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MAPPING FLAT CRACKS ONTO PENNY-SHAPED CRACKS: SHEAR LOADINGS

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ABSTRACT

Consider a three-dimensional homogeneous isotropic elastic solid containing a flat crack, Ω, subjected to a shear loading. The problem of finding the resulting stress distribution can be reduced to a pair of coupled hypersingular integral equations over Ω for the tangential components of the crack-opening displacement vector. Here, these equations are first written as a single equation for a complex displacement. This equation is then transformed into a similar equation over a circular region D, using a conformal mapping between Ω and D. This new equation is then regularized analytically by using an appropriate expansion method (Fourier series in the azimuthal direction and series of orthogonal polynomials in the radial direction). Analytical results for regions that are approximately circular are also obtained. These include formulae for the crack-opening displacement and the stress-intensity factors in terms of the conformal mapping or the shape of the crack.

1. INTRODUCTION

Consider a flat crack Ω in a three-dimensional elastic solid. If the crack is pressurized, the resulting discontinuity in the normal component of the displacement across the crack, $[u_3]$, can be found by solving a two-dimensional hypersingular integral equation over Ω. This equation is of the form

$$H_\Omega\{[u_3]\} \equiv \frac{1}{4\pi} \oint_\Omega [u_3(x,y)] \frac{\mathrm{d}\Omega}{R^3} = p(x_0, y_0), \quad \text{for} \quad (x_0, y_0) \in \Omega, \tag{1.1}$$

where $R = \{(x-x_0)^2 + (y-y_0)^2\}^{1/2}$ and p is proportional to the prescribed pressure opening the crack; (1.1) is to be solved subject to

$$[u_3(x,y)] = 0 \quad \text{for} \quad (x,y) \in \partial\Omega,$$

where $\partial\Omega$ is the boundary of Ω (the crack edge). The hypersingular integral in (1.1) can be defined in several equivalent ways; one natural definition in the context of boundary-value problems is

$$\oint_\Omega w(x,y) \frac{\mathrm{d}\Omega}{R^3} = \lim_{z_0 \to 0} \frac{\partial}{\partial z_0} \int_\Omega w(x,y) \left\{ \lim_{z \to 0} \frac{\partial}{\partial z} \left(\frac{1}{\sqrt{R^2 + (z-z_0)^2}} \right) \right\} \mathrm{d}\Omega.$$

If Ω is a circular disc (a penny-shaped crack), analytical solutions of (1.1) are available. In particular, expansion methods can be developed: use a Fourier decomposition in the azimuthal direction together with an expansion in terms of Gegenbauer polynomials in the radial direction. This method is effective because it incorporates the known behaviour of $[u_3]$ around the edge of Ω, and it allows the two-dimensional hypersingular integral to be evaluated analytically (see Section 3).

In a previous paper (Martin, 1994), we have developed a method for treating (1.1) when Ω is not circular. In order to use an expansion method, we mapped Ω onto a disc D, using a conformal mapping. This preserves the structure of the hypersingularity, allowing the use of the Fourier–Gegenbauer expansion method on the transformed integral equation.

The problem of a crack subjected to a shear loading is more complicated. It can be reduced to a pair of coupled hypersingular integral equations over Ω for the two tangential components of the discontinuity in the displacement vector across Ω. We start by complexifying these equations, yielding a single equation for a single (complex) unknown. For penny-shaped cracks, this equation has explicit solutions; in particular, expansion methods can be developed as for pressurized cracks (although the resulting formulae are more complicated). For non-circular cracks, we proceed as in Martin (1994) and map the crack onto a circular disc, using a conformal mapping. As well as leading to a viable numerical method, the combination of conformal mapping and two-dimensional hypersingular integral equations can be used to obtain analytical results for cracks Ω that are approximately circular. Specifically, we consider such cracks when they are subject to a constant uniform shear, and obtain relatively simple results in the transform domain for the crack-opening displacement and the stress-intensity factors.

The problem of the shear loading of an almost circular crack has been considered by Gao (1988), extending earlier work of Rice (1985, 1989) and Gao and Rice (1987) on pressurized cracks, and of Gao and Rice (1986) on the shear loading of a semi-infinite crack. They consider the general problem of calculating the change in the stress field around a flat crack when the crack front is perturbed in the plane of the crack. Their approach is based on the use of three-dimensional weight functions. There is also an extensive Russian literature on almost circular, pressurized cracks; see Panasyuk (1970, Chpt. IX) and Panasyuk *et al.* (1981). Martin (1994) has used the present method to confirm some of the results of Gao and Rice (1987) for pressurized cracks. Here, we compare with some results of Gao (1988) for the stress-intensity factors. Our results for the crack-opening displacement are new.

2. SHEAR LOADING OF FLAT CRACKS

Consider a three-dimensional homogeneous isotropic elastic solid containing an arbitrary flat crack Ω. We choose Cartesian coordinates x, y and x_3, with origin O, so that Ω lies in the plane $x_3 = 0$ (the xy-plane). We assume that Ω is a simply-connected region and that O is a point in Ω. Let us suppose that the crack is subjected to a shear loading; thus, we suppose that

$$\tau_{x3}(x, y, 0) = \frac{\mu}{1-\nu} q_x(x, y) \quad \text{and} \quad \tau_{y3}(x, y, 0) = \frac{\mu}{1-\nu} q_y(x, y)$$

for $(x, y) \in \Omega$, where τ_{ij} is the stress tensor, μ is the shear modulus, ν is Poisson's ratio, and $q_x(x, y)$ and $q_y(x, y)$ are prescribed.

The tangential components of the discontinuity in the displacement vector across the crack, $[u_x(x, y)]$ and $[u_y(x, y)]$, solve a pair of integral equations over Ω. One form of this pair is

$$\frac{-1}{4\pi} \oint_\Omega \left\{\alpha \frac{\partial}{\partial x}\left(\frac{1}{R}\right) + \beta \frac{\partial}{\partial y}\left(\frac{1}{R}\right)\right\} \mathrm{d}\Omega = q_x(x_0, y_0), \tag{2.1}$$

$$\frac{-1}{4\pi} \oint_\Omega \left\{\alpha \frac{\partial}{\partial y}\left(\frac{1}{R}\right) - \beta \frac{\partial}{\partial x}\left(\frac{1}{R}\right)\right\} \mathrm{d}\Omega = q_y(x_0, y_0), \tag{2.2}$$

for $(x_0, y_0) \in \Omega$, where $R = \{(x-x_0)^2 + (y-y_0)^2\}^{1/2}$,

$$\alpha = \frac{\partial[u_x]}{\partial x} + \frac{\partial[u_y]}{\partial y} \quad \text{and} \quad \beta = (1-\nu)\left(\frac{\partial[u_x]}{\partial y} - \frac{\partial[u_y]}{\partial x}\right).$$

These equations are to be solved subject to

$$[u_x(x, y)] = 0 \quad \text{and} \quad [u_y(x, y)] = 0 \quad \text{for} \quad (x, y) \in \partial\Omega, \tag{2.3}$$

where $\partial\Omega$ is the boundary of Ω (the crack edge); $\partial\Omega$ is a simple closed curve. The pair (2.1) and (2.2) was derived by Bui (1975) and by Guidera and Lardner (1975). It involves Cauchy principal-value integrals over Ω, and is known as the *traction BIE*. For its numerical treatment, see Polch *et al.* (1987) and Weaver (1977).

Integrating by parts, using (2.3), we obtain an alternative pair of equations:

$$\frac{1}{8\pi} \oint_\Omega \{(2-\nu+3\nu\cos 2\Theta)[u_x] + 3\nu[u_y]\sin 2\Theta\} \frac{\mathrm{d}\Omega}{R^3} = q_x(x_0, y_0), \tag{2.4}$$

$$\frac{1}{8\pi} \oint_\Omega \{3\nu[u_x]\sin 2\Theta + (2-\nu-3\nu\cos 2\Theta)[u_y]\} \frac{\mathrm{d}\Omega}{R^3} = q_y(x_0, y_0), \tag{2.5}$$

for $(x_0, y_0) \in \Omega$, where the angle Θ is defined by

$$x - x_0 = R\cos\Theta \quad \text{and} \quad y - y_0 = R\sin\Theta.$$

The pair (2.4) and (2.5) is a coupled system of hypersingular integral equations for $[u_x]$ and $[u_y]$; it is to be solved subject to (2.3).

It turns out to be convenient to complexify (2.4) and (2.5), so that we replace two real equations by one complex equation. As we will be using conformal mapping later, we need two non-interacting complex units, i and j, with $\mathrm{i}^2 = -1$ and $\mathrm{j}^2 = -1$; we will use i in the conformal mapping and j in the complexification of (2.4) and (2.5), which become

$$M_\Omega w \equiv \frac{1}{8\pi} ⨎_\Omega \{(2-\nu)w + 3\nu\bar{w}\,\mathrm{e}^{2\mathrm{j}\Theta}\}\frac{\mathrm{d}\Omega}{R^3} = q(x_0, y_0), \quad \text{for} \quad (x_0, y_0) \in \Omega, \tag{2.6}$$

where

$$w(x, y) = [u_x] + \mathrm{j}[u_y], \quad q(x_0, y_0) = q_x + \mathrm{j}q_y \tag{2.7}$$

and the overbar denotes complex conjugation with respect to j : $\bar{w} = [u_x] - \mathrm{j}[u_y]$. Note that (2.6) reduces to (1.1) when $\nu = 0$; this can provide a useful check on subsequent calculations. In the sequel, we will concentrate on (2.6).

