ELSEVIER

Wave Motion 24 (1996) 421–433

WAVE MOTION

On angular-spectrum representations for scattering by infinite rough surfaces

J.A. DeSanto, P.A. Martin *

Department of Mathematical and Computer Sciences, Colorado School of Mines, Golden, CO 80401-1887, USA

Received 24 April 1996; revised 24 June 1996

Abstract

A plane acoustic wave insonifies an infinite rough surface. The reflected field is written as an angular-spectrum representation (plane-wave expansion), with an unknown amplitude function A. It is pointed out that A must be considered as a generalized function, and not as a continuous function. Various decompositions of A are suggested and analysed. Energy considerations lead to relations between the coefficients in these decompositions, generalizing some known results for scattering by periodic surfaces (gratings). It is shown that the reflected field must include at least one propagating plane wave.

1. Introduction

Many wave problems can be phrased as boundary-value problems in a semi-infinite domain, nominally taken as $z > 0$. The simplest such problem is the reflection of a plane acoustic wave by a plane rigid boundary $z = 0$; then, the angle of reflection equals the angle of incidence, so that there is a single reflected plane wave propagating away from $z = 0$. Text-book solutions of such problems (for example, [1, Section 5.5] or [2, Section 3.1]) usually start by *assuming* that the reflected field is a plane wave with an unknown amplitude propagating in an unknown direction; these unknowns are then determined from the boundary condition on $z = 0$.

Now, consider more complicated problems, obtained by

- replacing the boundary $z = 0$ by a rough surface $z = s$; or
- introducing a finite gap in $z = 0$ through which waves can propagate; or
- replacing the rigid plane boundary by a non-uniform impedance boundary; or by
- some combination of the above.

For these problems, a standard procedure is to represent the scattered (reflected) field u as a linear combination of plane waves, including evanescent waves:

$$u(x, z) = \int_{-\infty}^{\infty} A(\mu) \exp\left\{ik\left(\mu x + z\sqrt{1-\mu^2}\right)\right\} d\mu. \tag{1}$$

* Corresponding author. Permanent address: Department of Mathematics, University of Manchester, Manchester M13 9PL, UK.

0165-2125/96/$15.00
PII S0165-2125(96)00028-5

This is known as an *angular-spectrum representation.* (References and further details are given below.) Use of such a representation reduces the problem to finding the amplitude A.

This paper is concerned with properties of $A(\mu)$, where μ is real. The simple case of a rigid plane boundary shows that A should be thought of as a generalized function, and not as a continuous function.

Some authors replace (1) by a contour integral in the complex μ-plane:

$$u(x, z) = \int_C A(\mu) \exp\left\{ik\left(\mu x + z\sqrt{1-\mu^2}\right)\right\} \mathrm{d}\mu; \tag{2}$$

$A(\mu)$ is usually assumed to be a meromorphic function of μ in a suitably cut plane. However, the solution for plane-wave reflection by a plane boundary cannot be represented in this way; there are also other physical problems for which A is not analytic (see, for example, [3, Section 5.B]).

For some problems, such as a finite gap in a plane rigid boundary, one can write

$$A(\mu) = A_0\delta(\mu - \mu_0) + B(\mu) \tag{3}$$

in (1), where δ is the Dirac delta function, $B(\mu)$ is continuous, and A_0 and μ_0 are known constants: the first term gives the reflected wave in the absence of the gap, and the second term gives the 'correction' due to the gap. The crucial observation is that the 'correction' really is small at large distances from the gap; the term involving B gives rise to a cylindrical wave, which is asymptotically negligible compared to the reflected plane wave.

The solution to the finite-gap problem can also be represented as (2), where $A(\mu)$ is analytic apart from a simple pole at $\mu = \mu_0$, and the contour C is indented below the pole. (In particular, $A(\mu)$ is not continuous at $\mu = \mu_0$.) Note that the reflected plane wave arises as a residue contribution from the pole. Further comments on the use of (2) will be made in Section 3.

For other problems, such as the problem of the scattering of a plane wave by an infinite rough surface, we do not have a rational way of making a decomposition like (3). One plausible possibility is to *assume* that we can write

$$A(\mu) = \sum_n A_n\delta(\mu - \mu_n) + B(\mu) \tag{4}$$

in (1), where B is continuous, as before; but we do not have a prescription for A_n and μ_n. Note that if the rough surface is a *periodic* surface, then we know μ_n (from the Bragg equation) and we know that $B \equiv 0$.

In this paper, we investigate some consequences of the assumption (4). We prove that, in general, it is impossible for $A(\mu)$ to be a continuous function of the real variable μ. Furthermore, we prove that (4) implies that at least one of the coefficients A_n, with $|\mu_n| < 1$, must be non-zero; physically, this means that there must be at least one reflected propagating plane wave. We also derive certain energy-based relations between A_n and B; these generalize some known results for scattering by periodic surfaces, and include a new relation akin to the optical theorem for obstacle scattering.

All of our results are derived for acoustic waves in two dimensions. However, we anticipate generalizations to three dimensions, to elastic waves, and to penetrable boundaries.

