wavenumber. A waveguide mode is incident from $x = -\infty$ in the region $0 < y < d$; it is partially reflected at the end of the septum and partially transmitted into the rest of the guide. The corresponding reflection and transmission coefficients can be determined exactly (Mittra & Lee [13] solve this problem in detail).

The same problem can be considered (and solved) when $h = \infty$. This corresponds to an open-ended waveguide; the geometry is sketched in Fig. 2 and is referred to as an *open region.* The connection between open-region and related closed-region problems is of interest because the latter are often easier to solve: for example, when the Wiener–Hopf technique is used, a certain function of a complex variable has to be factorized; this function is meromorphic in closed-region problems but has branch points in open-region problems. Mittra & Lee [13] show that the open-waveguide problem can be solved by taking the limit $h \to \infty$ of the bifurcated-waveguide problem, but only when the walls and septum are hard. The methods described below can be used to analyse such problems, even when the walls are impedance boundaries.

The geometry sketched in Fig. 1 has also been used by Linton & Evans [9] in the context of small-amplitude water waves. The governing partial differential equation is the modified Helmholtz equation

$$(\nabla^2 - l^2)\phi = 0,$$

where l is the positive real wavenumber in a direction perpendicular to the xy-plane. The semi-infinite plate and the bottom ($y = h$) are hard, whereas the boundary condition on the mean free surface $y = 0$ is an impedance condition,

$$K\phi + \partial\phi/\partial y = 0 \quad \text{on } y = 0,$$

where K is another positive real wavenumber. Two more wavenumbers, k and k_0, are defined to be the unique positive real roots of the dispersion relations

$$K = k \tanh kd \quad \text{and} \quad K = k_0 \tanh k_0 h, \tag{5a,b}$$

respectively, and then l is chosen to satisfy $K < k_0 < l < k$. Consequently, when a surface wave is incident from $x = -\infty$, it will be totally reflected by the end of the plate. Linton & Evans [9] gave an explicit formula for the argument of the (complex) reflection coefficient, which they used to estimate the frequencies of waves trapped above a long horizontal submerged plate. We shall examine their formula below, and extract the limiting formula for deep water ($h \to \infty$) (Section 5). Indeed, it was a study of the limiting problem (for the geometry sketched in Fig. 2) that originally motivated the present analysis.

2. Mellin transforms

In order to find asymptotic approximations for separable series (2), we use Mellin transforms. Given a function $f(x)$, its Mellin transform is defined by

$$\bar{f}(z) = \int_0^\infty f(x) x^{z-1}\,\mathrm{d}x,$$

where we shall always use the notation $z = \sigma + \mathrm{i}\tau$ for the transform variable z. Typically, $\bar{f}(z)$ will be an analytic function of z within a strip, $a < \sigma < b$, say; within this strip, we have

$$|\bar{f}(\sigma + \mathrm{i}\tau)| \to 0 \quad \text{as } |\tau| \to \infty. \tag{6}$$

The inverse Mellin transform is given by

$$f(x) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \bar{f}(z) x^{-z}\,\mathrm{d}z$$

for fixed $x > 0$, where $a < c < b$. We can obtain an asymptotic expansion of $f(x)$ for small x by moving the inversion contour to the left; each term arises as a residue contribution from an appropriate pole in the analytic continuation of $\bar{f}(z)$ into $\sigma \leqslant a$. Specifically, we have the following result.

Theorem 1 [14: p. 7] Suppose that $\bar{f}(z)$ is analytic in a left-hand plane, $\sigma \leqslant a$, apart from poles at $z = -a_m$ $(m = 0, 1, 2, \ldots)$; let the principal part of the Laurent expansion of $\bar{f}(z)$ about $z = -a_m$ be given by

$$\sum_{n=0}^{N(m)} A_{mn} \frac{(-1)^n n!}{(z + a_m)^{n+1}}.$$

Assume that (6) holds for $a' \leqslant \sigma \leqslant a$. Then, if a' can be chosen so that

$$-\mathrm{Re}\,(a_{M+1}) < a' < -\mathrm{Re}\,(a_M)$$

for some M, we have

$$f(x) = \sum_{m=0}^{M} \sum_{n=0}^{N(m)} A_{mn} x^{a_m} (\log x)^n + R_M(x),$$

where

$$R_M(x) = \frac{1}{2\pi \mathrm{i}} \int_{a'-\mathrm{i}\infty}^{a'+\mathrm{i}\infty} \bar{f}(z) x^{-z}\,\mathrm{d}z = \frac{x^{-a}}{2\pi} \int_{-\infty}^{\infty} \bar{f}(a' + \mathrm{i}\tau) x^{-\mathrm{i}\tau}\,\mathrm{d}\tau.$$

Furthermore, suppose that

$$\int_{-\infty}^{\infty} |\tilde{f}(a'+\mathrm{i}\tau)|\,\mathrm{d}\tau < \infty, \tag{7}$$

or, less restrictively,

$$\int_{-\infty}^{\infty} \tilde{f}(a'+\mathrm{i}\tau)\mathrm{e}^{\mathrm{i}\tau X}\,\mathrm{d}\tau < \infty \tag{8}$$

with $X = -\log x$. Then, the remainder $R_M(x)$ is $o(x^{\operatorname{Re}(a_M)})$, whence $f(x)$ has the asymptotic expansion

$$f(x) \sim \sum_{m=0}^{M} \sum_{n=0}^{N(m)} A_{mn} x^{a_m} (\log x)^n \quad \text{as } x \to 0+.$$

Note that (8) will be satisfied if $\tilde{f}(a'+\mathrm{i}\tau)$ is integrable for finite τ, and is $O(|\tau|^{-\delta})$ as $|\tau| \to \infty$ for some $\delta > 0$.

More information on Mellin transforms can be found in [2: Chap. 4; 3: §§ 12, 13; 16: Chap. 4; 18: Chap. 3]. In addition, we have used Mellin transforms previously to find asymptotic approximations for solutions to certain integral equations, near the end-points of the range of integration [11, 12].