3. PENNY-SHAPED CRACKS

Suppose that Ω is a penny-shaped crack of radius a; thus

$$\Omega = D_a \equiv \{(r, \theta) : 0 \leqslant r < a, -\pi \leqslant \theta < \pi\},$$

where r and θ are polar coordinates in the plane of the crack, $x = r\cos\theta$, $y = r\sin\theta$. We can express the loading as

$$q(x, y) = \sum_{n=-\infty}^{\infty} q_n(r/a)\,\mathrm{e}^{\mathrm{j}n\theta}, \tag{3.1}$$

where the Fourier components q_n are j-complex. Then, the j-complex crack-opening displacement has a similar expansion,

$$w(x, y) = a \sum_{n=-\infty}^{\infty} w_n(r/a)\,\mathrm{e}^{\mathrm{j}n\theta}.$$

It is known that (the dimensionless functions) w_n can be expressed as certain explicit integrals of q_m; see Guidera and Lardner (1975) and Martin (1982). These formulae simplify if we expand q_n as

$$q_n(r) = r^{|n|} \sum_{k=0}^{\infty} Q_k^n \frac{\Gamma(|n|+1/2)\Gamma(k+3/2)}{(|n|+k)!} \frac{C_{2k+1}^{|n|+1/2}(\sqrt{1-r^2})}{\sqrt{1-r^2}}, \tag{3.2}$$

where the j-complex coefficients Q_k^n are known and $C_m^\lambda(x)$ is a Gegenbauer polynomial of degree m and index λ (Erdélyi *et al.*, 1953, §10.9); these polynomials are orthogonal. Then, it can be shown that

$$w_n(r) = r^{|n|} \sum_{k=0}^{\infty} W_k^n \frac{\Gamma(|n|+1/2)k!}{\Gamma(|n|+k+3/2)} C_{2k+1}^{|n|+1/2}(\sqrt{1-r^2}). \tag{3.3}$$

The coefficients are related by the following formulae:

$$\left.\begin{aligned}
-2Q_0^{-n} &= (2-\nu)W_0^{-n}, && n \geqslant 0,\\
-2Q_k^1 &= (2-\nu)W_k^1+\nu\bar{W}_k^1, && k \geqslant 0,\\
-2Q_k^n &= (2-\nu)W_k^n-\nu\bar{W}_{k+1}^{2-n}, && n \geqslant 2, k \geqslant 0,\\
-2Q_k^{-n} &= (2-\nu)W_k^{-n}-\nu\bar{W}_{k-1}^{n+2}, && n \geqslant 0, k \geqslant 1.
\end{aligned}\right\} \tag{3.4}$$

These formulae are due, essentially, to Krenk (1979); see Appendix A. They answer the following question: what loading produces a given crack-opening displacement? Equivalently: given w, what is $M_\Omega w$ when $\Omega = D_a$?

As an example (which will be needed in Section 5), suppose that

$$[u_x] = (r/a)^m e^{im\theta}\sqrt{a^2-r^2} \quad \text{and} \quad [u_y] = 0 \tag{3.5}$$

with $m \geqslant 0$, so that $w = [u_x]$. Then, as $C_1^\lambda(x) = 2\lambda x$, we obtain

$$W_k^0 = \tfrac{1}{2}\delta_{0k}\delta_{0m} \quad \text{and} \quad W_k^{\pm n} = \tfrac{1}{4}(1 \mp \mathrm{ij})\delta_{0k}\delta_{nm}$$

for $n \geqslant 1$ and $k \geqslant 0$. As $C_3^\lambda(x) = \frac{2}{3}\lambda(\lambda+1)x\{2(\lambda+2)x^2-3\}$, we find that the crack-opening displacement (3.5) is produced by the loading $q = q^{(m)}$, say, where

$$q^{(m)} = -\tfrac{1}{2}(2-\nu)\frac{\Gamma(m+3/2)\Gamma(3/2)}{m!}(r/a)^m e^{im\theta}+\tfrac{1}{4}\nu(1+\mathrm{ij})\,e^{i(m-2)\theta}\hat{q}_m(r/a), \tag{3.6}$$

$\hat{q}_0 = 0, \hat{q}_1(r) = -\frac{3}{8}\pi r$ and

$$\hat{q}_m(r) = \Gamma(3/2)\left\{\frac{\Gamma(m+1/2)}{(m-2)!}-\frac{\Gamma(m+3/2)}{(m-1)!}r^2\right\}r^{m-2} \quad \text{for} \quad m \geqslant 2.$$

The equations (3.4) can be solved explicitly:

$$\left.\begin{aligned}
(2-\nu)W_0^{-n} &= -2Q_0^{-n}, && n \geqslant 0,\\
2(1-\nu)W_k^1 &= -(2-\nu)Q_k^1+\nu\bar{Q}_k^1, && k \geqslant 0,\\
2(1-\nu)W_k^n &= -(2-\nu)Q_k^n-\nu\bar{Q}_{k+1}^{2-n}, && n \geqslant 2, k \geqslant 0,\\
2(1-\nu)W_k^{-n} &= -(2-\nu)Q_k^{-n}-\nu\bar{Q}_{k-1}^{n+2}, && n \geqslant 0, k \geqslant 1.
\end{aligned}\right\} \tag{3.7}$$

These formulae answer the following question: what crack-opening displacement is induced by a given loading? Equivalently: given q, what is $M_\Omega^{-1}q$ when $\Omega = D_a$? They can also be expressed (more cumbersomely) in real form; see Appendix A.

As a simple example, suppose that the crack is loaded by a constant shear, $q = -\sigma$, say. Then, the only non-zero load coefficient is $Q_0^0 = -2\sigma/\pi$. Hence

$$w(x,y) = \frac{8\sigma}{\pi(2-\nu)}\sqrt{a^2-r^2}, \tag{3.8}$$

in agreement with Segedin (1951).

4. CONFORMAL MAPPING

Let us now consider mapping Ω onto a disc. Martin (1994) showed that it is appropriate and advantageous to choose a *conformal* mapping. Any other mapping will alter the singularity in the kernel in an essential way, so that the transformed integral equation is not "similar" to that for a penny-shaped crack. [For a general analysis of the effect of non-conformal mappings, see Èskin (1981, §19); such a mapping has been used recently by Penzel (1994).]

Let us assume for simplicity that Ω is star-shaped with respect to O, so that Ω is given by

$$\Omega = \{(r, \theta) : 0 \leqslant r < \rho(\theta), \quad -\pi \leqslant \theta < \pi\}; \tag{4.1}$$

thus, $\partial\Omega$ is given by $r = \rho(\theta)$. Next, let $z = x + \mathrm{i}y$ and $z_0 = x_0 + \mathrm{i}y_0$ be complex variables. Then, by the Riemann mapping theorem (Ahlfors, 1966, p. 222) there exists a conformal mapping of the unit disc, $|\zeta| < 1$, in the complex ζ-plane onto the region Ω in the z-plane. More precisely, let $\zeta = s\,\mathrm{e}^{\mathrm{i}\varphi}$, so that the unit disc is given by

$$D \equiv \{(s, \varphi) : 0 \leqslant s < 1, \quad -\pi \leqslant \varphi < \pi\}.$$

Then, given a length-scale a for Ω, we have

$$z = af(\zeta) \quad \text{for} \quad |\zeta| < 1, \tag{4.2}$$

as the conformal mapping of D onto Ω; this form is most convenient for our application. The analytic function f is known to exist for any simply-connected domain Ω; see Kober (1957) and Nehari (1952) for many examples. We assume that $|f'(\zeta)| \neq 0$ for all ζ with $|\zeta| < 1$.