2. Scattering by an infinite rough surface: Introduction

Consider the scattering of a plane wave by an infinite rough surface, S. We assume that the surface is one-dimensional, so that it can be described by

$$z = s(x), \quad -\infty < x < \infty$$

with $-h < s(x) \leq 0$ for some constant $h \geq 0$. The acoustic medium occupies $z > s$. For definiteness, we assume that S is a smooth, sound-hard surface, so that $s(x)$ is a differentiable function (although this condition can be weakened). Thus, we can write the total field as

$$u_{\text{tot}} = u_{\text{inc}} + u,$$

where u is the scattered field and

$$u_{\text{inc}}(r, \theta) = \mathrm{e}^{\mathrm{i}k(x \sin\theta_{\mathrm{i}} - z\cos\theta_{\mathrm{i}})} = \mathrm{e}^{-\mathrm{i}kr\cos(\theta+\theta_{\mathrm{i}})}, \quad |\theta_{\mathrm{i}}| < \tfrac{1}{2}\pi, \tag{5}$$

is the incident plane wave; k is the positive wave number, θ_{i} is the angle of incidence (it is the angle between the direction of propagation and the negative z-axis), and (r, θ) are plane polar coordinates: $x = r\sin\theta$ and $z = r\cos\theta$. The boundary condition is

$$\frac{\partial u_{\text{tot}}}{\partial n} = 0 \quad \text{on } S, \tag{6}$$

where $\partial/\partial n$ denotes normal differentiation *out* of the acoustic medium. The scattered field u must satisfy the Helmholtz equation,

$$(\nabla^2 + k^2)u = 0 \tag{7}$$

in $z > s$, and it must represent outgoing waves at infinity.

If S is flat ($s = 0$), we know that

$$u(r, \theta) = \mathrm{e}^{\mathrm{i}k(x\sin\theta_{\mathrm{i}} + z\cos\theta_{\mathrm{i}})} = \mathrm{e}^{\mathrm{i}kr\cos(\theta-\theta_{\mathrm{i}})}. \tag{8}$$

Note that this field does not satisfy the Sommerfeld radiation condition,

$$\sqrt{r}\left(\frac{\partial u}{\partial r} - \mathrm{i}ku\right) \to 0 \quad \text{as } r \to \infty, \tag{9}$$

and so this radiation condition is inappropriate for problems where a plane wave is scattered by an infinite (rough) surface.

More generally, consider the scattered field above the corrugations, $z > 0$. It is customary to write this field using an *angular-spectrum representation*,

$$u(x, z) = \int_{-\infty}^{\infty} F(\mu)\mathrm{e}^{\mathrm{i}k(\mu x + mz)} \frac{\mathrm{d}\mu}{m(\mu)}, \tag{10}$$

where $F(\mu)$ is the *spectral amplitude*, and

$$m(\mu) = \begin{cases} \sqrt{1-\mu^2}, & |\mu| < 1, \\ \mathrm{i}\sqrt{\mu^2 - 1}, & |\mu| > 1. \end{cases} \tag{11}$$

The integral is a superposition of plane waves; these are propagating, homogeneous plane waves when $|\mu| < 1$, and they are evanescent, inhomogeneous plane waves when $|\mu| > 1$. The definition (11) ensures that all the component waves propagate away from $z = 0$ or decay exponentially with increasing z.

Angular-spectrum representations are discussed at length by Ratcliffe [4], Clemmow [5] and Nieto-Vesperinas [6]. For applications to rough surfaces, see [6, Chap. 7; 7, 8]. For applications to surface water waves, see [9]. The three books [5,6,8] also consider three-dimensional problems for a half-space $z > 0$; the appropriate angular-spectrum representation is

$$u(x, y, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mu, \lambda)\, \mathrm{e}^{\mathrm{i}k(\mu x + \lambda y + mz)} \frac{\mathrm{d}\mu\, \mathrm{d}\lambda}{m}, \tag{12}$$

where $m \equiv m(\sqrt{\mu^2+\lambda^2})$ with $m(\mu)$ defined by (11). Analogous representations for time-dependent fields are discussed in [10,11].

It is worth noting that the free-space Green's function G (in two or three dimensions) has an angular-spectrum representation, known as the *Weyl representation*. For example, in two dimensions, we have [2, Section 2.9.5]

$$G(x,z;\xi,\zeta) = -\tfrac{1}{2}\mathrm{i}H_0^{(1)}\left(k\sqrt{(x-\xi)^2+(z-\zeta)^2}\right)$$
$$= \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \exp\{\mathrm{i}k[\mu(x-\xi)+m(\mu)|z-\zeta|]\}\,\frac{\mathrm{d}\mu}{m}. \tag{13}$$

So, for $z > \zeta$, G can be written as (10), with

$$F(\mu) = (2\pi\mathrm{i})^{-1}\exp\{-\mathrm{i}k(\mu\xi+m\zeta)\}.$$

3. Properties of angular-spectrum representations

The spectral amplitude F can be determined in terms of the field or its normal derivative on the plane $z = 0$. Thus, from (10), we have

$$N(x) \equiv \frac{1}{\mathrm{i}k}\frac{\partial u}{\partial z}(x,0) = \int_{-\infty}^{\infty} F(\mu)\mathrm{e}^{\mathrm{i}k\mu x}\,\mathrm{d}\mu,$$

which is a Fourier transform. Inverting this gives

$$F(\mu) = \frac{k}{2\pi}\int_{-\infty}^{\infty} N(x)\mathrm{e}^{-\mathrm{i}k\mu x}\,\mathrm{d}x. \tag{14}$$

Indeed, the standard derivation of (10) uses a Fourier transform of (7) with respect to x.