3. Separable series: A problem of Ramanujan

Consider the series (2), namely

$$f(x) = \sum_{n=1}^{\infty} c_n u(\mu_n x). \tag{9}$$

We shall find the asymptotic behaviour of $f(x)$ as $x \to 0+$ by calculating its Mellin transform. We have

$$\tilde{f}(z) = \sum_{n=1}^{\infty} c_n \int_0^{\infty} x^{z-1} u(\mu_n x)\,\mathrm{d}x = \tilde{u}(z) \sum_{n=1}^{\infty} \mu_n^{-z} c_n. \tag{10}$$

To make progress, we must be able to locate the singularities of $\tilde{u}(z)$ and of the sum on the right-hand side of (10). So, to fix ideas, consider the following example.

Example 1. Find the behaviour of

$$f_\nu(x) = \sum_{n=1}^{\infty} n^{\nu-1} \mathrm{e}^{-nx} \quad \text{as } x \to 0+,$$

where ν is a real parameter.

We can take $c_n = n^{\nu-1}$, $\mu_n = n$, and $u(x) = \mathrm{e}^{-x}$. Hence

$$\tilde{f}_\nu(z) = \zeta(z-\nu+1)\Gamma(z), \tag{11}$$

where $\Gamma(z)$ is the gamma function and $\zeta(z)$ is the Riemann zeta function. It is known that $\Gamma(z)$ is an analytic function of z, apart from simple poles at $z=-N$ $(N=0,1,2,\ldots)$; near $z=-N$,

$$\Gamma(z) \simeq (-1)^N[(z+N)^{-1}+\psi(N+1)]/N!,$$

where $\psi(z)=\Gamma'(z)/\Gamma(z)$. It is also known that $\zeta(z)$ is analytic for all z, apart from a simple pole at $z=1$; near $z=1$,

$$\zeta(z) \simeq (z-1)^{-1}+\gamma, \tag{12}$$

where $\gamma=0{\cdot}5772\cdots$ is Euler's constant.

Let us suppose that $0<\nu<1$. Then, $\tilde{f}_\nu(z)$ is analytic for $\sigma>\nu$. We choose the inversion contour along $\sigma=c$, with $c>\nu$. Moving the contour to the left, we pick up a residue contribution from the simple pole at $z=\nu$: this gives the leading contribution as

$$f_\nu(x) \sim x^{-\nu}\Gamma(\nu) \quad \text{as } x\to 0+ \quad \text{for } 0<\nu<1. \tag{13}$$

Mittra & Lee [13: § 1-4, eqn (4.1)] have obtained this result, using the Euler–Maclaurin sum formula.

If we move the inversion contour further to the left, we formally obtain Ramanujan's expansion [1: p. 306],

$$f_\nu(x) \sim x^{-\nu}\Gamma(\nu)+\sum_{m=1}^{\infty}\frac{(-x)^m}{m!}\zeta(1-\nu-m) \quad \text{as } x\to 0. \tag{14}$$

The fact that this is an *asymptotic* expansion follows from Theorem 1 and the known properties of $\zeta(z)$ and $\Gamma(z)$ as $|\tau|\to\infty$. Thus, from [17: p. 276], we have

$$\zeta(\sigma+\mathrm{i}\tau)=O\big(|\tau|^{\alpha(\sigma)}\log|\tau|\big) \quad \text{as } |\tau|\to\infty, \tag{15}$$

where

$$\alpha(\sigma)=\begin{cases} \frac{1}{2}-\sigma & (\sigma\leqslant 0), \\ \frac{1}{2} & (0\leqslant\sigma\leqslant\frac{1}{2}), \\ 1-\sigma & (\frac{1}{2}\leqslant\sigma\leqslant 1), \\ 0 & (\sigma\geqslant 1), \end{cases} \tag{16}$$

and the factor of $\log|\tau|$ can be omitted except when σ is close to 0 or 1; and

$$\Gamma(\sigma+\mathrm{i}\tau)=O\big(|\tau|^{\sigma-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}\pi|\tau|}\big) \quad \text{as } |\tau|\to\infty. \tag{17}$$

Hence, although $\zeta(\sigma+\mathrm{i}\tau)$ grows algebraically as $|\tau|\to\infty$, for $\sigma<1$, the exponential decay of $\Gamma(\sigma+\mathrm{i}\tau)$ ensures that (7) is satisfied for all values of a'.

The asymptotic formula (14) is valid for all values of ν, apart from $\nu=-N$. In these cases, there is a double pole at $z=-N$, giving a term proportional to $x^N\log x$. For example, when $\nu=0$, we obtain

$$\begin{aligned} f_0(x) &\sim -\log x+\sum_{m=1}^{\infty}\frac{(-x)^m}{m!}\zeta(1-m) \\ &= -\log x+\tfrac{1}{2}x-\sum_{n=1}^{\infty}\frac{x^{2n}\mathrm{B}_{2n}}{2n(2n)!} \quad \text{as } x\to 0, \end{aligned} \tag{18}$$

since $\zeta(0) = -\frac{1}{2}$, $\zeta(-2m) = 0$, and $\zeta(1-2m) = -\mathrm{B}_{2m}/(2m)$ $(m = 1, 2, \ldots)$, where B_n is a Bernoulli number. In fact, $f_0(x)$ can be found explicitly by integrating the geometric series to give

$$f_0(x) = -\log(1 - \mathrm{e}^{-x}),$$

which agrees completely with (18).

A related example is the following.

EXAMPLE 2. Find the behaviour of

$$g_\nu(x) = \sum_{n=1}^{\infty} (n - \tfrac{1}{2})^{\nu-1} \mathrm{e}^{-(n-\frac{1}{2})x} \quad \text{as } x \to 0+,$$

where ν is a real parameter.

We find that

$$\tilde{g}_\nu(z) = (2^{z-\nu+1} - 1)\tilde{f}_\nu(z),$$

where $\tilde{f}_\nu(z)$ is given by (11) and we have used

$$\sum_{n=1}^{\infty} (n - \tfrac{1}{2})^{-z} = (2^z - 1)\zeta(z). \tag{19}$$

So, if $0 < \nu < 1$, the leading contribution is again given by (13), although the subsequent terms are different.