First, we investigate the effect of the mapping (4.2) on the kernels of (2.6). We have

$$z - z_0 = a(f(\zeta) - f(\zeta_0)) = R\,\mathrm{e}^{\mathrm{i}\Theta} \tag{4.3}$$

and define S and Φ by

$$\zeta - \zeta_0 = S\,\mathrm{e}^{\mathrm{i}\Phi}.$$

Let us also define δ and δ_0 by

$$f'(\zeta) = |f'(\zeta)|\,\mathrm{e}^{\mathrm{i}\delta} \quad \text{and} \quad f'(\zeta_0) = |f'(\zeta_0)|\,\mathrm{e}^{\mathrm{i}\delta_0}, \tag{4.4}$$

respectively. Then, we have $R\,\mathrm{e}^{\mathrm{i}\Theta} \simeq aS|f'(\zeta_0)|\,\mathrm{e}^{\mathrm{i}(\Phi+\delta_0)}$ for small S. Hence, $R \simeq a|f'(\zeta_0)|S$ and $\Theta \simeq \Phi + \delta_0$; we also have, to the same order of accuracy, $R \simeq a|f'(\zeta)|S$ and $\Theta \simeq \Phi + \delta$. These approximations lead to the following exact expressions for the kernels in (2.6):

$$a^3 R^{-3} = |f'(\zeta)|^{-3/2}|f'(\zeta_0)|^{-3/2}\{S^{-3} + K^{(1)}(\zeta, \zeta_0)\}\,\mathrm{e}^{\mathrm{j}(\delta_0 - \delta)},$$

$$a^3 R^{-3}\,\mathrm{e}^{2\mathrm{j}\Theta} = |f'(\zeta)|^{-3/2}|f'(\zeta_0)|^{-3/2}\{S^{-3}\,\mathrm{e}^{2\mathrm{j}\Phi} + K^{(2)}(\zeta, \zeta_0)\}\,\mathrm{e}^{\mathrm{j}(\delta_0 + \delta)},$$

where $K^{(1)}$ and $K^{(2)}$ are defined by

$$K^{(1)}(\zeta, \zeta_0) = \frac{|f'(\zeta)|^{3/2}|f'(\zeta_0)|^{3/2}}{|f(\zeta) - f(\zeta_0)|^3}\,\mathrm{e}^{\mathrm{j}(\delta - \delta_0)} - \frac{1}{|\zeta - \zeta_0|^3}, \tag{4.5}$$

$$K^{(2)}(\zeta,\zeta_0) = \frac{|f'(\zeta)|^{3/2}|f'(\zeta_0)|^{3/2}}{|f(\zeta)-f(\zeta_0)|^3}\,e^{j(2\Theta-\delta-\delta_0)} - \frac{1}{|\zeta-\zeta_0|^3}\,e^{2j\Phi}. \tag{4.6}$$

In Appendix B, we show that $K^{(1)} = O(S^{-2})$ and $K^{(2)} = O(S^{-1})$ as $S \to 0$. Thus, $K^{(1)}$ has a Cauchy-type singularity but $K^{(2)}$ has only a weak singularity.

Next, consider the Jacobian of the transformation: it is $a^2|f'(\zeta)|^2$. Hence, if we put

$$\zeta = \xi + \mathrm{i}\eta = s\,e^{\mathrm{i}\varphi} \quad \text{and} \quad \zeta_0 = \xi_0 + \mathrm{i}\eta_0 = s_0\,e^{\mathrm{i}\varphi_0},$$

we find that $\mathrm{d}\Omega = \mathrm{d}x\,\mathrm{d}y = a^2|f'(\zeta)|^2\,\mathrm{d}\xi\,\mathrm{d}\eta = a^2|f'(\zeta)|^2 s\,\mathrm{d}s\,\mathrm{d}\varphi$.

Finally, setting

$$\begin{aligned} w(x(\zeta),y(\zeta)) &= a|f'(\zeta)|^{-1/2}\,e^{j\delta}W(\xi,\eta), \\ q(x(\zeta_0),y(\zeta_0)) &= |f'(\zeta_0)|^{-3/2}\,e^{j\delta_0}Q(\xi_0,\eta_0), \end{aligned} \tag{4.7}$$

we find that (2.6) becomes

$$(M+K)W = Q, \quad \text{for} \quad (\xi_0,\eta_0)\in D, \tag{4.8}$$

where $M \equiv M_D$ and all quantities are dimensionless. This equation is to be solved subject to $W = 0$ on $s = 1$. The operator K is defined by

$$KW = \frac{2-\nu}{8\pi}\,⨍_D W(\xi,\eta)K^{(1)}(\zeta,\zeta_0)\,\mathrm{d}\xi\,\mathrm{d}\eta + \frac{3\nu}{8\pi}\int_D \bar{W}(\xi,\eta)K^{(2)}(\zeta,\zeta_0)\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{4.9}$$

The hypersingular operator M [defined by (2.6) with $\Omega = D$] is precisely the operator for a shear-loaded penny-shaped crack. Consequently, as described in Section 3, we have an explicit expression for M^{-1}. Hence, we can write (4.8) equivalently as

$$(I+M^{-1}K)W = M^{-1}Q,$$

which is a regularized version of the hypersingular integral equation (4.8).

Computationally, the following schemes suggest themselves. First, expand W as

$$W(\xi,\eta) = \sum_{n=-N}^{N} s^{|n|}\,e^{jn\varphi}\sum_{l=0}^{L} W_l^n C_{2l+1}^{|n|+1/2}(\sqrt{1-s^2}).$$

Then, either use the orthogonality of the Gegenbauer polynomials and of the trigonometric functions, leading to a Petrov–Galerkin method, or simply collocate. The efficacy of this numerical method for the shear loading of arbitrary flat cracks will be investigated elsewhere. Here, we prefer to obtain analytical results for cracks Ω that are approximately circular.

5. SOMEWHAT CIRCULAR CRACKS

As in Martin (1994), we consider the conformal mapping

$$f(\zeta) = \zeta + \varepsilon g(\zeta), \tag{5.1}$$

where ε is a small dimensionless parameter and g is analytic; we assume that $g(0) = 0$. When (5.1) is combined with (4.2), we see that the unit disc in the ζ-plane is mapped into a domain Ω in the z-plane that is approximately a circle of radius a; moreover, $\zeta = 0$ is mapped to $z = 0$. Specific choices for g will be considered later.

Substitute (5.1) into (4.9) to give $K = \varepsilon\mathscr{K} + O(\varepsilon^2)$ as $\varepsilon \to 0$. The operator $\mathscr{K}$ is defined by

$$
\begin{aligned}
(\mathscr{K}W)(\zeta_0) &= \frac{1}{8\pi}\int_D \{(2-\nu)W(\xi,\eta) + 3\nu\bar{W}(\xi,\eta)\,\mathrm{e}^{2\mathrm{j}\Phi}\}K_1(\zeta,\zeta_0)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\quad + \frac{1}{8\pi}\,⨍_D W(\xi,\eta)K_2(\zeta,\zeta_0)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&= \mathscr{K}_1 W + \mathscr{K}_2 W,
\end{aligned}
$$

say, where the kernels K_1 and K_2 are defined by

$$
K_1(\zeta,\zeta_0) = \frac{1}{S^3}\,\mathrm{Re}\left\{(3+2\mathrm{ij})\left[\frac{1}{2}(g'(\zeta)+g'(\zeta_0)) - \frac{g(\zeta)-g(\zeta_0)}{\zeta-\zeta_0}\right]\right\},
$$

$$
K_2(\zeta,\zeta_0) = \frac{\nu-2}{S^3}\,\mathrm{Re}\left\{2\mathrm{ij}\left[g'(\zeta) - \frac{g(\zeta)-g(\zeta_0)}{\zeta-\zeta_0}\right]\right\};
$$

we use $\mathrm{Re}\,\{Z\}$ to denote the real part of an i-complex quantity Z.

If we now approximate W and Q by

$$
W(\xi,\eta) = W_0(\xi,\eta) + \varepsilon W_1(\xi,\eta) \quad \text{and} \quad Q(\xi,\eta) = Q_0(\xi,\eta) + \varepsilon Q_1(\xi,\eta),
$$

we find that W_0 solves

$$
MW_0 = Q_0 \tag{5.2}
$$

and then W_1 solves

$$
MW_1 = Q_1 - \mathscr{K}W_0. \tag{5.3}
$$

Both (5.2) and (5.3) are (j-complex) integral equations over the unit disc; they can be solved using the methods of Section 3. Then, from (4.7), using

$$
|f'(\zeta)| \simeq 1 + \varepsilon\,\mathrm{Re}\,\{g'(\zeta)\} \quad \text{and} \quad \mathrm{e}^{\mathrm{j}\delta} \simeq 1 - \varepsilon\,\mathrm{Re}\,\{\mathrm{ij}g'(\zeta)\}
$$

for small ε, we obtain

$$
w(x(\zeta), y(\zeta)) = aW_0 + a\varepsilon(W_1 - W_0\,\mathrm{Re}\,\{(\tfrac{1}{2}+\mathrm{ij})g'(\zeta)\}); \tag{5.4}
$$

this yields an approximation to the crack-opening displacement, via (2.7), correct to first order in ε.

In order to proceed, we must be able to calculate $\mathscr{K}W_0$. Assume that g is analytic in a domain that includes D, so that we can surround D by a simple closed contour C. Then, the residue calculus gives

$$K_1(\zeta,\zeta_0) = \frac{1}{2S^3}\operatorname{Re}\left\{\frac{3+2ij}{2\pi i}\int_C \frac{(\zeta-\zeta_0)^2}{(\omega-\zeta)^2(\omega-\zeta_0)^2} g(\omega)\,d\omega\right\},$$

$$K_2(\zeta,\zeta_0) = \frac{\nu-2}{S^3}\operatorname{Re}\left\{\frac{2ij}{2\pi i}\int_C \frac{\zeta-\zeta_0}{(\omega-\zeta)^2(\omega-\zeta_0)} g(\omega)\,d\omega\right\},$$

which are integral representations involving the values of g on C. Hence,

$$(\mathscr{K}_1 W_0)(\zeta_0) = \frac{1}{2}\operatorname{Re}\left\{\frac{3+2ij}{2\pi i}\int_C \frac{g(\omega)}{(\omega-\zeta_0)^2} L_1(\omega,\zeta_0)\,d\omega\right\},$$

$$(\mathscr{K}_2 W_0)(\zeta_0) = \frac{\nu-2}{2}\operatorname{Re}\left\{\frac{2ij}{2\pi i}\int_C \frac{g(\omega)}{\omega-\zeta_0} L_2(\omega,\zeta_0)\,d\omega\right\},$$

where

$$L_1(\omega,\zeta_0) = M\left\{\frac{(\zeta-\zeta_0)^2}{(\omega-\zeta)^2} W_0(\zeta)\right\}, \tag{5.5}$$

$$L_2(\omega,\zeta_0) = H\left\{\frac{\zeta-\zeta_0}{(\omega-\zeta)^2} W_0(\zeta)\right\}, \tag{5.6}$$

$H \equiv H_D$ and H_Ω is the basic hypersingular operator defined by (1.1) and studied by Martin (1994). Note that L_1 and L_2 are independent of the mapping g. To proceed further, we must specify the loading; we consider a simple non-trivial choice.