If $N(x)$ has compact support, so that it vanishes for all $|x| > X$, say, we know that $F(\mu)$ is an entire function of the complex variable μ. In particular, $F(\mu)$ is a continuous function of μ.

On the other hand, when a plane wave is scattered by an infinite surface, F *cannot* be a continuous function. To see this, consider again the flat surface ($s = 0$), for which u is given by (8): from (14) and

$$\delta(\mu) = \frac{k}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}k\mu x}\,\mathrm{d}x,$$

where δ is the Dirac delta-function, we obtain

$$F(\mu) = \delta(\mu - \sin\theta_\mathrm{i})\cos\theta_\mathrm{i}.$$

This shows that we should treat F as a generalized function; see also [9, p.387].

As we noted in Section 1, there are some applications of angular-spectrum representations in which $F(\mu)$ is considered as an analytic function of μ in an appropriate cut plane, and (10) is replaced by a contour integral. If a discrete set of plane waves is to be represented, $F(\mu)$ will have simple poles on the real μ-axis. However, if u consists *solely* of this discrete set of plane waves, $F(\mu)$ *cannot* be analytic apart from poles. In particular, as the flat-surface case shows, the assumption of analyticity is too restrictive for most of the problems of interest to us.

Some examples where F has a simple-pole singularity are discussed in [5, Section 3.3]; there are also situations where F has poles off the real axis.

Two questions arise naturally. First, given the spectral amplitude F, what is the far-field behaviour of u? Second, given the problem of plane-wave scattering by an infinite rough surface, what can be said about the smoothness of F? We consider these in turn.

3.1. Far-field behaviour

The first question is well known. If $F(\mu)$ is a *continuous* (bounded) function of the real variable μ, then u behaves like an outgoing cylindrical wave,

$$u(x, z) \sim \mathrm{e}^{\mathrm{i}kr}/\sqrt{r} \quad \text{as } r = \sqrt{x^2 + z^2} \to \infty.$$

Such fields, decaying with r and propagating outwards, do satisfy the Sommerfeld radiation condition (9). This result is derived in [5, Section 3.2]. In fact, if we neglect the evanescent components ($|\mu| > 1$), we have

$$u(r, \theta) \simeq \int_{-\pi/2}^{\pi/2} F(\sin\varphi)\, \mathrm{e}^{\mathrm{i}kr\cos(\theta-\varphi)}\, \mathrm{d}\varphi \sim \sqrt{\frac{2\pi}{kr}}\mathrm{e}^{\mathrm{i}(kr-\pi/4)}\, F(\sin\theta) \quad \text{as } kr \to \infty \tag{15}$$

using the method of stationary phase [12, p.220].

For three-dimensional problems, u (defined by (12)) behaves like an outgoing spherical wave,

$$u(x, y, z) \sim \mathrm{e}^{\mathrm{i}kR}/R \quad \text{as } R = \sqrt{x^2 + y^2 + z^2} \to \infty.$$

This result can be found in [5, p.44], but a much more complete and rigorous derivation has been given in [3]. (Results from [3] are quoted in [6, Section 2.12], but without mentioning the crucial continuity constraint on F).

If F has a singularity, the above results are not applicable. Various asymptotic results are available for certain types of singularity. For example, if $F(\mu)$ is an analytic function of μ, apart from poles, the method of steepest descent may be used; see [12, Chap. 7] or [5, Section 3.3].

3.2. Smoothness of F

For the second question, we start with a negative result.

Theorem 1. *Suppose that a plane wave, defined by*

$$u_{\mathrm{inc}}(r, \theta) = \mathrm{e}^{-\mathrm{i}kr\cos(\theta+\theta_{\mathrm{i}})} \quad \textit{with } |\theta_{\mathrm{i}}| < \tfrac{1}{2}\pi$$

is incident upon an infinite, sound-hard, rough surface. Then, the scattered field u does not have an angular-spectrum representation with a continuous spectral amplitude.

Proof. Assume that u can be represented as (10), where F is continuous. Thus, u is a cylindrical wave, with far-field asymptotics given by (15). We show that this assumption leads to a contradiction, using an energy argument.

The time-averaged flux of energy through a surface $\mathcal{S}$ is (proportional to)

$$E(\mathcal{S}) = \mathrm{Im} \int_{\mathcal{S}} u_{\mathrm{tot}} \frac{\partial u_{\mathrm{tot}}^*}{\partial n}\, \mathrm{d}s = \frac{1}{2\mathrm{i}} \int_{\mathcal{S}} \left(u_{\mathrm{tot}} \frac{\partial u_{\mathrm{tot}}^*}{\partial n} - u_{\mathrm{tot}}^* \frac{\partial u_{\mathrm{tot}}}{\partial n} \right) \mathrm{d}s,$$

where asterisk denotes complex conjugation. We take $\mathcal{S} = H_r \cup S_r$, a closed curve, where

$$S_r = \{(x, z) : z = s(x),\ |x| \le r\}$$

is a truncated rough surface and H_r is a semicircle of radius r. (H_r may be a little larger than a semicircle as the rough surface has $s(x) \leq 0$; however, this will not affect the argument below.)