4. Nonseparable series: A model problem

In this section, and the next, we consider some nonseparable series involving the roots of the transcendental equation (5b). Apart from the real roots $\pm k_0$, (5b) also has an infinite number of pure imaginary roots $\pm \mathrm{i}k_n$ $(n = 1, 2, \ldots)$. Thus, k_n are the positive real roots of

$$K + k_n \tan k_n h = 0 \quad (n = 1, 2, \ldots); \tag{20}$$

they are ordered so that $(n - \frac{1}{2})\pi < k_n h < n\pi$. In the context of water-wave problems, h is the constant water depth, and K is the positive real wavenumber. We are interested in the deep-water limit, $h \to \infty$. In dimensionless variables, we define

$$x = (Kh)^{-1} \quad \text{and} \quad \lambda_n(x) = k_n h,$$

so that

$$\cos \lambda_n(x) + x\lambda_n(x) \sin \lambda_n(x) = 0, \tag{21}$$

with

$$(n - \tfrac{1}{2})\pi < \lambda_n(x) < n\pi \quad (n = 1, 2, \ldots). \tag{22}$$

Later, we shall study some series involving $\lambda_n(x)$. Clearly, the convergence of these series will depend on the behaviour of $\lambda_n(x)$ as $n \to \infty$, although we are interested in the behaviour as $x \to 0$. It is straightforward to show that, in these limits, $\lambda_n(x) \sim n\pi$ as $n \to \infty$ for fixed x, but $\lambda_n(x) \sim (n - \frac{1}{2})\pi$ as $x \to 0$ for fixed n. These estimates can be refined:

$$\lambda_n(x) \sim n\pi - (n\pi x)^{-1} - (x - \tfrac{1}{3})(n\pi x)^{-3} \quad \text{as } n \to \infty \quad \text{for fixed } x, \tag{23}$$

$$\lambda_n(x) \sim (n - \tfrac{1}{2})\pi(1 + x + x^2) \quad \text{as } x \to 0 \quad \text{for fixed } n. \tag{24}$$

It is this nonuniform behaviour that causes difficulties.

To find some uniform approximations, we return to the definition (21). Write

$$\lambda_n(x) = \mu_n + \nu_n(x), \tag{25}$$

where

$$\mu_n = (n - \tfrac{1}{2})\pi$$

and $0 < \nu_n < \frac{1}{2}\pi$. Then, (21) gives

$$\sin \nu_n(x) - x[\mu_n + \nu_n(x)] \cos \nu_n(x) = 0. \tag{26}$$

Discarding the second term inside the square brackets (this is certainly reasonable for large n), we obtain

$$\nu_n(x) \simeq \tan^{-1}(\mu_n x) = \nu_n^{(1)}(x), \tag{27}$$

say, which is a separable approximation to $\nu_n(x)$. One can show that the approximation $\lambda_n(x) \simeq \mu_n + \nu_n^{(1)}(x)$ agrees with the first two terms in (23) and with the first two terms in (24).

We can obtain an improved approximation by iteration: replace $\nu_n(x)$ by $\nu_n^{(1)}(x)$ inside the braces in (26) to give

$$\nu_n(x) \simeq \tan^{-1}[\mu_n x + x \tan^{-1}(\mu_n x)] = \nu_n^{(2)}(x), \tag{28}$$

say. Then, the approximation $\lambda_n(x) \simeq \mu_n + \nu_n^{(2)}(x)$ agrees with the three-term asymptotics in (23) and in (24).

For a simple, but nontrivial, problem, we consider the following example.

EXAMPLE 3. Let

$$f(x) = \pi \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n(x)} - \frac{1}{n\pi} \right).$$

The series converges for all $x \geq 0$; in fact, using the bounds (22), we have

$$0 < f(x) < \pi \sum_{n=1}^{\infty} \left(\frac{1}{(n - \frac{1}{2})\pi} - \frac{1}{n\pi} \right) = 2 \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} = 2 \log 2$$

for $x > 0$. Since $\lambda_n(0) = (n - \frac{1}{2})\pi = \mu_n$, for all n, we write

$$f(x) = 2 \log 2 + S(x), \tag{29}$$

where

$$S(x) = \pi \sum_{n=1}^{\infty} s_n(x) \quad \text{and} \quad s_n(x) = \frac{1}{\lambda_n(x)} - \frac{1}{\mu_n}.$$

We have $S(x) \to 0$ as $x \to 0$ and $S(x)$ is bounded as $x \to \infty$, whence $\tilde{S}(z)$ is analytic in a strip $-\delta < \sigma < 0$, where $\delta > 0$. In fact, we note that $s_n(x) = O(x)$ as $x \to 0$ and is bounded as $x \to \infty$, whence $\tilde{s}_n(z)$ is analytic for $-1 < \sigma < 0$; thus, we expect that $\delta = 1$. However, we also note that formal differentiation of $S(x)$ results in a divergent series, suggesting that $S(x)$ does not behave like x as $x \to 0$.

We shall treat $S(x)$ using our separable approximations for $\nu_n(x)$. Since the latter may not be appropriate for small values of n, we split the sum and write

$$S(x) = \pi \sum_{n=1}^{M} s_n(x) + \pi \sum_{n=M+1}^{\infty} s_n(x) = S_M(x) + S_M^{\infty}(x), \tag{30}$$

say, where M is fixed. For S_M, we can use (24) to give

$$S_M(x) \sim \pi \sum_{n=1}^{M} \mu_n^{-1}[(1 + x + x^2)^{-1} - 1] = -\pi x \sum_{n=1}^{M} \mu_n^{-1} + O(x^3) \tag{31}$$

as $x \to 0$. For S_M^{∞}, we start with

$$s_n(x) = \frac{1}{\mu_n}\left[\left(1 + \frac{\nu_n}{\mu_n}\right)^{-1} - 1\right] \simeq -\frac{1}{\mu_n^2}\left(\nu_n - \frac{\nu_n^2}{\mu_n}\right),$$

since $|\nu_n/\mu_n|$ is small. Next, we approximate ν_n by $\nu_n^{(2)}$ and ν_n^2 by $(\nu_n^{(1)})^2$, where $\nu_n^{(1)}$ and $\nu_n^{(2)}$ are defined by (27) and (28), respectively. Finally, since $|\nu_n^{(1)}/\mu_n|$ is small, we can approximate $\nu_n^{(2)}$ using the Taylor approximation

$$\tan^{-1}(X + H) \simeq \tan^{-1} X + H(1 + X^2)^{-1} \tag{32}$$

for small H; the result is

$$s_n(x) \simeq -\mu_n^{-2}\{\nu_n^{(1)}(x) + x\nu_n^{(1)}(x)[1 + (\mu_n x)^2]^{-1} - \mu_n^{-1}[\nu_n^{(1)}(x)]^{-2}\} = s_n^{(1)}(x),$$

say. This is our final separable approximation for $s_n(x)$. We find that the error $|s_n - s_n^{(1)}|$ is $O(n^{-4})$ as $n \to \infty$ for fixed x, and is $O(x^3)$ as $x \to 0$ for fixed n.