6. UNIFORM SHEAR OF A SOMEWHAT CIRCULAR CRACK

Suppose that the crack is subjected to a constant shear,

$$\tau_{x3}(x,y,0) = -\tau \quad \text{and} \quad \tau_{y3}(x,y,0) = 0 \quad \text{for} \quad (x,y)\in\Omega, \tag{6.1}$$

where τ is a constant. Then, $q = -(1-\nu)\tau/\mu = -\sigma$, say, whence

$$Q_0 = -\sigma \quad \text{and} \quad Q_1 = -\tfrac{1}{2}\sigma\operatorname{Re}\{(3+2ij)g'(\zeta_0)\}.$$

From (3.8), we obtain

$$W_0 = M^{-1}Q_0 = \frac{8\sigma}{\pi(2-\nu)}\sqrt{1-s^2}.$$

The calculation of $W_1 = M^{-1}(Q_1 - \mathscr{K}W_0)$ is complicated. In outline, it proceeds as follows (the details are given in Appendix C).

(1) Evaluate L_1, defined by (5.5), by expanding in powers of ζ and then using the formulae for $M\{\zeta^n\sqrt{1-s^2}\}$, namely (3.5) and (3.6). The resulting infinite series can be summed to give (C.3).
(2) Evaluate L_2, defined by (5.6). To do this, use the formula for $H\{\zeta^n\sqrt{1-s^2}\}$ given by Martin (1994); the result is (C.4).

(3) At this stage, we have expressed $\mathscr{K} W_0$ as a contour integral around C, so we do the same for Q_1. This yields a contour integral for $Q_1 - \mathscr{K} W_0$, (C.5).
(4) Next, apply M^{-1}. To do this, expand the integrand in powers of ζ_0, and then use (3.2), (3.3) and (3.7). Again, the resulting infinite series can be summed.
(5) Evaluate the contour integral around C. It turns out that the integrand has a double pole at $\omega = 0$ and a double pole at $\omega = \zeta$.

The final expression for W_1 is (C.7). When this is substituted into (5.4), we obtain

$$w(x(\zeta), y(\zeta)) = \frac{8(1-\nu)\tau a}{\pi\mu(2-\nu)} \sqrt{1-s^2}\{1+\varepsilon \operatorname{Re}[h(\zeta)+(1+\mathrm{ij})\mathscr{W}(\zeta)]\}, \tag{6.2}$$

where

$$\mathscr{W} = \frac{\nu}{2(2-\nu)} \left\{ \frac{2h'(\zeta)-h'(0)}{\zeta} + \frac{h(0)-h(\zeta)}{\zeta^2} \right\}. \tag{6.3}$$

Here, we have found it convenient to introduce the function h, defined by

$$h(\zeta) = \zeta^{-1} g(\zeta); \tag{6.4}$$

h is analytic within C, with $h(0) = g'(0)$. Note that j occurs in only one place in (6.2), whence

$$[u_x] = \frac{8(1-\nu)\tau a}{\pi\mu(2-\nu)} \{1+\varepsilon \operatorname{Re}[h(\zeta)+\mathscr{W}(\zeta)]\} \sqrt{1-s^2},$$

$$[u_y] = \frac{8(1-\nu)\tau a}{\pi\mu(2-\nu)} \varepsilon \operatorname{Re}[\mathrm{i}\mathscr{W}(\zeta)] \sqrt{1-s^2}.$$

Equation (6.2) is our approximation for the (complex) crack-opening displacement; it is correct to first order in ε. Note that $\mathscr{W}$ vanishes when $\nu = 0$; in this case, (6.2) reduces to a result of Martin (1994). At the "crack centre", $z = 0$, we find that

$$w(0,0) = \frac{8(1-\nu)\tau a}{\pi\mu(2-\nu)} \left\{ 1+\varepsilon \operatorname{Re}\left[h(0)+\frac{3\nu(1+\mathrm{ij})}{4(2-\nu)} h''(0) \right] \right\}. \tag{6.5}$$

Let us express (6.2) in terms of the original variables. From (4.2) and (5.1), with $z = r\,\mathrm{e}^{\mathrm{i}\theta}$ and $\zeta = s\,\mathrm{e}^{\mathrm{i}\varphi}$, we have

$$r = as\,\mathrm{e}^{\mathrm{i}(\varphi-\theta)} + a\varepsilon\,\mathrm{e}^{-\mathrm{i}\theta} g(\zeta).$$

The imaginary part of this equation gives $\theta = \varphi + O(\varepsilon)$, whence the real part gives

$$r = as(1+\varepsilon \operatorname{Re}\{h(\zeta)\}).$$

In particular, $\partial\Omega$ corresponds to $s = 1$:

$$r = \rho(\theta) = a(1+\varepsilon \operatorname{Re}\{h(\mathrm{e}^{\mathrm{i}\varphi})\}).$$

Hence, discarding terms of $O(\varepsilon^2)$,

$$a\sqrt{1-s^2} = \sqrt{\rho^2-r^2}\{1-\varepsilon(1-s^2)^{-1}\,\mathrm{Re}\,[h(\mathrm{e}^{\mathrm{i}\varphi})-s^2h(\zeta)]\}.$$

Then, eliminating $\sqrt{1-s^2}$ from (6.2), we obtain

$$w(x,y) = \frac{8(1-\nu)\tau}{\pi\mu(2-\nu)}\sqrt{\rho^2-r^2}\left\{1+\varepsilon\,\mathrm{Re}\left[\frac{h(\zeta)-h(\mathrm{e}^{\mathrm{i}\varphi})}{1-s^2}+(1+\mathrm{ij})\mathscr{W}(\zeta)\right]\right\}, \quad (6.6)$$

which is well defined as $s \to 1$:

$$\lim_{r\to\rho}\frac{w(x,y)}{\sqrt{\rho-r}} = \frac{8(1-\nu)\tau\sqrt{2a}}{\pi\mu(2-\nu)}\{1+\varepsilon(w_1+\mathrm{Re}\,[(1+\mathrm{ij})\mathscr{W}(\mathrm{e}^{\mathrm{i}\varphi})])\}, \quad (6.7)$$

where

$$w_1 = \tfrac{1}{2}\mathrm{Re}\,\{h(\mathrm{e}^{\mathrm{i}\varphi})-\mathrm{e}^{\mathrm{i}\varphi}h'(\mathrm{e}^{\mathrm{i}\varphi})\}.$$

This formula can be used to extract the stress-intensity factors. Thus, using the same notation and normalisation as Gao (1988), we find that

$$K_2(\theta) = \alpha\sqrt{\pi a}\{\cos\theta+\varepsilon(w_1\cos\theta+\gamma\sin\theta+\mathrm{Re}\,[\mathrm{e}^{\mathrm{i}\varphi}\mathscr{W}(\mathrm{e}^{\mathrm{i}\varphi})])\}, \quad (6.8)$$

$$K_3(\theta) = -(1-\nu)\alpha\sqrt{\pi a}\{\sin\theta+\varepsilon(w_1\sin\theta-\gamma\cos\theta-\mathrm{Re}\,[\mathrm{i}\mathrm{e}^{\mathrm{i}\varphi}\mathscr{W}(\mathrm{e}^{\mathrm{i}\varphi})])\}, \quad (6.9)$$

where $\alpha = 4\tau/(\pi(2-\nu))$ and

$$\varepsilon\gamma(\theta) = \rho'(\theta)/\rho(\theta) = \varepsilon\,\mathrm{Re}\{\mathrm{i}\mathrm{e}^{\mathrm{i}\varphi}h'(\mathrm{e}^{\mathrm{i}\varphi})\}\,; \quad (6.10)$$

γ arises because the normal to $\partial\Omega$ at a point P on $\partial\Omega$ (in the plane of the crack) is not in the direction of the position vector of P with respect to O.

6.1. *The flat elliptical crack*

This geometry provides a simple check on the theory, as the exact solution is known; see Appendix D. The conformal mapping from the interior of the unit disc, D, onto the interior of an ellipse, Ω, is complicated; it involves a Jacobian elliptic function (Nehari, 1952, p. 296; Kober, 1957, p. 177). However, for ellipses of small eccentricity, there is a simple approximation (Nehari, 1952, p. 265, Ex. 2), namely

$$z = f(\zeta) = a(\zeta+\tfrac{1}{2}\varepsilon\zeta(1+\zeta^2)), \quad (6.11)$$

so that $h(\zeta) = \frac{1}{2}(1+\zeta^2)$. This maps D onto

$$\Omega = \{(x,y): x^2+(1+\varepsilon)^2y^2 < a^2\}$$

approximately, which is an ellipse with major and minor axes of length $2a(1+\varepsilon)$ and $2a$, respectively. Substituting into (6.3) yields $\mathscr{W} = \frac{3}{4}\nu/(2-\nu)$, which is real, whence $[u_y] = 0$ as expected. Substitution into (6.6) then gives $[u_x]$, in agreement with the exact result, (D.1).

