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Scattering of water waves by a submerged disc using a hypersingular integral equation

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Abstract

The three-dimensional interaction between water waves and a submerged disc, in deep water, is considered. The problem is reduced to a hypersingular integral equation over the surface of the disc. The integral equation is solved numerically using an expansion–collocation method, generalizing a method used previously by Parsons and Martin for several two-dimensional water-wave problems. This method is shown to be very effective: it incorporates the known behaviour near the edge of the disc and it permits all hypersingular (finite-part) inegrals to be evaluated analytically. Numerical results are presented, with emphasis on the scattering properties of the submerged disc. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords: Water waves; Submerged disc; Hypersingular integral equation

1. Introduction

Nowadays, the interaction of surface water waves with immersed structures is often calculated using sophisticated boundary-element codes. Such codes are based on boundary integral equations of a well-studied type. However, these codes are not well suited to structures that have some thin components such as plates, which are subject to fluid loading on both of their sides: thin plates inevitably imply hypersingular integral equations.

In this paper, we consider the interaction of water waves with a thin rigid plate, in three dimensions. The plate is modelled as an open surface S, and is submerged in deep water. The velocity potential ϕ is discontinuous across S by an amount $[\phi]$, which is unknown; it is $[\phi]$ that solves a hypersingular integral equation, as shown in Section 2.

Before describing our treatment of this equation, let us mention previous work. First, there are several papers on *dock problems*, where the plate is flat and located in the undisturbed free surface, so that it is only wetted on one side; see MacCamy [1], Kim [2], Miles and Gilbert [3], Garrett [4], Miles [5,6], Maeda [7] and Farina [8]. Dock problems can be reduced to the solution of a boundary integral equation for ϕ; this equation is a Fredholm integral equation of the second kind.

Second, Parsons and Martin have considered some analogous two-dimensional water-wave problems, leading to one-dimensional hypersingular integral equations for $[\phi]$: these problems are scattering by flat [9] and curved [10] submerged plates, and by surface-piercing plates [10], and the trapping of water waves by submerged plates [11]. They used an expansion–collocation method to solve the one-dimensional hypersingular integral equations, in which $[\phi]$ is expanded using Chebyshev polynomials of the second kind. This method is very effective, and its convergence has been proved by Golberg [12,13] and by Ervin and Stephan, [14] in various function spaces. Ervin and Stephan [14] obtained the rate of convergence in appropriate Sobolev spaces. See also Frenkel [15] and Kaya and Erdogan [16].

Third, there are four articles on submerged plates. Yu and Chwang [17] have used matched eigenfunction expansions for scattering by a submerged horizontal circular disc, in water of finite depth. This work was extended to elliptical discs by Zhang and Williams [18,19]. The method of matched eigenfunction expansions is limited to horizontal discs, and does not incorporate the edge condition ($[\phi] = 0$ around the edge of S) explicitly. These papers contain results for the surface elevation in the vicinity of the disc. Zhang and Williams [19] have also calculated the exciting force and moment.

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Martin and Farina [20] have described a rigorous method for axisymmetric motions of a submerged horizontal circular disc, in deep water. They transformed the governing hypersingular integral equation for $[\phi]$ into a one-dimensional Fredholm integral equation of the second kind for a new unknown function; the new equation is a generalization of Love's integral equation, familiar from the electrostatics of a circular-plate capacitor [21]. Numerical results for the added mass and damping of a heaving disc were obtained.