The Mellin transform of $s_n^{(1)}(x)$ is given by

$$\tilde{s}_n^{(1)}(z) = -\mu_n^{-z-2}\tilde{u}_1(z) + \mu_n^{-z-3}\tilde{u}_2(z), \tag{33}$$

where

$$\begin{aligned} \tilde{u}_1(z) &= \int_0^{\infty} x^{z-1} \tan^{-1} x \, dx, \\ \tilde{u}_2(z) &= \int_0^{\infty} x^{z-1}[\tan^{-1} x - x(1 + x^2)^{-1}] \tan^{-1} x \, dx. \end{aligned} \tag{34}$$

Here $\tilde{u}_1(z)$ is analytic for $-1 < \sigma < 0$; within this range, we can integrate by parts, giving

$$\tilde{u}_1(z) = -\frac{1}{2z}\int_0^{\infty} y^{\frac{1}{2}(z-1)} \frac{dy}{1 + y} = \frac{\pi}{2z \sin[\frac{1}{2}\pi(z - 1)]}, \tag{35}$$

using a standard integral. Also, since the integrand in (34) is $O(x^{z+3})$ as $x \to 0$, we see that $\tilde{u}_2(z)$ is analytic for $-4 < \sigma < 0$.

Summing over n, using (30) and (33), gives

$$\tilde{S}_M^\infty(z) \simeq -\psi_M(z+2)\tilde{u}_1(z) + \psi_M(z+3)\tilde{u}_2(z), \tag{36}$$

where, by definition,

$$\psi_M(z) = \pi \sum_{n=M+1}^{\infty} \mu_n^{-z} = \pi^{1-z}(2^z - 1)\zeta(z) - \pi \sum_{n=1}^{M} \mu_n^{-z}, \tag{37}$$

and we have used (19). Here $\psi_M(z)$ is analytic for all z, apart from a simple pole at $z = 1$;

$$\psi_M(z) \simeq (z-1)^{-1} + \gamma + \log(4/\pi) - \pi \sum_{n=1}^{M} \mu_n^{-1} \quad \text{near } z = 1. \tag{38}$$

Note that $\tilde{S}_M^\infty(\sigma + i\tau)$ decays exponentially as $|\tau| \to \infty$, whence Theorem 1 will yield an asymptotic expansion for $S_M^\infty(x)$. In order to invert $\tilde{S}_M^\infty(z)$, we start with the inversion contour to the left of $z = 0$, and then move it further to the left; thus, we are interested in singularities in $\sigma < 0$. Consider the first term on the right-hand side of (36). From (35), we see that $\tilde{u}_1(z)$ has simple poles at $z = -1, -3, \ldots$ (and other poles in $\sigma \geqslant 0$); near $z = -1$, we have

$$\tilde{u}_1(z) \simeq (z+1)^{-1} + 1. \tag{39}$$

Hence, $\psi_M(z+2)\tilde{u}_1(z)$ has a double pole at $z = -1$; (38) and (39) give

$$-\psi_M(z+2)\tilde{u}_1(z) \simeq -(z+1)^{-2} - (z+1)^{-1}\left(1 + \gamma + \log(4/\pi) - \pi \sum_{n=1}^{M} \mu_n^{-1}\right)$$

near $z = -1$, giving terms proportional to $x \log x$ and x in $S_M^\infty(x)$. The next singularity at $z = -3$ gives a term in x^3, but we have already made errors of this order when we replaced $s_n(x)$ by $s_n^{(1)}(x)$. The second term on the right-hand side of (36) is analytic for $-4 < \sigma < 0$, apart from a simple pole at $z = -2$, and so this gives a term proportional to x^2. Combining these results, using Theorem 1, gives

$$S_M^\infty(x) = x \log x - x\left(1 + \gamma + \log(4/\pi) - \pi \sum_{n=1}^{M} \mu_n^{-1}\right) + O(x^2)$$

as $x \to 0$. Finally, using (30) and (31), we obtain

$$S(x) = x \log x - x[1 + \gamma + \log(4/\pi)] + O(x^2) \tag{40}$$

as $x \to 0$, and then $f(x)$ is given by (29). Note that, as expected, this result does not depend on M (see (30)).

5. Nonseparable series: A problem of Linton & Evans [9]

In this section, we consider a water-wave problem described in Section 1 and solved by Linton & Evans [9]. The geometry is shown in Fig. 1. They calculate a certain complex reflection coefficient; its argument is proportional to the right-hand side of their equation (3.34), which we write as follows:

$$E(h) = \tan^{-1}[\alpha^{-1}(l^2 - k_0^2)^{\frac{1}{2}}] - \tan^{-1}(l/\alpha) - \tfrac{1}{2}\pi - (\alpha/\pi)L_0 + T, \tag{41}$$

where $L_0 = c \log(h/c) + d \log(h/d)$ and

$$T = \sum_{n=1}^{\infty} \left[\tan^{-1}\left(\frac{\alpha}{(l^2 + n^2\pi^2/c^2)^{\frac{1}{2}}}\right) - \tan^{-1}\left(\frac{\alpha}{(l^2 + k_n^2)^{\frac{1}{2}}}\right) + \tan^{-1}\left(\frac{\alpha}{(l^2 + \kappa_n^2)^{\frac{1}{2}}}\right)\right].$$

The parameters d, l, and K are fixed. The quantity k_0 is defined by (5b) and $\alpha = (k^2 - l^2)^{\frac{1}{2}}$, where k is defined by (5a). We have $c = h - d > 0$ and $K < k_0 < l < k$. The quantities k_n solve (20), whereas κ_n are the positive real roots of

$$K + \kappa_n \tan \kappa_n d = 0 \quad (n = 1, 2, \ldots)$$

satisfying $(n - \frac{1}{2})\pi < \kappa_n d < n\pi$. Note that we use k_0, k_n, and κ_n where Linton & Evans [9] use κ, κ_{n+1}, and k_n, respectively; also, there is an additional term of $-\frac{1}{2}\pi$ in (41) which was omitted by Linton & Evans [9] (Linton, private communication).