Applying Green's theorem in the region bounded by $\mathcal{S}$ to u_{tot} and u^*_{tot} shows that

$$E(\mathcal{S}) = E(H_r) + E(S_r) = 0.$$

But the homogeneous boundary condition on S_r, namely (6), implies that $E(S_r) = 0$. (Other common boundary conditions, such as $u_{\text{tot}} = 0$, could also be imposed.) Hence,

$$0 = E(H_r) = [u_{\text{tot}}; u_{\text{tot}}],$$

where, by definition,

$$[u; v] = \frac{1}{2\mathrm{i}} \int_{H_r} \left(u \frac{\partial v^*}{\partial r} - v^* \frac{\partial u}{\partial r} \right) r \, \mathrm{d}\theta \simeq \frac{r}{2\mathrm{i}} \int_{-\pi/2}^{\pi/2} \left(u \frac{\partial v^*}{\partial r} - v^* \frac{\partial u}{\partial r} \right) \mathrm{d}\theta. \tag{16}$$

As $u_{\text{tot}} = u_{\text{inc}} + u$, we obtain

$$0 = [u_{\text{inc}}; u_{\text{inc}}] + [u; u_{\text{inc}}] + [u_{\text{inc}}; u] + [u; u]. \tag{17}$$

Substituting from (5) and integrating over θ gives

$$[u_{\text{inc}}; u_{\text{inc}}] = 2kr \cos \theta_{\text{i}}, \tag{18}$$

which is unbounded as the radius of $H_r, r \to \infty$. (Recall that $|\theta_{\text{i}}| < \frac{1}{2}\pi$, so that we do not consider grazing incidence.)

Now, assume that u behaves like (15) for large r, where F is a continuous function. We find that

$$[u; u] \sim -2\pi \int_{-\pi/2}^{\pi/2} |F(\sin\theta)|^2 \, \mathrm{d}\theta \quad \text{as } r \to \infty. \tag{19}$$

Similarly,

$$[u; u_{\text{inc}}] \sim \sqrt{\frac{\pi kr}{2}} \mathrm{e}^{\mathrm{i}(kr - \pi/4)} \int_{-\pi/2}^{\pi/2} F(\sin\theta)[\cos(\theta + \theta_{\text{i}}) - 1] \mathrm{e}^{\mathrm{i}kr\cos(\theta+\theta_{\text{i}})} \, \mathrm{d}\theta \tag{20}$$

for large kr. The integral can be estimated using the method of stationary phase. The stationary-phase points are given by

$$\theta + \theta_{\text{i}} = 0, \pm\pi,$$

provided that θ is in the range of integration. When $\theta = -\theta_{\text{i}}$, we have $\cos(\theta + \theta_{\text{i}}) = 1$ which implies a contribution of $\mathrm{O}((kr)^{-1})$ to the integral. If $\theta = -\theta_{\text{i}} \pm \pi$ is in the range of integration, it gives a contribution of $\mathrm{O}((kr)^{-1/2})$; for (20), it is not, and so

$$[u; u_{\text{inc}}] = [u_{\text{inc}}; u]^* = \mathrm{o}(1) \quad \text{as } kr \to \infty.$$

When this result, (18) and (19) are substituted into the right hand side of (17), we find that (17) cannot be satisfied. Thus, we have a contradiction. □

3.3. Discussion

General results, based on energy considerations, are well known in scattering theory. Waterman [13] gives a systematic study for periodic surfaces. Voronovich [8, Section 2.3] considers infinite rough surfaces, but his analysis is incomplete: his closed curve S comprises S_r, a straight line L_r at height $z = z_0$, and two line segments at $x = \pm r$; the contribution due to these segments is supposed to vanish as $r \to \infty$, assuming that u and u_{inc} decay suitably – but these fields do not decay, and so a more careful analysis is required.

For scattering by a *bounded* obstacle, consideration of (17) leads to a known result. Thus, we replace H_r by a large circle surrounding the obstacle. We find that $[u_{\text{inc}}; u_{\text{inc}}] = 0$, and that one stationary-phase point contributes to (20). The result is

$$\operatorname{Re}\{F(\sin\theta_{\mathrm{i}})\} + \tfrac{1}{2}\int_{-\pi}^{\pi} |F(\sin\theta)|^2 \, d\theta = 0, \tag{21}$$

which is known as the *optical theorem* or the *forward-scattering theorem.*

3.4. A conjecture

Theorem 1 suggests that we examine the following conjecture.

Conjecture : *Suppose that a plane wave, defined by* (5), *is incident upon an infinite, sound-hard, rough surface. Then, the scattered field u contains at least one propagating, plane-wave component.*

If this conjecture is true, Theorem 1 would follow as a corollary. Physically, the conjecture seems to be obvious: the energy in the incident wave, coming from infinity, must be reflected back to infinity as it cannot pass through the rough surface. However, some books on scattering by rough surfaces suggest that only the incoherent field is worthy of our attention. (For example, in [14, p.5], one finds: 'For very rough surfaces the field is totally diffuse'.)