The methods described in the previous two paragraphs are special and limited in scope. For a completely general method, we could develop a boundary element method for the numerical solution of our two-dimensional hypersingular integral equation over S. Instead, we take an intermediate path, and develop an expansion–collocation method, preserving the nice features of the Chebyshev method for one-dimensional hypersingular integral equations. At this stage, we suppose that S is a flat circular disc. Then, we expand $[\phi]$ in terms of certain functions that are orthogonal over the unit disc: we use a Fourier series in the azimuthal angle, with the Fourier coefficients expanded in terms of associated Legendre functions. This expansion has two virtues: all hypersingular integrals are evaluated analytically and the edge condition is satisfied automatically. Numerical results for two problems have been obtained. For the radiation of waves by a heaving horizontal disc, we find excellent agreement with the results of Martin and Farina [20]; these results are not repeated here. For the scattering problem, where a regular wavetrain is scattered by a fixed disc, we have computed the total scattering cross-section Q and the differential cross-section. (These are measures of the scattered wave energy.) Interesting results are found when the disc is close to the free surface: for example, the graph of Q against wavenumber has sharp peaks; this quasi-resonant behaviour is examined numerically, and is similar to that observed by Martin and Farina [20] in the added-mass curves for a heaving disc.

When using the expansion–collocation method, care must be taken in choosing the collocation points. We describe our experience with this choice, so as to obtain a well-conditioned matrix. At present, we do not have proof that the numerical method is convergent; some remarks on this are made in Section 4.4. In the paper, we limit ourselves to horizontal flat circular plates. The method is easily extended to flat circular plates at other orientations. Moreover, we can also use the method for flat plates of other shapes; the key to this is a knowledge of the conformal mapping between the plate and a circular disc [22]. These extensions will be described elsewhere.

Finally, let us make a few additional remarks. First, we note that submerged plates may find application in coastal engineering, perhaps as components in a breakwater or in a wave-focussing device. Second, we emphasise that we are using an inviscid model. It may be possible to quantify the effects of viscosity in thin-plate problems, but we have not pursued this. We prefer to think of our solution as the limiting solution for scattering by a submerged spheroid as the spheroid degenerates into a disc. Third, we are not aware of any published experimental results for three-dimensional thin-plate problems. However, we can cite the interesting review by Miles [23] of (theoretical and) experimental results for the related problem of scattering by a submerged truncated vertical circular cylinder, in water of finite depth.

2. Formulation

Consider a thin rigid plate, S, completely submerged beneath the free surface of deep water. We assume that S is represented by a smooth open surface with a smooth edge ∂S. We take Cartesian coordinates (x, y, z) with the origin in the mean free surface; the water occupies the region $z < 0$. Linear water-wave theory is employed. Thus, under the usual conditions, the time-harmonic velocity potential is

$$\mathrm{Re}\{\phi(x,y,z)\mathrm{e}^{-i\omega t}\}$$

where ϕ satisfies Laplace's equation in the water,

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0 \tag{1}$$

the linearized free surface condition

$$\frac{\partial\phi}{\partial x^2}-K\phi=0, \text{ on } z=0 \tag{2}$$

and a boundary condition on the plate,

$$\frac{\partial\phi}{\partial n}=V \tag{3}$$

where V is prescribed and $K=\omega^2/g$ is the wavenumber. We also require that ϕ satisfies a radiation condition at infinity,

$$\lim_{r\to\infty} r^{1/2}\left(\frac{\partial\phi}{\partial r}-iK\phi\right)=0 \tag{4}$$

where $r=(x^2+y^2)^{1/2}$.

Let us now introduce the Green's function G, given by

$$G(P;Q)\equiv G(x,y,z;\xi,\eta,\zeta)$$

$$=[R^2+(z-\zeta)^2]^{-1/2}+⨍_0^\infty\frac{k+K}{k-K}\mathrm{e}^{k(z+\zeta)}J_0(kR)\,\mathrm{d}k \tag{5}$$

where $R=[(x-\xi)^2+(y-\eta)^2]^{1/2}$ and J_0 is a Bessel function. This fundamental solution to our problem satisfies Eq. (1), except at $P=Q$ where it has a singularity. G also satisfies Eqs. (2) and (4). By using Green's theorem it is possible to represent ϕ as

$$\phi(P)=\frac{1}{4\pi}\int_S[\phi(q)]\frac{\partial}{\partial n_q}G(P,q)\,\mathrm{d}S_q \tag{6}$$

where P is an arbitrary point in the water. Here

$$[\phi(q)] = \phi(q^+) - \phi(q^-)$$

is the discontinuity in ϕ across the plate, where $q \in S$, q^+ and q^- are corresponding points on S^+ and S^-, respectively, $S^{\pm}$ are two sides of the plate, and $\partial/\partial n_q$, denotes normal differentiation at q in the direction from S^+ into the water. Applying the boundary condition on S^+ gives

$$\frac{1}{4\pi}\frac{\partial}{\partial n_p}\int_S [\phi(q)]\frac{\partial}{\partial n_q}G(P,q)\,\mathrm{d}S_q = V(p^+),\ p \in S \quad (7)$$