For the stress-intensity factors, we find that $w_1 = \frac{1}{2}\sin^2\theta$ and $\gamma = -\sin 2\theta$. Substitution into (6.8) and (6.9) then gives results in agreement with the exact results, (D.2) and (D.3).

6.2. Harmonic wave-form perturbations

Consider the choice

$$h(\zeta) = \zeta^n,$$

where n is an integer. This leads to a crack Ω with boundary $\partial\Omega$ given by

$$r = \rho(\theta) = a(1+\varepsilon\cos n\theta),$$

with an error of $O(\varepsilon^2)$. Such geometries were considered by Gao (1988) in his study of the configurational stability of a crack with an oscillatory edge.

It turns out that the case $n = 1$ is special: w_1 and $\mathscr{W}$ both vanish identically, whence

$$[u_x(x,y)] = \frac{8(1-\nu)\tau}{\pi\mu(2-\nu)}\sqrt{\rho^2-r^2}\left\{1-\varepsilon\frac{\cos\theta}{1+r/a}\right\} \quad \text{and} \quad [u_y] = 0.$$

The corresponding stress-intensity factors agree with those obtained by Gao [1988, equation (52)].

For higher values of n, we can easily calculate w, K_2 and K_3. We find that $[u_y] = 0$ if $n = 2$, but not for $n > 2$. We also find complete agreement with Gao [1988, equation (53)] for the stress-intensity factors.

6.3. Mapping independent formulae

One objection to our formulae for w, K_2 and K_3 might be that they involve the conformal mapping h. However, for domains that are perturbations of a circle, there is an explicit formula for h, correct to first order in ε. Thus, it was shown by Nehari [1952, p. 265, equation (146)] that $f(\zeta) = a\zeta(1+\varepsilon h(\zeta))$, with

$$\begin{aligned} h(\zeta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\psi}+\zeta}{e^{i\psi}-\zeta}p(\psi)\,d\psi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-s^2+2is\sin(\varphi-\psi)}{1+s^2-2s\cos(\varphi-\psi)}p(\psi)\,d\psi, \end{aligned}$$

maps $|\zeta| < 1$ onto the domain Ω, given by (4.1) with $\rho(\theta) = a(1+\varepsilon p(\theta))$. Differentiating,

$$h'(\zeta) = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{e^{i\psi}}{(e^{i\psi}-\zeta)^2}p(\psi)\,d\psi = \frac{-i}{\pi}\int_{-\pi}^{\pi}\frac{p'(\psi)}{e^{i\psi}-\zeta}\,d\psi$$

after an integration by parts. We can then substitute into (6.2) and (6.3) to obtain a formula for w; in particular, we obtain

$$\mathscr{W}(\zeta) = \frac{\nu}{2\pi(2-\nu)}\int_{-\pi}^{\pi}\frac{3-\zeta\,e^{-i\psi}}{(e^{i\psi}-\zeta)^2}p(\psi)\,d\psi.$$

However, care is needed when calculating $h(e^{i\varphi})$ and $h'(e^{i\varphi})$, for the resulting integrals are singular: the basic result needed is

$$\lim_{s\to 1-}\int_{-\pi}^{\pi}\frac{F(\psi)}{e^{i\psi}-s\,e^{i\varphi}}\,d\psi = \fint_{-\pi}^{\pi}\frac{F(\psi)}{e^{i\psi}-e^{i\varphi}}\,d\psi+\pi e^{-i\varphi}F(\varphi),$$

which can be derived from the Sokhotski–Plemelj formula (Kress, 1989, Theorem 7.6). Thus,

$$h(e^{i\varphi}) = p(\varphi)+\frac{1}{2\pi}\fint_{-\pi}^{\pi}\frac{e^{i\psi}+e^{i\varphi}}{e^{i\psi}-e^{i\varphi}}p(\psi)\,d\psi$$

$$= p(\varphi)+\frac{i}{2\pi}\fint_{-\pi}^{\pi}p(\psi)\cot\left(\frac{\varphi-\psi}{2}\right)d\psi,$$

$$h'(e^{i\varphi}) = -ie^{-i\varphi}p'(\varphi)-\frac{i}{\pi}\fint_{-\pi}^{\pi}\frac{p'(\psi)}{e^{i\psi}-e^{i\varphi}}\,d\psi.$$

In particular, we have

$$\mathrm{Re}\{h(e^{i\varphi})\} = p(\varphi) \quad \text{and} \quad \mathrm{Re}\{ie^{i\varphi}h'(e^{i\varphi})\} = p'(\varphi).$$

Now, suppose we fix θ. Then, we can choose to set the scale a as $a = \rho(\theta)$. We then have

$$\varepsilon p(\psi) = [\rho(\psi)]/[\rho(\theta)]-1$$

from the definition of p, so that $p(\varphi) \simeq p(\theta) = 0$ and $p'(\varphi) = \gamma(\theta)$ [which is consistent with (6.10)]. Hence, substituting into (6.6), we obtain

$$w(x,y) = \frac{8(1-\nu)\tau}{\pi\mu(2-\nu)}\sqrt{[\rho(\theta)]^2-r^2}\left\{1+w_0+\frac{\nu}{2-\nu}\mathrm{Re}\,[(1+ij)\mathscr{W}_1]\right\}, \qquad (6.12)$$

where

$$w_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\rho(\theta)[\rho(\psi)-\rho(\theta)]}{[\rho(\theta)]^2+r^2-2r\rho(\theta)\cos(\theta-\psi)}\,d\psi,$$

$$\mathscr{W}_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{3\rho(\theta)-r\,e^{i(\theta-\psi)}}{[\rho(\theta)\,e^{i\psi}-r\,e^{i\theta}]^2}[\rho(\psi)-\rho(\theta)]\,d\psi.$$

The formula (6.12) is new. When $\nu = 0$, it reduces to a formula obtained previously by Gao and Rice (1987) and Martin (1994).

Finally, consider the stress-intensity factors, defined by (6.8) and (6.9). We find that

$$w_1 = p(\varphi)-\frac{1}{4\pi}\fint_{-\pi}^{\pi}p'(\psi)\cot\left(\frac{\varphi-\psi}{2}\right)d\psi.$$

Choosing $a = \rho(\theta)$ so that $p(\varphi) = 0$, as before, we can integrate by parts to give

$$\varepsilon w_1 = \frac{1}{8\pi} \int_{-\pi}^{\pi} \frac{\rho(\psi)-\rho(\theta)}{\rho(\theta)\sin^2[(\theta-\psi)/2]} \mathrm{d}\psi.$$

We also have

$$\varepsilon \mathrm{e}^{\mathrm{i}\varphi} \mathscr{W}(\mathrm{e}^{\mathrm{i}\varphi}) = \left\{ -2\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta}\gamma(\theta) + \frac{1}{\pi} \int_{-\pi}^{\pi} \gamma(\psi)\,\mathrm{e}^{-\mathrm{i}\psi} \cot\left(\frac{\theta-\psi}{2}\right) \mathrm{d}\psi \right.$$

$$\left. + \frac{1}{2\pi} \mathrm{e}^{-\mathrm{i}\theta} \int_{-\pi}^{\pi} \left\{1 - \mathrm{i}\cot\left(\frac{\theta-\psi}{2}\right)\right\} \frac{\rho(\psi)-\rho(\theta)}{\rho(\theta)} \mathrm{d}\psi \right\}.$$

We can then verify that the resulting expressions for K_2 and K_3 agree with a special case of the formulae obtained by Gao [1988, equation (31)].

7. DISCUSSION

In this paper, we have done three things. First, we have drawn attention to the simplifying consequences of complexifying the well-known pair of integral equations for the shear loading of an arbitrary flat crack; the result is a single integral equation for a complex displacement discontinuity. Then, in Section 4, we showed how to transform this hypersingular integral equation over the crack into a similar equation over a circular region, using an appropriate conformal mapping. The new equation is then amenable to expansion–collocation methods, which are known to yield efficient schemes for the numerical treatment of integral equations over circular domains. In particular, this method could be applied to the shear loading of a rectangular crack. Finally, we obtained various analytical results for cracks that are almost circular: these results are much simpler if they are expressed in terms of the conformal mapping, they include new formulae for the crack-opening displacement under uniform shear, and they give stress-intensity factors in agreement with Gao (1988).

The method will generalize to dynamic problems, such as the scattering of elastic waves by a flat crack. This is because the singular part of the elastodynamic fundamental solution (Kupradze matrix) is the same as for the static problem, and so the regularization method of Section 4 is applicable. For somewhat circular cracks, the method of Section 5 could then be used to study low-frequency diffraction problems.