EXAMPLE 4. Find

$$\lim_{h \to \infty} E(h) = E_\infty, \tag{42}$$

say, where $E(h)$ is defined by (41). This corresponds, physically, to solving the same water-wave problem as Linton & Evans [9] but for the geometry shown in Fig. 2, in which the water is infinitely deep.

Note that, as h varies, so too do k_0, k_n, and c; all other parameters remain unchanged. To begin with, (5b) shows that

$$k_0 h \sim Kh(1 + 2e^{-2Kh}) \quad \text{as } Kh \to \infty,$$

so we can replace k_0 by K in the first term of $E(h)$ as $h \to \infty$. It is elementary to show that $L_0 = d(\log h + 1 - \log d) + o(1)$ as $h \to \infty$. For T, we note that the arguments of the three inverse tangents behave like

$$\frac{\alpha c}{n\pi}, \quad \frac{\alpha h}{n\pi}, \quad \frac{\alpha d}{n\pi},$$

respectively, as $n \to \infty$, and so we can write $T = T_1 - T_2 + T_3$, where

$$T_1 = \sum_{n=1}^{\infty} \left[\tan^{-1}\left(\frac{\alpha}{(l^2 + n^2\pi^2/c^2)^{\frac{1}{2}}}\right) - \frac{\alpha c}{n\pi}\right], \tag{43}$$

$$T_2 = \sum_{n=1}^{\infty} \left[\tan^{-1}\left(\frac{\alpha}{(l^2 + k_n^2)^{\frac{1}{2}}}\right) - \frac{\alpha h}{n\pi}\right], \tag{44}$$

$$T_3 = \sum_{n=1}^{\infty} \left[\tan^{-1}\left(\frac{\alpha}{(l^2 + \kappa_n^2)^{\frac{1}{2}}}\right) - \frac{\alpha d}{n\pi}\right], \tag{45}$$

and we have used $c - h + d = 0$. We note that T_1 is a separable series, T_2 is a nonseparable series, and T_3 is independent of h. So, at this stage, we have

$$E(h) = -\tan^{-1}[\alpha(l^2 - K^2)^{-\frac{1}{2}}] - \tan^{-1}(l/\alpha) - (\alpha d/\pi)(\log h + 1 - \log d) + T_1 - T_2 + T_3 + o(1) \tag{46}$$

as $h \to \infty$, since $\tan^{-1} X + \tan^{-1}(1/X) = \frac{1}{2}\pi$. We examine T_1 and T_2 in turn.

5.1 *Deep-water behaviour of* T_1

From (43), we have $T_1 = f(\pi/c)$, where $f(x)$ is defined by (9) with $c_n = 1$, $\mu_n = n$, and

$$u(x) = \tan^{-1}\left(\frac{\alpha}{(l^2+x^2)^{\frac{1}{2}}}\right) - \frac{\alpha}{x}. \tag{47}$$

Proceeding as in Section 3, we obtain $\tilde{f}(z) = \zeta(z)\tilde{u}(z)$, where $\tilde{u}(z)$ is analytic for $1 < \sigma < 3$. We must find the singularities of $\tilde{u}(z)$ in $0 \leqslant \sigma \leqslant 1$; singularities in $\sigma < 0$ will lead to terms that are $o(1)$ as $h \to \infty$.

For $1 < \sigma < 3$, we integrate by parts to remove the inverse tangent, giving

$$\tilde{f}(z) = (\alpha/z)\zeta(z)\tilde{u}_1(z), \tag{48}$$

where

$$\tilde{u}_1(z) = \int_0^\infty x^z\left(\frac{x}{(x^2+l^2)^{\frac{1}{2}}(x^2+k^2)} - \frac{1}{x^2}\right)\mathrm{d}x \tag{49}$$

and we have used the relation $k^2 = \alpha^2 + l^2$. The function $\tilde{u}_1(z)$ is also analytic for $1 < \sigma < 3$. To find the singularities of $\tilde{u}_1(z)$ in $\sigma \leqslant 1$, write

$$\tilde{u}_1(z) = \tilde{u}_2(z) + \tilde{u}_3(z) + \tilde{u}_4(z), \tag{50}$$

where

$$\tilde{u}_2(z) = \int_0^\beta \frac{x^{z+1}}{(x^2+l^2)^{\frac{1}{2}}(x^2+k^2)}\mathrm{d}x, \qquad \tilde{u}_3(z) = -\int_0^\beta x^{z-1}\,\mathrm{d}x,$$

$$\tilde{u}_4(z) = \int_\beta^\infty x^z\left(\frac{x}{(x^2+l^2)^{\frac{1}{2}}(x^2+k^2)} - \frac{1}{x^2}\right)\mathrm{d}x,$$

and β is an arbitrary positive number. Here $\tilde{u}_2(z)$ is analytic for $\sigma > -2$. The function $\tilde{u}_3(z)$ is analytic for $\sigma > 1$, and can be continued analytically into the whole plane, apart from a simple pole at $z = 1$; near $z = 1$, we have

$$\tilde{u}_3(z) \simeq -(z-1)^{-1} - \log\beta.$$

The function $\tilde{u}_4(z)$ is analytic for $\sigma < 3$. Hence, $\tilde{u}_1(z)$ is analytic for $-2 < \sigma < 3$, apart from a simple pole at $z = 1$; near $z = 1$, we have

$$\tilde{u}_1(z) \simeq -(z-1)^{-1} + Q,$$

where

$$Q = -\log\beta + \tilde{u}_2(1) + \tilde{u}_4(1). \tag{51}$$

We can evaluate Q explicitly (see Appendix A):