The conjecture is true trivially for a flat surface, when the scattered field is simply the specular plane wave (8). It is also true for *periodic* surfaces (gratings); such surfaces can support a finite number of propagating plane waves,

$$u_n(r, \theta) = e^{ikr\cos(\theta - \theta_n)},$$

where the angles θ_n satisfy $|\theta_n| \le \frac{1}{2}\pi$ and the Bragg equation

$$\sin\theta_n = \sin\theta_{\mathrm{i}} + 2n\pi/(kd),$$

and d is the period of the surface: $s(x + jd) = s(x)$ for any integer j. In general, there will be $(N_1 + N_2 + 1)$ real values of θ_n, with $\theta_0 = \theta_{\mathrm{i}}$, $N_1 \ge 0$, $N_2 \ge 0$ and $-N_1 \le n \le N_2$, so that we can write the propagating part of the scattered field as

$$\sum_{n=-N_1}^{N_2} A_n u_n(r, \theta),$$

where A_n are coefficients. The values of N_1 and N_2 will depend on θ_{i} and kd. However, there is always at least one propagating wave, namely the specular wave, u_0. For more information on scattering by periodic surfaces, see [13] and references therein.

The difficulty in establishing the conjecture, rigorously, for general surfaces $s(x)$ is that we need to know something about the far-field behaviour of u which in turn implies that we have a complete representation for u.

In Section 4, we assume a rather general representation for u. It implies that the conjecture is true. It also leads to some constraints on the terms in the representation, generalizing some results of Waterman [13] and others for scattering by periodic surfaces.

4. Scattering by an infinite rough surface

Let us represent the field in $z > 0$ above a rough surface $z = s(x) \le 0$ using an angular-spectrum representation. We assume that the total field u_{tot} can be decomposed into four parts,

$$u_{\mathrm{tot}} = u_{\mathrm{inc}} + u_{\mathrm{pr}} + u_{\mathrm{ev}} + u_{\mathrm{con}}. \tag{22}$$

The second part consists of N reflected homogeneous plane waves u_n (propagating in N discrete directions θ_n, where $-N_1 \le n \le N_2$, $N = N_1 + N_2 + 1$, and $|\theta_n| < \frac{1}{2}\pi$). Thus

$$u_{\mathrm{pr}}(r,\theta) = \sum_{n=-N_1}^{N_2} A_n u_n(r,\theta),$$

where

$$u_n(r,\theta) = \mathrm{e}^{ikr\cos(\theta-\theta_n)} = \exp\left\{ikr\left(\mu_n \sin\theta + \sqrt{1-\mu_n^2}\,\cos\theta\right)\right\},$$

$\mu_n = \sin\theta_n$ and $|\mu_n| < 1$. Without loss of generality, we take

$$\theta_0 = \theta_{\mathrm{i}}.$$

The third part in (22) consists of a discrete sum of M evanescent waves,

$$u_{\mathrm{ev}}(r,\theta) = \sum_{n=-M_1}^{-N_1-1} B_n v_n(r,\theta) + \sum_{n=N_2+1}^{M_2} B_n v_n(r,\theta), \tag{23}$$

where $M = M_1 + M_2 - N + 1$,

$$v_n(r,\theta) = \exp\left\{kr\left(\mathrm{i}\mu_n \sin\theta - \sqrt{\mu_n^2 - 1}\,\cos\theta\right)\right\} \quad \text{and } |\mu_n| > 1. \tag{24}$$

Note that, for simplicity, we do not include grazing waves in u_{pr} or u_{ev} ($|\mu_n| = 1$). (Our notation in (23) seems cumbersome, but it will facilitate comparison later with known results for periodic surfaces.)

The fourth part in (22) consists of a continuous spectrum of plane waves,

$$u_{\mathrm{con}}(r,\theta) = \int_{-\infty}^{\infty} F(\mu)\mathrm{e}^{ikr(\mu\sin\theta + m\cos\theta)}\,\frac{\mathrm{d}\mu}{m(\mu)}.$$

The coefficients A_n and B_n, and the continuous function F are all unknown. Moreover, for a non-periodic surface, we know neither the integers N and M nor the real numbers μ_n. (For a periodic surface, these are given by the Bragg equation.) Nevertheless, we can deduce some constraints that must be satisfied by these unknowns, using energy arguments.

Exactly as in Section 3.2, we obtain

$$
\begin{aligned}
0 = [u_{\text{tot}}; u_{\text{tot}}] &= [u_{\text{inc}}; u_{\text{inc}}] + [u_{\text{pr}}; u_{\text{pr}}] + [u_{\text{ev}}; u_{\text{ev}}] + [u_{\text{con}}; u_{\text{con}}] \\
&\quad + 2\,\text{Re}\{[u_{\text{inc}}; u_{\text{pr}}] + [u_{\text{pr}}; u_{\text{ev}}] + [u_{\text{ev}}; u_{\text{con}}] + [u_{\text{con}}; u_{\text{inc}}] + [u_{\text{inc}}; u_{\text{ev}}] + [u_{\text{pr}}; u_{\text{con}}]\},
\end{aligned} \tag{25}
$$

where $[u; v]$ is defined by (16). As F is continuous, we know that $u_{\text{con}}(r, \theta)$ will behave like (15) for large kr, whence

$$
[u_{\text{con}}; u_{\text{con}}] = -2\pi \int_{-\pi/2}^{\pi/2} |F(\sin\theta)|^2 \, \mathrm{d}\theta + \mathrm{o}(1) \quad \text{as } kr \to \infty. \tag{26}
$$