The same equation is obtained by applying the boundary condition on S^-; $V(p^-) = -V(p^+)$ as the plate is rigid. The integro-differential equation, Eq. (7), is to be solved subject to the edge condition

$$[\phi] = 0 \text{ on } \partial S \quad (8)$$

ϕ is discontinuous across the plate only.

Interchanging the order of integration and normal differentiation in Eq. (7) produces a hypersingular integral. Such a procedure is proper as long as the resulting integral is then interpreted as a finite-part integral. We obtain

$$\frac{1}{4\pi}\oint_S [\phi]\frac{\partial^2 G(p,q)}{\partial n_p \partial n_q}\,\mathrm{d}S_q = V(p),\ p \in S \quad (9)$$

which is to be solved subject to Eq. (8). The cross indicates that the integral is a finite-part integral; these are defined by Martin and Farina [20] in their Appendix A.

3. The kernel for flat plates

The hypersingular integral equation, Eq. (9), is applicable to smooth plates S of any shape. However, considerable simplification is obtained if S is *flat*. Denote the kernel of Eq. (9) by

$$H = \frac{\partial^2 G}{\partial n_p \partial n_q}$$

It is an explicit but complicated function. Decompose G into its singular and regular parts by $G = G_s + G_r$, where

$$G_s = [R^2 + (z-\zeta)^2]^{-1/2} \text{ and } G_r = G - G_s$$

It will be useful to decompose H similarly as $H = H_s + H_r$.

Let $\mathbf{n}(p) = (n_1, n_2, n_3)$ be the unit normal vector at $p \in S^+$. As S^+ is flat, $\mathbf{n}(q) = \mathbf{n}(p)$. Then we find that

$$\frac{\partial^2 G_s}{\partial n_p \partial n_q} = \frac{1}{|\mathbf{p}-\mathbf{q}|^3} - \frac{3}{|\mathbf{p}-\mathbf{q}|^5}\{\mathbf{n}\cdot(\mathbf{p}-\mathbf{q})\}^2$$

where $\mathbf{p}$ and $\mathbf{q}$ are the position vectors of p and q, respectively. But $\mathbf{p}-\mathbf{q}$ is a vector in the plane of the plate, whence $\mathbf{n}\cdot(\mathbf{p}-\mathbf{q}) = \mathbf{0}$ and

$$H_s = |\mathbf{p}-\mathbf{q}|^{-3} \quad (10)$$

The result of Eq. (10) holds for flat plates with arbitrary orientation. We can calculate H_r for such plates, but the calculation is much simpler when the plate is *horizontal*, as we henceforth assume. In this case, $|\mathbf{p}-\mathbf{q}| = R$.

G_r can be written in the form

$$G_r = \int_0^\infty \frac{k+K}{k-K}\mathrm{e}^{k(z+\zeta)}J_0(kR)\,\mathrm{d}k + 2\pi i K\mathrm{e}^{K(z+\zeta)}J_0(KR) \quad (11)$$

where the integral must be interpreted as a Cauchy principal value. Define dimensionless coordinates X and Z by

$$X = KR \text{ and } Z = -K(z+\zeta) \quad (12)$$

Note that since z and ζ are negative, both X and Z are non-negative. Then, a simple change to the integration variable in Eq. (11) gives

$$G_r = KF(X,Z) + 2\pi i K\mathrm{e}^{-Z}J_0(X) \quad (13)$$

where

$$F(X,Z) = \int_0^\infty \frac{\nu+1}{\nu-1}\mathrm{e}^{-\nu Z}J_0(\nu X)\,\mathrm{d}\nu \quad (14)$$