$$Q = \log(2/l) + (k/\alpha)\log[(k-\alpha)/l]. \tag{52}$$

Returning to (48), we see that $\tilde{f}(z)$ has a double pole at $z = 1$, a simple pole at $z = 0$, and is otherwise analytic in $-2 < \sigma < 3$; near $z = 1$,

$$\tilde{f}(z) \simeq \alpha(1+w)^{-1}(w^{-1}+\gamma)(-w^{-1}+Q) \simeq \alpha[-w^{-2}+w^{-1}(Q-\gamma+1)],$$

where $w = z - 1$, whereas near $z = 0$,

$$\tilde{f}(z) \simeq (\alpha/z)\zeta(0)\tilde{u}_1(0) = -\tfrac{1}{2}z^{-1}\tan^{-1}(\alpha/l),$$

after evaluating $\tilde{u}_1(0)$ (see Appendix B). Moreover, it is clear that the analytic continuation of $\tilde{u}_1(z)$ is bounded as $|\tau| \to \infty$ for $-2 < \sigma < 3$. Hence, (48) and (15) imply that we can move the inversion contour to the left of $z = 0$, so that Theorem 1 yields the asymptotic approximation

$$f(x) = (\alpha/x)\log x + (\alpha/x)(Q-\gamma+1) - \tfrac{1}{2}\tan^{-1}(\alpha/l) + o(1)$$

as $x \to 0$. Replacing x by π/c, with $c = h - d$, and expanding for large h gives

$$T_1 = (\alpha/\pi)\{-h\log h + h(\log\pi + Q - \gamma + 1) + d[\log(h/\pi) - Q - \gamma]\} - \tfrac{1}{2}\tan^{-1}(\alpha/l) + o(1) \quad (53)$$

as $h \to \infty$, where Q is given by (52).

5.2 *Deep-water behaviour of T_2*

The series T_2, defined by (44), is nonseparable. As in Example 3, we expect the leading behaviour to be given by (44) with $k_n h$ replaced by $\mu_n = (n - \tfrac{1}{2})\pi$. So, we consider

$$T_\infty(1/h) = \sum_{n=1}^{\infty}\left[\tan^{-1}\left(\frac{\alpha}{(l^2+\mu_n^2/h^2)^{\frac{1}{2}}}\right) - \frac{\alpha h}{n\pi}\right]. \quad (54)$$

We have

$$T_\infty(x) = \sum_{n=1}^{\infty}\left[\tan^{-1}\left(\frac{\alpha}{(l^2+\mu_n^2x^2)^{\frac{1}{2}}}\right) - \frac{\alpha}{\mu_n x}\right] + \frac{\alpha}{x}\sum_{n=1}^{\infty}\left(\frac{1}{\mu_n} - \frac{1}{n\pi}\right);$$

the second sum is $(2/\pi)\log 2$. Hence,

$$T_\infty(x) = [2\alpha/(\pi x)]\log 2 + f(x),$$

where $f(x)$ is the separable series (9), with $c_n = 1$, $\mu_n = (n - \tfrac{1}{2})\pi$, and $u(x)$ again given by (47). We obtain

$$\tilde{f}(z) = \pi^{-z}(2^z - 1)\zeta(z)\tilde{u}(z) = (\alpha/z)\pi^{-z}(2^z-1)\zeta(z)\tilde{u}_1(z),$$

where $\tilde{u}_1(z)$ is defined by (49) and we have used (19). Note that, unlike the function defined by (48), here, $\tilde{f}(z)$ does not have a pole at $z = 0$ (because $2^z - 1$

has a simple zero at $z = 0$). However, it does have a double pole at $z = 1$; near $z = 1$,

$$\begin{aligned}\tilde{f}(z) &\simeq \alpha(1+w)^{-1}\pi^{-1}(1 - w\log\pi)(1 + 2w\log 2)(w^{-1} + \gamma)(-w^{-1} + Q)\\ &\simeq (\alpha/\pi)[-w^{-2} + w^{-1}(Q - \gamma + 1 + \log\pi - 2\log 2)],\end{aligned}$$

where $w = z - 1$. Hence

$$T_\infty(x) = (\alpha/\pi)[x^{-1}\log x + x^{-1}(Q - \gamma + 1 + \log\pi)] + o(1)$$

as $x \to 0$, and so, as $h \to \infty$, we obtain

$$T_\infty(1/h) = -(\alpha/\pi)h\log h + (\alpha/\pi)h(Q - \gamma + 1 + \log\pi) + o(1). \tag{55}$$

We now examine the difference between T_2 and $T_\infty(1/h)$. Using (44) and (54), we define

$$T_4 = T_2 - T_\infty(1/h) = \sum_{n=1}^{\infty} t_n, \tag{56}$$

where

$$t_n = \tan^{-1}\left(\frac{\alpha}{(l^2 + k_n^2)^{\frac{1}{2}}}\right) - \tan^{-1}\left(\frac{\alpha}{(l^2 + \mu_n^2/h^2)^{\frac{1}{2}}}\right).$$

Clearly, $t_n = o(1)$ as $h \to \infty$, for fixed n, so we have

$$T_4 = \sum_{n=M+1}^{\infty} t_n + o(1) \quad \text{as } h \to \infty,$$

where M is fixed (cf. (30)). Writing $k_n h = \mu_n + \nu_n$, as in (25), we have

$$l^2 + k_n^2 \simeq \Delta_n^2 + 2\nu_n\mu_n/h^2$$

as $|\nu_n/\mu_n|$ is small, where $\Delta_n^2 = l^2 + \mu_n^2/h^2$. Hence

$$\frac{\alpha}{(l^2 + k_n^2)^{\frac{1}{2}}} \simeq \frac{\alpha}{\Delta_n}\left(1 - \frac{\nu_n\mu_n}{\Delta_n^2 h^2}\right).$$