Similarly,

$$
\begin{aligned}
[u_{\text{con}}; u_{\text{pr}}] &\sim -\sqrt{\frac{\pi kr}{2}} \mathrm{e}^{\mathrm{i}(kr-\pi/4)} \sum_n A_n^* \int_{-\pi/2}^{\pi/2} F(\sin\theta)\,[\cos(\theta-\theta_n)+1]\,\mathrm{e}^{-\mathrm{i}kr\cos(\theta-\theta_n)}\,\mathrm{d}\theta \\
&= -2\pi \sum_{n=-N_1}^{N_2} A_n^* F(\sin\theta_n) + \mathrm{o}(1) \quad \text{as } kr \to \infty
\end{aligned} \tag{27}
$$

using the method of stationary phase. From Section 3.2, we also have

$$
[u_{\text{inc}}; u_{\text{inc}}] = 2kr\cos\theta_{\text{i}} \quad \text{and} \quad [u_{\text{con}}; u_{\text{inc}}] = \mathrm{o}(1)
$$

as $kr \to \infty$. An integration by parts shows that

$$
[u_{\text{ev}}; u_{\text{con}}] = \mathrm{O}((kr)^{-1/2}) \quad \text{as } kr \to \infty.
$$

The five remaining terms in (25) can be evaluated exactly. We find that (see the Appendix)

$$
[u_m; u_n] = -(\delta_m + \delta_n)\, \mathcal{S}(kr, \mu_m - \mu_n), \tag{28}
$$

$$
[v_m; u_n] = -(\mathrm{i}\gamma_m + \delta_n)\, \mathcal{S}(kr, \mu_m - \mu_n), \tag{29}
$$

$$
[v_m; v_n] = -\mathrm{i}(\gamma_m - \gamma_n)\, \mathcal{S}(kr, \mu_m - \mu_n), \tag{30}
$$

$$
[u_m; u_{\text{inc}}] = -(\delta_m - \cos\theta_{\text{i}})\, \mathcal{S}(kr, \mu_m - \sin\theta_{\text{i}}), \tag{31}
$$

$$
[v_m; u_{\text{inc}}] = -(\mathrm{i}\gamma_m - \cos\theta_{\text{i}})\, \mathcal{S}(kr, \mu_m - \sin\theta_{\text{i}}), \tag{32}
$$

where $\gamma_m = \sqrt{\mu_m^2 - 1}$ and $\delta_n = \sqrt{1 - \mu_n^2} = \cos\theta_n$ are real and positive,

$$
\mathcal{S}(kr, \mu) = \mu^{-1}\sin(kr\mu) \quad \text{and} \quad \mathcal{S}(kr, 0) = kr. \tag{33}
$$

Note that

$$
[u_n; u_n] = -2kr\cos\theta_n, \qquad [v_n; v_n] = 0 \qquad [u_0; u_{\text{inc}}] = 0.
$$

Hence,

$$
[u_{\text{pr}}; u_{\text{pr}}] = -2kr \sum_n |A_n|^2 \cos\theta_n + \sum_{\substack{m,n \\ m \neq n}} A_m A_n^* \,[u_m; u_n],
$$

$$
[u_{\text{ev}}; u_{\text{ev}}] = \sum_{m,n} B_m B_n^* \,[v_m; v_n], \qquad [u_{\text{pr}}; u_{\text{ev}}] = \sum_{m,n} A_m B_n^* \,[u_m; v_n],
$$

$$
[u_{\text{pr}}; u_{\text{inc}}] = \sum_m A_m \,[u_m; u_{\text{inc}}], \qquad [u_{\text{ev}}; u_{\text{inc}}] = \sum_m B_m \,[v_m; u_{\text{inc}}].
$$

Let us substitute all these results into (25). For the terms of $O(kr)$ to balance, we must have

$$\cos\theta_{\mathrm{i}} - \sum_{n=-N_1}^{N_2} |A_n|^2 \cos\theta_n = 0. \tag{34}$$

This result is well known for scattering by periodic surfaces [13]. Moreover, if all the reflected propagating plane waves are absent ($A_n = 0$ for all n), (34) cannot be satisfied ($|\theta_{\mathrm{i}}| < \frac{1}{2}\pi$), implying that the conjecture in Section 3.4 is true.

The terms of $O(1)$ will balance provided that

$$\sum_{n=-N_1}^{N_2} \mathrm{Re}\,\{A_n^* F(\sin\theta_n)\} + \tfrac{1}{2}\int_{-\pi/2}^{\pi/2} |F(\sin\theta)|^2\,\mathrm{d}\theta = 0, \tag{35}$$

which is reminiscent of the optical theorem (21) for obstacle scattering. Note that this relation, which appears to be new, is satisfied trivially for periodic surfaces ($F \equiv 0$).