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APPENDIX A: COMPLEX SOLUTIONS FOR PENNY-SHAPED CRACKS

Polynomial loadings of penny-shaped cracks are considered in Krenk (1979) and Martin (1982, 1986). We start by writing the loading as

$$q_x(x,y) = \sum_{n=0}^{\infty} t_n(r/a)\cos n\theta + \sum_{n=1}^{\infty} \tilde{t}_n(r/a)\sin n\theta,$$

$$q_y(x,y) = \sum_{n=1}^{\infty} s_n(r/a)\sin n\theta + \sum_{n=0}^{\infty} \tilde{s}_n(r/a)\cos n\theta.$$

Then, the crack-opening displacement can be expanded similarly,

$$[u_x(x,y)] = a\sum_{n=0}^{\infty} u_n(r/a)\cos n\theta + a\sum_{n=1}^{\infty} \tilde{u}_n(r/a)\sin n\theta,$$

$$[u_y(x,y)] = a\sum_{n=1}^{\infty} v_n(r/a)\sin n\theta + a\sum_{n=0}^{\infty} \tilde{v}_n(r/a)\cos n\theta.$$

Next, expand t_n as

$$t_n(r) = r^n \sum_{k=0}^{\infty} T_k^n \frac{\Gamma(n+1/2)\Gamma(k+3/2)}{(n+k)!}\,\frac{C_{2k+1}^{n+1/2}(\sqrt{1-r^2})}{\sqrt{1-r^2}},$$

with similar expansions for s_n, $\tilde{t}_n$ and $\tilde{s}_n$, involving coefficients S_k^n, $\tilde{T}_k^n$ and $\tilde{S}_k^n$, respectively. Then, it can be shown that

$$u_n(r) = r^n \sum_{k=0}^{\infty} U_k^n \frac{\Gamma(n+1/2)k!}{\Gamma(n+k+3/2)} C_{2k+1}^{n+1/2}(\sqrt{1-r^2}),$$

with similar expansions for v_n, $\tilde{u}_n$ and $\tilde{v}_n$, involving coefficients V_k^n, $\tilde{U}_k^n$ and $\tilde{V}_k^n$, respectively. The coefficients U_k^n and V_k^n are related to T_k^n and S_k^n by the following formulae (Martin, 1986):

$$\left.\begin{aligned}
-2T_0^0 &= (2-\nu)U_0^0, && \\
-4T_k^0 &= 2(2-\nu)U_k^0 - \nu(U_{k-1}^2 + V_{k-1}^2), && k \geqslant 1,\\
-(T_k^1 + S_k^1) &= U_k^1 + V_k^1, && k \geqslant 0,\\
-2(T_k^2 + S_k^2) &= (2-\nu)(U_k^2 + V_k^2) - 2\nu U_{k+1}^0, && k \geqslant 0,\\
-2(T_k^n + S_k^n) &= (2-\nu)(U_k^n + V_k^n) - \nu(U_{k+1}^{n-2} - V_{k+1}^{n-2}), && n \geqslant 3, k \geqslant 0,\\
-2(T_0^n - S_0^n) &= (2-\nu)(U_0^n - V_0^n), && n \geqslant 1,\\
-2(T_k^n - S_k^n) &= (2-\nu)(U_k^n - V_k^n) - \nu(U_{k-1}^{n+2} + V_{k-1}^{n+2}), && n \geqslant 1, k \geqslant 1.
\end{aligned}\right\} \tag{A.1}$$

Similarly, the coefficients $\tilde{U}_k^n$ and $\tilde{V}_k^n$ are related to $\tilde{T}_k^n$ and $\tilde{S}_k^n$ by the following formulae, obtained by following the procedure given on p. 527 of (Martin (1986):

$$\left.\begin{aligned}
-2\tilde{S}_0^0 &= (2-\nu)\tilde{V}_0^0, && \\
-4\tilde{S}_k^0 &= 2(2-\nu)\tilde{V}_k^0 + \nu(\tilde{V}_{k-1}^2 - \tilde{U}_{k-1}^2), && k \geqslant 1,\\
-(\tilde{S}_k^1 - \tilde{T}_k^1) &= (1-\nu)(\tilde{V}_k^1 - \tilde{U}_k^1), && k \geqslant 0,\\
-2(\tilde{S}_k^2 - \tilde{T}_k^2) &= (2-\nu)(\tilde{V}_k^2 - \tilde{U}_k^2) + 2\nu\tilde{V}_{k+1}^0, && k \geqslant 0,\\
-2(\tilde{S}_k^n - \tilde{T}_k^n) &= (2-\nu)(\tilde{V}_k^n - \tilde{U}_k^n) + \nu(\tilde{V}_{k+1}^{n-2} + \tilde{U}_{k+1}^{n-2}), && n \geqslant 3, k \geqslant 0,\\
-2(\tilde{S}_0^n + \tilde{T}_0^n) &= (2-\nu)(\tilde{V}_0^n + \tilde{U}_0^n), && n \geqslant 1,\\
-2(\tilde{S}_k^n + \tilde{T}_k^n) &= (2-\nu)(\tilde{V}_k^n + \tilde{U}_k^n) + \nu(\tilde{V}_{k-1}^{n+2} - \tilde{U}_{k-1}^{n+2}), && n \geqslant 1, k \geqslant 1.
\end{aligned}\right\} \tag{A.2}$$

The complexified form of these equations can be obtained by using the definition (2.7). Specifically, we have

$$Q_k^0 = T_k^0 + \mathrm{j}\tilde{S}_k^0 \quad\text{and}\quad Q_k^{\pm n} = \tfrac{1}{2}(T_k^n \pm S_k^n) + \tfrac{1}{2}\mathrm{j}(\tilde{S}_k^n \mp \tilde{T}_k^n),$$

for $n \geqslant 1$ and $k \geqslant 0$, with similar expressions for W_k^n.

For the record, we note that (A.1) and (A.2) can be solved explicitly:

$$\begin{aligned}
(2-\nu)U_0^0 &= -2T_0^0, && \\
4(1-\nu)U_k^0 &= -2(2-\nu)T_k^0 - \nu(T_{k-1}^2 + S_{k-1}^2), && k \geqslant 1,\\
(2-\nu)(U_0^n - V_0^n) &= -2(T_0^n - S_0^n), && n \geqslant 1,\\
2(1-\nu)(U_k^n - V_k^n) &= -(2-\nu)(T_k^n - S_k^n) - \nu(T_{k-1}^{n+2} + S_{k-1}^{n+2}), && n \geqslant 1, k \geqslant 1,\\
U_k^1 + V_k^1 &= -(T_k^1 + S_k^1), && k \geqslant 0,\\
2(1-\nu)(U_k^2 + V_k^2) &= -(2-\nu)(T_k^2 + S_k^2) - 2\nu T_{k+1}^0, && k \geqslant 0,\\
2(1-\nu)(U_k^n + V_k^n) &= -(2-\nu)(T_k^n + S_k^n) - \nu(T_{k+1}^{n-2} - S_{k+1}^{n-2}), && n \geqslant 3, k \geqslant 0.\\
(2-\nu)\tilde{V}_0^0 &= -2\tilde{S}_0^0, && \\
4(1-\nu)\tilde{V}_k^0 &= -2(2-\nu)\tilde{S}_k^0 + \nu(\tilde{S}_{k-1}^2 - \tilde{T}_{k-1}^2), && k \geqslant 1,
\end{aligned}$$

$$
\begin{aligned}
(1-\nu)(\tilde{V}_k^1-\tilde{U}_k^1) &= -(\tilde{S}_k^1-\tilde{T}_k^1), && k \geqslant 0, \\
2(1-\nu)(\tilde{V}_k^2-\tilde{U}_k^2) &= -(2-\nu)(\tilde{S}_k^2-\tilde{T}_k^2)+2\nu\tilde{S}_{k+1}^0, && k \geqslant 0, \\
2(1-\nu)(\tilde{V}_k^n-\tilde{U}_k^n) &= -(2-\nu)(\tilde{S}_k^n-\tilde{T}_k^n)+\nu(\tilde{S}_{k+1}^{n-2}+\tilde{T}_{k+1}^{n-2}), && n \geqslant 3, k \geqslant 0, \\
(2-\nu)(\tilde{V}_0^n+\tilde{U}_0^n) &= -2(\tilde{S}_0^n+\tilde{T}_0^n), && n \geqslant 1, \\
2(1-\nu)(\tilde{V}_k^n+\tilde{U}_k^n) &= -(2-\nu)(\tilde{S}_k^n+\tilde{T}_k^n)+\nu(\tilde{S}_{k-1}^{n+2}-\tilde{T}_{k-1}^{n+2}), && n \geqslant 1, k \geqslant 1.
\end{aligned}
$$

These results simplify greatly if they are expressed in terms of w and q, defined by (2.7); see Section 3.

APPENDIX B: THE KERNELS $K^{(1)}$ AND $K^{(2)}$

The kernels $K^{(1)}$ and $K^{(2)}$ are defined by (4.5) and (4.6), respectively; we expand them for small S. Write

$$\mathscr{F} = (\zeta-\zeta_0)\frac{f''(\zeta_0)}{f'(\zeta_0)} = \mathscr{F}_1+\mathrm{i}\mathscr{F}_2 = O(S) \quad \text{as} \quad S \to 0.$$

Hence, $f'(\zeta) \simeq f'(\zeta_0)\{1+\mathscr{F}\}$ and $f'(\zeta_0) \simeq f'(\zeta)\{1-\mathscr{F}\}$.