Then, using the Taylor approximation (32), we find that

$$t_n \simeq \frac{-\alpha\nu_n\mu_n}{h^2\Delta_n(\Delta_n^2 + \alpha^2)}.$$

Finally, we use the approximation (27), $\nu_n \simeq \nu_n^{(1)} = \tan^{-1}(\mu_n/Kh)$, giving

$$t_n \simeq -(\alpha/h)t_n^{(1)}(1/h),$$

where

$$t_n^{(1)}(y) = \frac{\mu_n y\tan^{-1}(\mu_n y/K)}{(l^2 + \mu_n^2 y^2)^{\frac{1}{2}}(k^2 + \mu_n^2 y^2)} \tag{57}$$

and we have used $k^2 = \alpha^2 + l^2$. So, we have approximated T_4 by a separable series:

$$T_4 = -(\alpha/h)T_M^\infty(1/h) + o(1) \tag{58}$$

as $h \to \infty$, where

$$T_M^\infty(x) = \sum_{n=M+1}^{\infty} t_n^{(1)}(x)$$

and $t_n^{(1)}(x)$ is defined by (57). We now take the Mellin transform of $T_M^\infty(x)$. Since $t_n^{(1)}(x) \sim \frac{1}{2}\pi(\mu_n x)^{-2}$ as $x \to \infty$, we see that $\tilde{T}_M^\infty(z)$ is analytic in a strip $\beta < \sigma < 2$, for some β, so we can take the inversion contour just to the left of $\sigma = 2$. We have

$$\tilde{T}_M^\infty(z) = \pi^{-1}\psi_M(z)\tilde{u}(z).$$

where $\psi_M(z)$ is defined by (37) and

$$\tilde{u}(z) = \int_0^\infty \frac{y^z \tan^{-1}(y/K)}{(y^2 + l^2)^{\frac{1}{2}}(y^2 + k^2)} \mathrm{d}y$$

is analytic for $-2 < \sigma < 2$. Hence, $\tilde{T}_M^\infty(z)$ is analytic for $-2 < \sigma < 2$, apart from a simple pole at $z = 1$; using (38), we have

$$\tilde{T}_M^\infty(z) \simeq L\pi^{-1}(z-1)^{-1}$$

near $z = 1$, where

$$L = \tilde{u}(1) = \int_0^\infty \frac{y \tan^{-1}(y/K)}{(y^2 + l^2)^{\frac{1}{2}}(y^2 + k^2)} \mathrm{d}y. \tag{59}$$

(The evaluation of L is discussed below.) Hence, as the conditions of Theorem 1 are easily seen to be satisfied, we obtain

$$T_M^\infty(x) = L/\pi x + o(1)$$

as $x \to 0$, whence (58) gives $T_4 = -(\alpha/\pi)L + o(1)$ as $h \to \infty$. Finally, we combine this result with (55) and (56) to give

$$T_2 = (\alpha/\pi)[-h \log h + h(Q - \gamma + 1 + \log \pi) - L] + o(1) \quad \text{as } h \to \infty. \tag{60}$$

5.3 *Evaluation of L*

The quantity L, defined by the integral (59), cannot be evaluated in terms of elementary functions. However, it can be expressed in terms of dilogarithms [8]. We find that (see Appendix C)

$$\alpha L = \tfrac{1}{4}\pi^2 - \tfrac{1}{2}A \log(k + K) + \tfrac{1}{2}A \log(k - K) - \delta \tan^{-1}(\psi/\alpha) - \mathscr{L}, \tag{61}$$

where $\psi = (l^2 - K^2)^{\frac{1}{2}}$ and

$$\mathscr{L} = \mathrm{Li}_2(\mathrm{e}^{-A}, \delta) - \mathrm{Li}_2(\mathrm{e}^{-A}, \pi + \delta), \tag{62}$$

$$A = \sinh^{-1}(\alpha/l) = -\log[(k-\alpha)/l], \qquad \delta = \tan^{-1}(\psi/K). \tag{63a,b}$$

Here, the dilogarithm is defined by

$$\mathrm{Li}_2(z) = -\int_0^z \log(1-w)\frac{\mathrm{d}w}{w} \tag{64}$$

for complex z, and

$$\mathrm{Li}_2(r, \theta) = \mathrm{Re}[\mathrm{Li}_2(r\mathrm{e}^{\mathrm{i}\theta})] = -\tfrac{1}{2}\int_0^r \log(1 - 2x\cos\theta + x^2)\frac{\mathrm{d}x}{x}.$$

5.4 *Synthesis*

From (53) and (60), we have

$$T_1 - T_2 = (\alpha/\pi)[L + d(\log h - \log\pi - Q + \gamma)] - \tfrac{1}{2}\tan^{-1}(\alpha/l) + o(1) \tag{65}$$

as $h \to \infty$, so that the terms involving h and $h\log h$ in (53) and (60) cancel. Moreover, when (65) is substituted into (46), we see that the terms in $\log h$ cancel, leaving only bounded terms as $h \to \infty$. Specifically, from (42), we obtain

$$E_\infty = (\alpha d/\pi)[\log(d/\pi) + \gamma - 1 - Q] \\ + \alpha L/\pi + T_3 - \tfrac{1}{2}\pi - \tan^{-1}(\alpha/\psi) + \tfrac{1}{2}\tan^{-1}(\alpha/l).$$

Now, substituting for Q and L from (52) and (61), respectively, we obtain our final expression for E_∞, namely

$$E_\infty = \frac{\alpha d}{\pi}\left(\log\frac{ld}{2\pi} + \gamma - 1\right) - \frac{kd}{\pi}\log\frac{k-\alpha}{l} - \tan^{-1}\frac{\alpha}{\psi} - \tfrac{1}{2}\tan^{-1}\frac{l}{\alpha} \\ + T_3 - \frac{1}{\pi}\mathscr{L} + \frac{1}{2\pi}\log\frac{k-\alpha}{l}\log\frac{k+K}{k-K} - \frac{1}{\pi}\tan^{-1}\frac{\psi}{K}\tan^{-1}\frac{\psi}{\alpha}, \tag{66}$$

where $\psi = (l^2 - K^2)^{\frac{1}{2}}$, T_3 is the series (45), and $\mathscr{L}$ is the combination of dilogarithms given by (62).

The above expression for E_∞ bears little resemblance to $E(h)$; indeed, it is perhaps surprising to see terms involving products of logarithms and products of inverse tangents. Nevertheless, the result can be checked by solving the deep-water problem (using the geometry in Fig. 2) directly. This has been done by Parsons [15], using the Wiener–Hopf technique, as used for a similar problem by Greene & Heins [7]; the two approaches yield the same result. In fact, the present work began with the intention of confirming (66), but it seems that the methods devised may have wider applicability.