Note that the semi-infinite integral in Eq. (11), which is related to the main task of the evaluation of G_r, is now expressed as a function F of the two variables X and Z. Using a Laplace transform, it is not difficult to show that [24]

$$F(X,Z) = (X^2+Z^2)^{-1/2} - \pi\mathrm{e}^{-Z}(\mathbf{H}_0(X) + Y_0(X)) - 2\int_0^Z \mathrm{e}^{t-Z}(X^2+t^2)^{-1/2}\,\mathrm{d}t \quad (15)$$

where $\mathbf{H}_0$ is a Struve function and Y_0 is a Bessel function of the second kind.

As the plate is horizontal, we have $\mathbf{n}(p) = \mathbf{n}(q) = (0,0,1)$, whence

$$H_r = \frac{\partial^2 G_r}{\partial z^2} = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)G_r$$

As $J_0(kR)$ satisfies the two-dimensional Helmholtz equation, we find that

$$H_r = \oint_0^\infty \frac{k+K}{k-K}\mathrm{e}^{k(z+\zeta)}k^2 J_0(kR)\,\mathrm{d}k$$

Next, as $k^2 = K^2 + (k-K)(k+K)$, we see that

$$H_r = K^2 G_r + \int_0^\infty (k^2 + 2Kk + K^2)\mathrm{e}^{k(z+\zeta)}J_0(kR)\,\mathrm{d}k$$

(This result can also be obtained by differentiating Eq. (11) twice with respect to z.) The remaining integral can be calculated from

$$\int_0^\infty \mathrm{e}^{-kY}J_0(kR)\,\mathrm{d}k = (R^2+Y^2)^{-1/2},\ Y > 0$$

by differentiation with respect to Y. The result is

$$H_r = K_2 G_r + K^3 \left\{ \frac{3Z^2}{(X^2+Z^2)^{5/2}} + \frac{2Z-1}{(X^2+Z^2)^{3/2}} + \frac{1}{(X^2+Z^2)^{1/2}} \right\} \tag{16}$$

In summary, the hypersingular integral equation, Eq. (9), can be written as

$$\frac{1}{4\pi} \oint_S [\phi(q)] \left\{ \frac{1}{R^3} + H_r(p,q) \right\} \mathrm{d}S_q = V(p), \; p \in S \tag{17}$$

where S is a flat horizontal plate of any shape and H_r is given by Eq. (16). Eq. (17) is to be solved subject to the edge condition, Eq. (8).

4. The expansion–collocation method

4.1. Review of the one-dimensional theory

In two dimensions, many wave problems involving thin plates can be reduced to an equation of the form

$$\oint_{-1}^{1} \left\{ \frac{1}{(x-t)^2} + H(x,t) \right\} v(t) \, \mathrm{d}t = f(x) \text{ for } -1 < x < 1 \tag{18}$$

supplemented by two boundary conditions, which we take to be $v(-1) = v(1) = 0$. Here, v is the unknown function, f is prescribed and the kernel H is known. Assuming that f is sufficiently smooth, the solution v has square-root zeros at the end-points. This suggests that we write

$$v(x) = \sqrt{1-x^2} u(x)$$

Then, we expand u using a set of orthogonal polynomials; a good choice is to use Chebyshev polynomials of the second kind, U_n, defined by

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}, \; n = 0, 1, 2, \cdots$$

This is a good choice because of the formula

$$\frac{1}{\pi} \oint_{-1}^{1} \frac{\sqrt{1-t^2} U_n(t)}{(x-t)^2} \mathrm{d}t = -(n+1) U_n(x) \tag{19}$$

Thus, we approximate u by

$$\sum_{n=0}^{N} a_n U_n(x)$$

substitute into Eq. (18) and evaluate the hypersingularity analytically, using Eq. (19). To find the $(N+1)$ coefficients a_n, we collocate at $(N+1)$ points; good choices are the zeros of T_{N+1} or U_{N+1}, where T_n is a Chebyshev polynomial of the first kind.