Consider $K^{(1)}$. From (4.4), we have

$$\mathrm{e}^{\mathrm{i}(\delta-\delta_0)} = \frac{f'(\zeta)}{f'(\zeta_0)}\frac{|f'(\zeta_0)|}{|f'(\zeta)|} \simeq \frac{1+\mathscr{F}}{|1+\mathscr{F}|} \simeq (1+\mathscr{F})(1-\mathscr{F}_1) \simeq 1+\mathrm{i}\mathscr{F}_2,$$

giving $\delta-\delta_0 \simeq \mathscr{F}_2 = O(S)$. Now, Martin (1994, Appendix B) showed that

$$\frac{|f'(\zeta)|^{3/2}|f'(\zeta_0)|^{3/2}}{|f(\zeta)-f(\zeta_0)|^3} - \frac{1}{|\zeta-\zeta_0|^3} = O(S^{-1}) \quad \text{as} \quad S \to 0.$$

Hence, $K^{(1)} \simeq \mathrm{j}\mathscr{F}_2 S^{-3}$, which is $O(S^{-2})$ as $S \to 0$.

Next, consider $K^{(2)}$. From (4.3), we have $R\,\mathrm{e}^{\mathrm{i}\Theta} \simeq a(\zeta-\zeta_0)f'(\zeta_0)\{1+\frac{1}{2}\mathscr{F}\}$, whence

$$\mathrm{e}^{\mathrm{i}(\Theta-\Phi-\delta_0)} \simeq (1+\tfrac{1}{2}\mathscr{F})|1+\tfrac{1}{2}\mathscr{F}|^{-1}.$$

Also, $R\,\mathrm{e}^{\mathrm{i}\Theta} \simeq a(\zeta-\zeta_0)f'(\zeta)\{1-\mathscr{F}\}\{1+\frac{1}{2}\mathscr{F}\}$, whence

$$\mathrm{e}^{\mathrm{i}(\Theta-\Phi-\delta)} \simeq (1-\tfrac{1}{2}\mathscr{F})|1-\tfrac{1}{2}\mathscr{F}|^{-1}.$$

Hence

$$\mathrm{e}^{\mathrm{i}(2\Theta-2\Phi-\delta-\delta_0)} \simeq 1+O(S^2)$$

as $S \to 0$. It follows that $K^{(2)}$ is $O(S^{-1})$ as $S \to 0$.

APPENDIX C: EVALUATION OF W_1

To begin, we evaluate L_1, defined by (5.5). We have

$$\frac{(\zeta-\zeta_0)^2}{(\omega-\zeta)^2} = \sum_{m=0}^{\infty}\frac{(m+1)}{\omega^{m+2}}(\zeta^{m+2}-2\zeta^{m+1}\zeta_0+\zeta^m\zeta_0^2)$$

for $|\zeta/\omega| < 1$ (which is satisfied since $|\zeta| < 1$ and $|\omega| > 1$). But $M\{\zeta^n\sqrt{1-s^2}\}$ can be evaluated using (3.5) and (3.6); hence

$$L_1(\omega,\zeta_0) = -\frac{4\sigma}{\pi}\sum_{m=0}^{\infty}(m+1)Z_0^{m+2}\Gamma(3/2)l_m + \frac{2\sigma\nu(1+\mathrm{ij})}{\pi(2-\nu)}\sum_{m=0}^{\infty}\frac{m+1}{\omega^{m+2}}N_m(s_0)\,\mathrm{e}^{\mathrm{i}m\varphi_0}, \tag{C.1}$$

where

$$l_m = \frac{\Gamma(m+7/2)}{(m+2)!} - 2\frac{\Gamma(m+5/2)}{(m+1)!} + \frac{\Gamma(m+3/2)}{m!} = -\frac{\Gamma(m+3/2)}{4(m+2)!},$$

$$N_m(r) = \hat{q}_{m+2}(r) - 2r\hat{q}_{m+1}(r) + r^2\hat{q}_m(r)$$

$$= \tfrac{3}{4}\Gamma(3/2)\left\{\frac{\Gamma(m+1/2)}{m!} - r^2\frac{\Gamma(m+3/2)}{(m+1)!}\right\}r^m$$

and $Z_0 = \zeta_0/\omega$. The sums in (C.1) can be evaluated, using

$$(1-Z_0)^{-\alpha} = \sum_{m=0}^{\infty} \frac{\Gamma(m+\alpha)}{\Gamma(\alpha)}\frac{Z_0^m}{m!}, \quad |Z_0| < 1: \tag{C.2}$$

$$L_1(\omega,\zeta_0) = \tfrac{1}{2}\sigma\{(1-Z_0)^{-1/2}+(1-Z_0)^{1/2}-2\} + \frac{3\sigma\nu(1+\mathrm{ij})}{8(2-\nu)\omega^2}\{(1-Z_0)^{-1/2}+(1-s_0^2)(1-Z_0)^{-3/2}\}. \tag{C.3}$$

To evaluate L_2, defined by (5.6), we note from Martin (1994) that

$$H\{\zeta^n\sqrt{1-s^2}\} = -\Gamma(n+3/2)\Gamma(3/2)\zeta_0^n/n!\,;$$

hence,

$$L_2(\omega,\zeta_0) = -\frac{\sigma Z_0}{(2-\nu)\omega}(1-Z_0)^{-3/2}. \tag{C.4}$$

Next, consider the right-hand side of (5.3). We have expressed $\mathscr{K}W_0$ as a contour integral around C, so it is convenient to do the same for Q_1:

$$Q_1(\zeta_0) = -\sigma\,\mathrm{Re}\left\{(\tfrac{3}{2}+\mathrm{ij})\frac{1}{2\pi\mathrm{i}}\int_C \frac{g(\omega)}{(\omega-\zeta_0)^2}\,\mathrm{d}\omega\right\}.$$

Hence

$$Q_1 - \mathscr{K}W_0 = -\mathrm{Re}\left\{\frac{1}{2\pi\mathrm{i}}\int_C \frac{g(\omega)}{(\omega-\zeta_0)^2}\mathscr{L}(\omega,\zeta_0)\,\mathrm{d}\omega\right\} \tag{C.5}$$

where

$$\begin{aligned}\mathscr{L} &= (\tfrac{3}{2}+\mathrm{ij})(\sigma+L_1) - (2-\nu)\mathrm{ij}(\omega-\zeta_0)L_2\\ &= \sigma\{(\tfrac{3}{4}+\tfrac{3}{2}\mathrm{ij})(1-Z_0)^{-1/2} + (\tfrac{3}{4}-\tfrac{1}{2}\mathrm{ij})(1-Z_0)^{1/2}\}\\ &\quad + \frac{15\sigma\nu(1+\mathrm{ij})}{16(2-\nu)\omega^2}\{(1-Z_0)^{-1/2}+(1-s_0^2)(1-Z_0)^{-3/2}\}.\end{aligned}$$

The next step is to apply M^{-1}, using (3.2), (3.3) and (3.7). We write (C.5) as

$$Q_1 - \mathscr{K}W_0 = -\left(\frac{\sigma}{\pi}\right)\mathrm{Re}\left\{\frac{1}{2\pi\mathrm{i}}\int_C \frac{g(\omega)}{\omega^2}(\mathscr{L}_1+\mathscr{L}_2)\,\mathrm{d}\omega\right\},$$

where

$$\begin{aligned}\mathscr{L}_1(s_0,\varphi_0;\omega) &= \pi(\tfrac{3}{4}+\tfrac{3}{2}\mathrm{ij})(1-Z_0)^{-5/2} + \pi(\tfrac{3}{4}-\tfrac{1}{2}\mathrm{ij})(1-Z_0)^{-3/2},\\ \mathscr{L}_2(s_0,\varphi_0;\omega) &= \frac{15\pi\nu(1+\mathrm{ij})}{16(2-\nu)\omega^2}\{(1-Z_0)^{-5/2}+(1-s_0^2)(1-Z_0)^{-7/2}\}.\end{aligned}$$