The remaining terms in (25) involve the function $\mathcal{S}(kr, \mu)$ with various values of $\mu \neq 0$. Explicitly, these terms combine to give

$$0 = \sum_{\substack{m,n \\ m\neq n}} A_m A_n^* [u_m; u_n] + \sum_{m,n} B_m B_n^* [v_m; v_n]$$
$$+ 2\,\mathrm{Re}\left\{\sum_n A_n [u_n; u_{\mathrm{inc}}] + \sum_{m,n} A_m B_n^* [u_m; v_n] + \sum_n B_n [v_n; u_{\mathrm{inc}}]\right\}.$$

To simplify this equation, write

$$C_n = \begin{cases} A_n, & -N_1 \le n \le N_2, \\ B_n, & \text{otherwise}, \end{cases} \qquad w_n = \begin{cases} u_n, & -N_1 \le n \le N_2, \\ v_n, & \text{otherwise}, \end{cases}$$

whence

$$0 = 2\,\mathrm{Re}\sum_{\substack{n \\ n\neq 0}} C_n [w_n; u_{\mathrm{inc}}] + \sum_{\substack{m,n \\ m\neq n}} C_m C_n^* [w_m; w_n],$$

where we have used $[u_m; v_n]^* = [v_n; u_m]$. We have

$$[w_m; w_n] = c_{mn}\,\mathcal{S}(kr, \mu_m - \mu_n) \quad \text{and} \quad [w_n; u_{\mathrm{inc}}] = c_n^{\mathrm{i}}\,\mathcal{S}(kr, \mu_n - \mu_0),$$

where the coefficients c_{mn} and c_n^{i} can be read off from (28)–(32). As $\mathcal{S}(kr, \mu)$ is an even function of μ, we have

$$0 = 2\,\mathrm{Re}\sum_{n=1}(c_n^{\mathrm{i}} C_n\,\mathcal{S}(kr, \mu_n - \mu_0) + c_{-n}^{\mathrm{i}} C_{-n}\,\mathcal{S}(kr, \mu_0 - \mu_{-n}))$$
$$+ \sum_{\substack{m,n \\ m>n}} (c_{mn} C_m C_n^* + c_{nm} C_n C_m^*)\mathcal{S}(kr, \mu_m - \mu_n). \tag{36}$$

We now reorder the summation. Assume that $\mu_m > \mu_n$ whenever $m > n$. Let

$$\mathcal{E} = \{\zeta_l > 0\colon \zeta_l \doteq \mu_m - \mu_n \text{ for some } m \text{ and } n, m > n\}$$

be the set of all allowable distinct values of $\mu_m - \mu_n$, $m > n$. This set will have L elements, ζ_l, $l = 1, 2, \ldots, L$. Then, we can rewrite (36) as

$$0 = \sum_{l=1}^{L} Q_l \, \mathcal{S}(kr, \zeta_l), \tag{37}$$

where Q_l is a quadratic combination of C_m. As (37) must hold for all kr, we deduce that $Q_l = 0$ for each l, yielding L constraints on the coefficients A_n and B_n. We cannot make these constraints more explicit without specifying the numbers μ_n. Below, we give a simple example with $N = 3$ and $M = 0$, and the specialization to periodic surfaces.

4.1. A simple example

For simplicity, suppose $N = 3$ and $M = 0$, with

$$-1 < \mu_{-1} < \mu_0 = \sin\theta_{\mathrm{i}} < \mu_1 < 1,$$

so that there are three reflected propagating plane waves and no evanescent waves. Let us assume that the numbers

$$\zeta_1 = \mu_1 - \mu_0, \qquad \zeta_2 = \mu_1 - \mu_{-1} \qquad \zeta_3 = \mu_0 - \mu_{-1}$$

are distinct ($L = 3$). Then, we find the following *three* constraints:

$$(\cos\theta_n - \cos\theta_{\mathrm{i}})\,\mathrm{Re}\,\{A_n\} + (\cos\theta_n + \cos\theta_{\mathrm{i}})\mathrm{Re}\,\{A_n A_0^*\} = 0 \quad \text{for } n = \pm 1 \quad \text{and} \quad \mathrm{Re}\,\{A_1 A_{-1}^*\} = 0; \tag{38}$$

these are to be supplemented with (34). Thus, in this (non-degenerate) case, we have found four real constraints on the three complex coefficients, A_{-1}, A_0 and A_1.

If $\mu_{-1} = -\mu_1$, then we obtain only *two* constraints from (37): we have

$$\zeta_1 = \mu_1 - \mu_0 \quad \text{and} \quad \zeta_2 = 2\mu_1 \qquad (L = 2)$$

giving (38) and

$$(\cos\theta_1 - \cos\theta_{\mathrm{i}})\mathrm{Re}\,\{A_{-1} + A_1\} + (\cos\theta_1 + \cos\theta_{\mathrm{i}})\mathrm{Re}\,\{(A_{-1} + A_1)A_0^*\} = 0.$$

4.2. Periodic surfaces

For periodic surfaces, the Bragg equation gives

$$\mu_n = \mu_0 + n\lambda, \qquad \text{where } \lambda = 2\pi/(kd).$$

Hence, $\zeta_l = l\lambda, l = 1, 2, \ldots$ so that $L = \infty$. Thus, we obtain an infinite number of constraints, $Q_l = 0$, where

$$Q_l = 2\,\mathrm{Re}\,(c_l^{\mathrm{i}} C_l + c_{-l}^{\mathrm{i}} C_{-l}) + \sum_{n=-\infty}^{\infty} (c_{n+l,n} C_{n+l} C_n^* + c_{n,n+l} C_n C_{n+l}^*).$$

We suspect that these constraints are known, although we have been unable to deduce them from Waterman's results [13].

5. Discussion

Another way to solve the problem of the scattering of a plane wave by an infinite rough surface is to use a boundary integral equation (BIE). To derive such a BIE is not straightforward. For obstacle scattering, there are two well-known possibilities, both of which may be used here.