4.2. The two-dimensional method

We now describe the method employed for solving the hypersingular integral equation, Eq. (17), when S is a horizontal circular disc. Introduce cylindrical polar coordinates (r, θ, z), so that $x = r\cos\theta$ and $y = r\sin\theta$. Then, the disc is given by

$$S = \{(r, \theta, z) : \; 0 \le r \le a, \; -\pi \le \theta < \pi, \; z = -d\} \tag{20}$$

It has radius a and is submerged at a distance d below the free surface; we can take $a = 1$ without loss of generality.

We will use an expansion–collocation method where the unknown function is expanded into its Fourier series in θ, and then the Fourier components (which depend on r) are expanded using Legendre functions. This approach can be viewed as a generalization of the one-dimensional method for solving one-dimensional hypersingular integral equations using Chebyshev polynomials of the second kind, described above.

If we write $\xi = s\cos\alpha$, $\eta = s\sin\alpha$ and $\zeta = -d$, we have

$$R^3 = [r^2 + s^2 - 2rs\cos(\theta - \alpha)]^{3/2}$$

Hence, we can write Eq. (17) as

$$\frac{1}{4\pi} \oint_S [\phi(s,\alpha)] \left\{ \frac{1}{R^3} + H_r(r,\theta; s,\alpha; d, K) \right\} s \, \mathrm{d}s \mathrm{d}\alpha = V(r,\theta),$$

$$(r, \theta) \in S \tag{21}$$

subject to $[\phi] = 0$ on $r = 1$. Note that the hypersingular part, R^{-3}, does not depend on the submergence depth (or orientation) of the plate. Moreover, all wave effects are included in H_r.

For simplicity, assume that $V(r, \theta)$ is an even function of θ. Then, the integral equation, Eq. (21), implies that $[\phi(r, \theta)]$ is an even function of θ. We shall expand it using the basis functions B_k^m, defined by

$$B_k^m(r,\theta) = P_{m+2k+1}^m(\sqrt{1-r^2})\cos m\theta, \; k, m = 0, 1, \cdots$$

where P_n^m is an associated Legendre function. The radial part of these basis functions can also be expressed in terms of Gegenbauer polynomials.

The functions $\{B_k^m\}$ are orthogonal over the unit disc with respect to the weight $(1 - r^2)^{-1/2}$:

$$\int_S B_k^m(r,\theta) B_l^n(r,\theta) \frac{r \mathrm{d}r \mathrm{d}\theta}{\sqrt{1-r^2}}$$

$$= 2\sigma_m \delta_{mn} \int_0^1 P_{m+2k+1}^m(\rho) P_{m+2l+1}^m(\rho) \, \mathrm{d}\rho$$

$$= \frac{\sigma_m \delta_{mn} \delta_{kl}}{m + 2k + 3/2} O_k^m$$

where δ_{ij} is the Kronecker delta,

$$O_k^m = \frac{(2m+2k+1)!}{(2k+1)!}$$

$\sigma_m = \pi/2$ if $m > 0$ and $\sigma_0 = \pi$; in the last step, we used the fact that the integrand is an even function of ρ and orthogonality relations for the associated Legendre functions [see Erdélyi et al. [25], eqns 3.11(19) and 3.11(21)].

The next formula due to Krenk, [26–28] is essential in the construction of the method:

$$\frac{1}{4\pi}\oint_S \frac{1}{R^3} B_k^m(s,\alpha) s \, ds \, d\alpha = C_k^m \frac{B_k^m(r,\theta)}{\sqrt{1-r^2}} \tag{22}$$

where

$$C_k^m = -\frac{\pi}{4}[P_{m+2k+1}^{m+1}(0)]^2 / O_k^m$$

Eq. (22) is the two-dimensional analogue of Eq. (19). It allows us to evaluate the hypersingular integrals analytically.