Appendix A

Evaluation of Q

From the definition (51), we have

$$Q = -\log\beta + \int_0^\beta \frac{y^2}{(y^2+l^2)^{\frac{1}{2}}(y^2+k^2)}\,dy + \int_\beta^\infty \left(\frac{y^2}{(y^2+l^2)^{\frac{1}{2}}(y^2+k^2)} - \frac{1}{y}\right) dy$$
$$= -\log\beta + \int_0^\beta \frac{dy}{(y^2+l^2)^{\frac{1}{2}}} + \int_\beta^\infty \left(\frac{1}{(y^2+l^2)^{\frac{1}{2}}} - \frac{1}{y}\right) dy - k^2 Q_1$$
$$= \log(2/l) - k^2 Q_1,$$

where

$$Q_1 = \int_0^\infty \frac{dy}{(y^2+l^2)^{\frac{1}{2}}(y^2+k^2)} = \frac{1}{l^2}\int_0^\infty \frac{d\theta}{\cosh\theta + a},$$

using the substitution $y = l\sinh\frac{1}{2}\theta$, with

$$a = 2k^2/l^2 - 1; \tag{A.1}$$

$a > 1$ since $k^2 > l^2$. Hence, using [6: eqn 3.513(2)],

$$Q_1 = -(\alpha k)^{-1}\log[(k-\alpha)/l],$$

whence the result (52) follows.

Appendix B

Evaluation of $\tilde{u}_1(0)$

From (50), and the analytic continuation of $\tilde{u}_3(z)$,

$$\tilde{u}_1(0) = \beta^{-1} + \tilde{u}_2(0) + \tilde{u}_4(0) = \int_0^\infty \frac{y}{(y^2+l^2)^{\frac{1}{2}}(y^2+k^2)}\,dy$$
$$= \int_l^\infty \frac{dx}{x^2+\alpha^2} = \frac{1}{\alpha}\tan^{-1}\frac{\alpha}{l},$$

where we have made the substitution $y = (x^2 - l^2)^{\frac{1}{2}}$.

Appendix C

Evaluation of L

The quantity L is defined by (59). Noting that the integrand is even, and putting $y = l\sinh\theta$, we obtain

$$L = \tfrac{1}{2}\int_{-\infty}^\infty g(\theta)\tan^{-1}\left(\frac{l}{K}\sinh\theta\right) d\theta,$$

where $g(\theta) = l \sinh \theta \, (l^2 \sinh^2 \theta + k^2)^{-1}$. Now, consider the identity

$$\tan^{-1}(pt) + \tan^{-1}(p/t) = \tan^{-1}\left(\frac{p(t+1/t)}{1-p^2}\right).$$

Put $t = \mathrm{i}e^{\theta}$ and $p = -\mathrm{i}q$, where $q = (K/l) + \mathrm{i}(1 - K^2/l^2)^{\frac{1}{2}} = e^{\mathrm{i}\delta}$, say (so that $|q| = 1$ and δ is given by (63b)). Hence

$$L = \tfrac{1}{2}\int_{-\infty}^{\infty} g(\theta) \tan^{-1}(qe^{\theta})\, d\theta + \tfrac{1}{2}\int_{-\infty}^{\infty} g(\theta) \tan^{-1}(-qe^{-\theta})\, d\theta.$$

The substitution $\theta = -\varphi$ shows that these two integrals are equal, whence

$$L = \int_{-\infty}^{\infty} g(\theta) \tan^{-1}(qe^{\theta})\, d\theta.$$

(One can check that L is real, even though q is not.) Put $z = e^{\theta}$, whence

$$L = \frac{2}{l}\int_0^{\infty} \frac{(z^2-1)\tan^{-1}(qz)}{z^4 + 2az^2 + 1}\, dz,$$

where a is defined by (A.1). If we define A by $\alpha = l \sinh A$ (so that $k = l \cosh A$), we find that the denominator vanishes at $z^2 = -e^{\pm 2A}$. Then, splitting into partial fractions gives $L = (L_+ - L_-)/\alpha$, where

$$L_{\pm} = \int_0^{\infty} \frac{e^{\pm A} \tan^{-1}(qz)}{z^2 + e^{\pm 2A}}\, dz.$$

We have $L_- = I(qe^{-A})$, where

$$I(X) = \int_0^{\infty} \frac{\tan^{-1}(Xy)}{y^2+1}\, dy = \int_0^{\frac{1}{2}\pi} \tan^{-1}(X \tan\theta)\, d\theta.$$

For L_+, note that $\tan^{-1}(qz) = \frac{1}{2}\pi - \tan^{-1}(\bar{q}/z)$, since $1/q = \bar{q}$, the complex conjugate of q; then, the substitution $z = 1/w$ shows that

$$L_+ = \tfrac{1}{4}\pi^2 - I(\bar{q}e^{-A}),$$

whence

$$\alpha L = \tfrac{1}{4}\pi^2 - I(qe^{-A}) - I(\bar{q}e^{-A}) = \tfrac{1}{4}\pi^2 - 2\,\mathrm{Re}\,[I(qe^{-A})]. \tag{C.1}$$

To evaluate $I(x)$, we follow Lewin [8: p. 224]. Differentiation gives

$$I'(x) = (x^2-1)^{-1} \log x.$$

Then, splitting into partial fractions and integrating, we obtain

$$I(x) = -\tfrac{1}{2}\int_0^{x} \left(\frac{1}{1-y} + \frac{1}{1+y}\right) \log y \, dy,$$

since $I(0) = 0$. An integration by parts then gives

$$I(x) = -\tfrac{1}{2}\{\log x \log[(1+x)/(1-x)] - \mathrm{Li}_2(x) + \mathrm{Li}_2(-x)\},$$

where the dilogarithm is defined by (64). We obtain the final result (61) after using this result in (C.1); note that

$$\log\left(\frac{1+q\mathrm{e}^{-A}}{1-q\mathrm{e}^{-A}}\right) = \tfrac{1}{2}\log\frac{k+K}{k-K} + \mathrm{i}\tan^{-1}\left(\frac{(l^2-K^2)^{\frac{1}{2}}}{\alpha}\right).$$

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