First, one might seek a solution for u as a distribution of wave sources over the surface S,

$$u(x,z)=\int_{-\infty}^{\infty}\nu(\xi)G(x,z;\xi,s(\xi))\,\mathrm{d}\xi,$$

where ν is an unknown source density and G is defined by (13); application of the boundary condition on S leads to a BIE for ν. However, we see immediately that non-uniform behaviour must be expected, for if ν has compact support, u will behave as a cylindrical wave in the far field whereas we know that u must include a propagating plane wave.

Second, one might attempt to derive a BIE for the boundary values of u (analogous to the Helmholtz integral equation for obstacle scattering). This implies that one has to address the problem of the radiation condition. In a subsequent paper, we will do this, making use of our results for angular-spectrum representations.

Acknowledgements

PAM acknowledges receipt of a Fulbright Scholarship Grant. He also thanks the Department of Mathematical and Computer Sciences, Colorado School of Mines, for its kind hospitality.

Appendix

The evaluations of (28)–(32) are similar, so we describe one of them here, namely $[v_m;v_n]$. $[u;v]$ and v_n are defined by (16) and (24), respectively. Hence,

$$[v_m;v_n]=\frac{1}{2\mathrm{i}}\int_{-\pi/2}^{\pi/2}\Psi(\theta)\mathrm{e}^{\Phi(\theta)}\,\mathrm{d}\theta,$$

where

$$\Phi(\theta)=kr\{\mathrm{i}(\mu_m-\mu_n)\sin\theta-(\gamma_m+\gamma_n)\cos\theta\},$$
$$\Psi(\theta)=kr\{-\mathrm{i}(\mu_m+\mu_n)\sin\theta+(\gamma_m-\gamma_n)\cos\theta\},$$

$\gamma_m=\sqrt{\mu_m^2-1}$ and $\gamma_n=\sqrt{\mu_n^2-1}$. We have

$$\begin{aligned}\Phi'(\theta)&=kr\{\mathrm{i}(\mu_m-\mu_n)\cos\theta+(\gamma_m+\gamma_n)\sin\theta\}\\&=\mathrm{i}kr\frac{\gamma_m+\gamma_n}{\mu_m+\mu_n}\left\{-\mathrm{i}(\mu_m+\mu_n)\sin\theta+\frac{\mu_m^2-\mu_n^2}{\gamma_m+\gamma_n}\cos\theta\right\}.\end{aligned}$$

But $\gamma_m^2-\gamma_n^2=\mu_m^2-\mu_n^2$, whence

$$\Phi'(\theta)=\mathrm{i}\frac{\gamma_m+\gamma_n}{\mu_m+\mu_n}\Psi(\theta).$$

Thus

$$\begin{aligned}[v_m; v_n] &= \frac{-1}{2}\frac{\mu_m+\mu_n}{\gamma_m+\gamma_n}\left(\exp\left\{\Phi\left(\tfrac{1}{2}\pi\right)\right\} - \exp\left\{\Phi\left(-\tfrac{1}{2}\pi\right)\right\}\right) \\ &= -\mathrm{i}\frac{\mu_m+\mu_n}{\gamma_m+\gamma_n}\sin\left(kr(\mu_m-\mu_n)\right) \\ &= -\mathrm{i}(\gamma_m-\gamma_n)\mathcal{S}(kr, \mu_m-\mu_n),\end{aligned}$$

where $\mathcal{S}$ is defined by (33).

References

[1] J.D. Achenbach, *Wave Propagation in Elastic Solids*, North-Holland, Amsterdam (1973).
[2] J.A. DeSanto, *Scalar Wave Theory*, Springer, Berlin (1992).
[3] G.C. Sherman, J.J. Stamnes and É. Lalor, "Asymptotic approximations to angular-spectrum representations", *J. Math. Phys. 17*, 760–776 (1976).
[4] J.A. Ratcliffe, "Some aspects of diffraction theory and their application to the ionosphere", *Repts. Progr. Phys. 19*, 188–267 (1956).
[5] P.C. Clemmow, *The Plane Wave Spectrum Representation of Electromagnetic Fields*, Pergamon Press, Oxford (1966).
[6] M. Nieto-Vesperinas, *Scattering and Diffraction in Physical Optics*, Wiley, New York (1991).
[7] J.A. DeSanto, "Exact spectral formalism for rough-surface scattering", *J. Opt. Soc. Amer. A 2*, 2202–2207 (1985).
[8] A.G. Voronovich, *Wave Scattering from Rough Surfaces*, Springer, Berlin (1994).
[9] R.A. Dalrymple and J.T. Kirby, "Angular spectrum modeling of water waves", *Rev. Aquatic Sciences 6*, 383–404 (1992).
[10] A.J. Devaney and G.C. Sherman, "Plane-wave representations for scalar wave fields", *SIAM Rev. 15*, 765–786 (1973).
[11] E. Heyman, "Time-dependent plane-wave spectrum representations for radiation from volume source distributions", *J. Math. Phys. 37*, 658–681 (1996).
[12] N. Bleistein and R.A. Handelsman, *Asymptotic Expansions of Integrals*, Dover, New York (1986).
[13] P.C. Waterman, "Scattering by periodic surfaces", *J. Acoust. Soc. Amer. 57*, 791–802 (1975).
[14] J.A. Ogilvy, *Theory of Wave Scattering from Random Rough Surfaces*, Adam Hilger, Bristol (1991).