To make use of Eq. (22), we expand $[\phi]$ in terms of the functions B_k^m. For brevity, we write

$$[\phi] \approx \sum_{k,m}^{N} a_k^m B_k^m := \sum_{k=0}^{N_1} \sum_{m=0}^{N_2} a_k^m B_k^m \tag{23}$$

Substituting Eq. (23) in the integral equation, Eq. (21), and then evaluating the hypersingular integrals analytically using Eq. (22), we obtain

$$\sum_{k,m}^{N} a_k^m \left\{ C_k^m \frac{B_k^m(r,\theta)}{\sqrt{1-r^2}} + \frac{1}{4\pi} \int_S B_k^m(s,\alpha) H_r(r,\theta;s,\alpha;d,K) s \, ds \, d\alpha \right\} = V(r,\theta),$$
$$(r,\theta) \in S \tag{24}$$

It remains to determine the unknown coefficients a_k^m.

One possible approach is to use a Galerkin method: multiply Eq. (24) by $B_l^n(r,\theta)$ and integrate over S to give

$$a_l^n \frac{\sigma_n C_l^n O_l^n}{n+2l+3/2} + \frac{1}{4\pi} \sum_{k,m}^{N} a_k^m \int_S B_l^n(r,\theta)$$
$$\times \int_S B_k^m(s,\alpha) H_r(r,\theta;s,\alpha;d,K) \, s \, ds \, d\alpha \, r \, dr \, d\theta$$
$$= \int_S V B_l^n \, dS$$

The main disadvantage of this method is the quadruple integral; it is possible to evaluate some of these integrals analytically for certain simple configurations, but we are interested in developing a more general method. Hence, we will use a collocation method, in which evaluation of Eq. (24) at $(N_1+1)(N_2+1)$ points on the disc gives a linear system for the coefficients a_k^m.

Before discussing the collocation method itself, we comment on some other computational aspects of the method. The associated Legendre function can be defined by

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} (d/dx)^m P_l(x) \tag{25}$$

In order to evaluate P_l^m for $l > m$ it is preferable to use a recurrence relation. Since most recurrences on m are unstable, the following recurrence on l, which is stable, is adopted:

$$(l-m)P_l^m(x) = (2l-1)xP_{l-1}^m(x) - (l+m-1)P_{l-2}^m(x) \tag{26}$$

This formula is convenient since we can use the closed-form expression for the starting value,

$$P_m^m(x) = (-1)^m (2m-1)!!(1-x^2)^{m/2} \tag{27}$$

where $n!!$ denotes the product of all odd integers less than or equal to n. Moreover from Eq. (25) it is seen that $P_{m-1}^m = 0$. Then using Eq. (26) with $l = m+1$ gives

$$P_{m+1}^m(x) = (2m+1)xP_m^m(x) \tag{28}$$

which can be used in conjunction with Eq. (27) to provide the two starting values needed for Eq. (26). For more information on the evaluation of Legendre functions, see Press et al. [29], Olver and Smith [30], and Alpert and Rokhlin [31].

The functions $\mathbf{H}_0$ and Y_0 are computed by approximating Chebyshev polynomials. This procedure produces a very efficient way of evaluating these special functions since polynomials of sixth-degree are sufficient to give an accurate approximation [32].

The method described above was implemented and a FORTRAN program was produced. This code uses LAPACK routine `cgerfs` for solving the linear system and NAG routine D01GCF to evaluate the double integrals in Eq. (24).

4.3. Collocation points and numerical results

In choosing the collocation points we look for a scheme which makes the matrix of the linear system for a_k^m, in Eq. (24), well conditioned.

We have assumed in Eq. (23) that the solution is symmetric about $\theta = 0$. This means that we can assume that the collocation points lie about a semi-disc given by $\{(r,\theta): 0 \leq r < 1, 0 \leq \theta \leq \pi\}$; collocating at symmetric points on the other semi-disc would give no further information for obtaining the solution of the linear system.