Consider $\mathscr{L}_1$; using (C.2), we obtain [cf. (3.1) and (3.2)]

$$\mathscr{L}_1(s,\varphi;\omega) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{j}n\varphi}s^{|n|}Q_0^n\frac{\Gamma(|n|+1/2)\Gamma(3/2)}{(|n|)!}\frac{C_1^{|n|+1/2}(\sqrt{1-s^2})}{\sqrt{1-s^2}},$$

where $Q_0^0 = 3+2\mathrm{ij}$, $Q_0^n = \tfrac{1}{2}(1-\mathrm{ij})(1-n)\omega^{-n}$, $Q_0^{-n} = \tfrac{1}{2}(1+\mathrm{ij})(5+3n)\omega^{-n}$, $n > 0$ and we have used $(1+\mathrm{ij})\mathrm{e}^{\mathrm{j}n\varphi} = (1+\mathrm{ij})\mathrm{e}^{-\mathrm{i}n\varphi}$ to pass between $\mathrm{e}^{\mathrm{j}n\varphi}$ and $\mathrm{e}^{\mathrm{i}n\varphi}$. Then,

$$M^{-1}\mathscr{L}_1 = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{j}n\varphi}s^{|n|}\sum_{k=0}^{\infty} W_k^n\frac{\Gamma(|n|+1/2)k!}{\Gamma(|n|+k+3/2)}C_{2k+1}^{|n|+1/2}(\sqrt{1-s^2}), \tag{C.6}$$

where the only non-zero coefficients are given by (3.7) as $(2-\nu)W_0^{-n} = -2Q_0^{-n}$ for $n \geqslant 0$, $2(1-\nu)W_0^n = -(2-\nu)Q_0^n$ for $n \geqslant 2$, and $2(1-\nu)W_1^{-n} = -\nu Q_0^{n+2}$ for $n \geqslant 0$. Substituting into (C.6) gives

$$M^{-1}\mathscr{L}_1 = \sqrt{1-s^2}\left\{\frac{(2-\nu)(1-\mathrm{ij})}{2(1-\nu)}S_1 - \frac{4(3+2\mathrm{ij})}{2-\nu} - \frac{2(1+\mathrm{ij})}{2-\nu}S_2 + \frac{\nu(1+\mathrm{ij})}{3(1-\nu)\omega^2}S_3\right\},$$

where $\zeta = s\,\mathrm{e}^{\mathrm{i}\varphi}$, $Z = \zeta/\omega$,

$$S_1 = \sum_{m=1}^{\infty}(m-1)Z^m = \frac{Z^2}{(1-Z)^2}, \quad S_2 = \sum_{m=1}^{\infty}(3m+5)Z^m = \frac{Z(8-5Z)}{(1-Z)^2},$$

$$S_3 = \sum_{m=0}^{\infty}(m+1)\{(m+\tfrac{5}{2})(1-s^2)-\tfrac{3}{2}\}Z^m = \frac{2Z+2+s^2(Z-5)}{2(1-Z)^3}$$

and we have summed the series using the geometric series and its derivatives.

A similar calculation succeeds for $\mathscr{L}_2$. We obtain

$$\mathscr{L}_2(s,\varphi;\omega) = \sum_{n=0}^{\infty}\mathrm{e}^{-\mathrm{j}n\varphi}s^n\Gamma(n+1/2)\sum_{k=0}^{1}Q_k^{-n}\frac{\Gamma(k+3/2)}{(n+k)!}\frac{C_{2k+1}^{n+1/2}(\sqrt{1-s^2})}{\sqrt{1-s^2}},$$

where

$$Q_0^{-n} = \frac{\nu(1+\mathrm{ij})(2n+3)}{(2-\nu)\omega^{n+2}} \quad \text{and} \quad Q_1^{-n} = \frac{\nu(1+\mathrm{ij})(n+1)}{2(2-\nu)\omega^{n+2}}, \quad n \geqslant 0.$$

Then, $M^{-1}\mathscr{L}_2$ is given by (C.6), where, now, the only non-zero coefficients are given by (3.7) as $(2-\nu)W_0^{-n} = -2Q_0^{-n}$ for $n \geqslant 0$, $2(1-\nu)W_0^n = -\nu\bar{Q}_1^{2-n}$ for $n \geqslant 2$, and $2(1-\nu)W_1^{-n} = -(2-\nu)Q_1^{-n}$ for $n \geqslant 0$. Substituting into (C.6) gives

$$M^{-1}\mathscr{L}_2 = -\sqrt{1-s^2}\left\{\frac{\nu^2(1-\mathrm{ij})}{2(1-\nu)(2-\nu)}S_1 + \frac{\nu(1+\mathrm{ij})}{3(1-\nu)\omega^2}S_3 + \frac{4\nu(1+\mathrm{ij})}{(2-\nu)^2\omega^2}S_4\right\},$$

where

$$S_4 = \sum_{m=0}^{\infty}(2m+3)Z^m = \frac{3-Z}{(1-Z)^2}.$$

Combining the above results, we have $M^{-1}(\mathscr{L}_1+\mathscr{L}_2) = \sqrt{1-s^2}\mathscr{M}$, where

$$\mathscr{M} = \frac{2}{2-\nu}\left\{(1-\mathrm{ij})S_5 - (1+\mathrm{ij})S_6 - \frac{2\nu(1+\mathrm{ij})}{(2-\nu)\omega^2}S_4\right\},$$

$$S_5 = S_1 - 1 = \frac{2Z-1}{(1-Z)^2}, \quad S_6 = S_2 + 5 = \frac{5-2Z}{(1-Z)^2}.$$

Next, we substitute $\mathscr{M}$ into

$$W_1 = -\left(\frac{\sigma}{\pi}\right)\sqrt{1-s^2}\,\mathrm{Re}\{I\} \quad \text{with} \quad I = \frac{1}{2\pi\mathrm{i}}\int_C \frac{g(\omega)}{\omega^2}\mathscr{M}\,\mathrm{d}\omega \tag{C.7}$$

and evaluate the contour integral over C. The integrand has a double pole at $\omega = 0$ and a double pole at $\omega = \zeta$; evaluating the residues at these poles gives

$$I = \frac{2(1-\mathrm{ij})}{2-\nu}\{\zeta h'(\zeta) - h(\zeta)\} - \frac{2(1+\mathrm{ij})}{2-\nu}\{3\zeta h'(\zeta) + 5h(\zeta)\} - \frac{4\nu(1+\mathrm{ij})}{(2-\nu)^2}\left\{\frac{2h'(\zeta)-h'(0)}{\zeta} + \frac{h(0)-h(\zeta)}{\zeta^2}\right\},$$

where h is defined by (6.4).

APPENDIX D: THE FLAT ELLIPTICAL CRACK

Consider a flat elliptical crack,

$$\Omega = \{(x,y):(x/A)^2+(y/B)^2 < 1\}$$

subjected to the shear loading (6.1). Then, it was shown by Eshelby (1957, 1963) that the crack-opening displacement is given exactly by $[u_y] = 0$ and

$$[u_x(x,y)] = \frac{-2(1-\nu)k^2 B\tau}{\mu\Delta}\sqrt{1-\beta^2},$$

where $\Delta = (\nu-k^2)E(k)-\nu k'^2K(k)$, $k' = B/A$, $k^2 = 1-(B/A)^2$, $\beta^2 = (x/A)^2+(y/B)^2$, and K and E are complete elliptic integrals of the first and second kind, respectively. The two stress-intensity factors were obtained by Kassir and Sih (1966); they are given by

$$K_2 = -(\pi B/A)^{1/2}\mathscr{A}^{-1/4}(k^2/\Delta)\tau B\cos\phi,$$

$$K_3 = (1-\nu)(\pi B/A)^{1/2}\mathscr{A}^{-1/4}(k^2/\Delta)\tau A\sin\phi,$$

where $x = A\beta\cos\phi$, $y = B\beta\sin\phi$ and $\mathscr{A} = A^2\sin^2\phi+B^2\cos^2\phi$.

We can approximate the above formulae for an ellipse of small eccentricity. Thus, suppose that $A = a(1+\varepsilon)$ and $B = a$, where ε is small. As $k^2 \simeq 2\varepsilon$ is small also, we can use the power-series expansions for $K(k)$ and $E(k)$ to give $\Delta = \frac{1}{4}\pi k^2\{\nu-2+\frac{1}{8}k^2(\nu+4)\}+O(k^6)$ as $k\to 0$, so that

$$\frac{k^2}{\Delta} \simeq \frac{-4}{\pi(2-\nu)}\left\{1+\varepsilon\frac{4+\nu}{4(2-\nu)}\right\}.$$

Also, as $\rho^2 \simeq a^2(1+2\varepsilon x^2/r^2)$, we have

$$a^2(1-\beta^2) \simeq a^2-r^2+2\varepsilon x^2 \simeq \rho^2-2\varepsilon a^2(x/r)^2-r^2+2\varepsilon x^2 \simeq (\rho^2-r^2)(1-2\varepsilon x^2/r^2),$$

whence $a\sqrt{1-\beta^2} \simeq \sqrt{\rho^2-r^2}\,(1-\varepsilon x^2/r^2)$. Hence

$$[u_x(x,y)] = \frac{8(1-\nu)\tau}{\pi\mu(2-\nu)}\sqrt{\rho^2-r^2}\left\{1+\varepsilon\left(\frac{4+\nu}{4(2-\nu)}-\cos^2\theta\right)\right\} \tag{D.1}$$

correct to first order in ε. For the stress-intensity factors, we have $\sqrt{B/A} \simeq 1-\frac{1}{2}\varepsilon$, $\mathscr{A} \simeq a^2(1+2\varepsilon\sin^2\theta)$, $B\cos\phi \simeq (1-\varepsilon\sin^2\theta)a\cos\theta$, $A\sin\phi \simeq (1+\varepsilon(1+\cos^2\theta))a\sin\theta$. Hence, the stress-intensity factors are given approximately by

$$K_2 = \frac{4\tau\sqrt{\pi a}}{\pi(2-\nu)}\left\{1+\varepsilon\left[\frac{3\nu}{4(2-\nu)}-\frac{3}{2}\sin^2\theta\right]\right\}\cos\theta, \tag{D.2}$$

$$K_3 = \frac{-4(1-\nu)\tau\sqrt{\pi a}}{\pi(2-\nu)}\left\{1+\varepsilon\left[\frac{3\nu}{4(2-\nu)}+2-\frac{3}{2}\sin^2\theta\right]\right\}\sin\theta. \tag{D.3}$$