Consider a *tensor-product collocation*: the collocation points are taken as the intersection of concentric circles (radius r_l) with equally separated rays emanating from the origin (angle θ_n). Precisely, the collocation points are

$$(r_l, \theta_n), \quad l = 0, 1, \cdots, N_1, \quad n = 0, 1, \cdots, N_2$$

where $\{r_l\}$ is a certain set of distinct points in (0, 1) and

$$\theta_n = (2n+1)\pi/(2N_2+2), \quad n = 0, 1, \cdots, N_2$$

are the zeros of $\cos(N_2+1)\theta$ in $(0, \pi)$. However care must be taken when choosing the distribution of the numbers r_l. For instance, choosing equally spaced numbers in [0, 1]

Table 1
Condition numbers for two collocation schemes and N collocation points. Here, $N_1 = N_2$ so that $N = (N_1 + 1)^2$, $K = 0$ and $d = 0.5$

N	Equally spaced r_l	Chebyshev r_l
4	7.5	5.7
9	90.6	16.1
16	3.0×10^3	38.9
25	2.2×10^5	92.1
36	1.6×10^7	132.3

gives a badly conditioned system. We adopt a *Chebyshev tensor-product* collocation, which means that the points in the radial variable are zeros of Chebyshev polynomials of the first kind, $T_{2N_1+2}(r)$ in [0, 1]; explicitly,

$$r_l = \cos[(2l+1)\pi/(4N_1+4)],\ l = 0, 1, \cdots, N_1$$

Table 1 shows the condition numbers associated with two different tensor-product collocation schemes; they were obtained with $K = 0$ and $d = 0.5$. The second column gives the results obtained by employing equally-spaced points in the radial direction. The third column gives the results obtained with Chebyshev tensor-product collocation. The table illustrates that there is a considerable difference between the conditioning of these two schemes. Henceforth, all results associated with the expansion–collocation method are obtained using Chebyshev tensor-product collocation.

In Table 2, the complex coefficients a_k^m (in parentheses) of the expansion, Eq. (23), of the solution $[\phi]$ are shown for a disc submerged to a depth given by $d = 0.1$, at $K = 0.3$. In this example, the potential solves a scattering problem where V is given by Eq. (32) below.

The coefficients change little as N_1 and N_2 increase and they decay rapidly; only the first terms have significant values. This indicates that the series converges rapidly and the solution is very stable.

4.4. Convergence

We do not have a proof that the expansion–collocation method is convergent. We can adapt the arguments of Golberg [12] formulating the problem in a weighted-L_2 space. Then, the collocation method can be viewed as a projection method; it will converge if the corresponding interpolation polynomials converge (see p. 187 of Kress [33]). Such convergence results are well known in one dimension, but the question seems to be open in two dimensions. It is closely related to the existence of convergent bivariate quadrature (cubature) on the semi-disc with respect to the weight w. In particular, if there exists a Gaussian-type cubature formula (that is, a formula that preserves polynomials of the highest degree) on the semi-disc with respect to w, convergence in the mean of the corresponding interpolation polynomials can be shown, as in Xu [34,35]. However, this specific Gaussian-type cubature is not yet known to exist. The existence of such a cubature is equivalent to the existence common zeros of a family of orthogonal polynomials. The latter is characterized through certain nonlinear matrix equations.

Table 2
Expansion coefficients a_k^m for $d = 0.1$ and $K = 0.3$

m	k	$N_1 = N_2 = 2$	$N_1 = N_2 = 3$	$N_1 = N_2 = 4$
0	0	(−1.0393, 0.32806)	(−1.29326, 0.32237)	(−1.29162, 0.32149)
0	1	(0.50626, −0.12737)	(0.50629, −0.12620)	(0.50517, −0.12599)
0	2	(−0.00596, 0.00150)	(−0.00903, 0.00225)	(−0.00546, 0.00136)
0	3		(−0.00349, 0.00086)	(−0.00347, 0.00086)
0	4			(−0.00393, 0.00098)
1	0	(−0.00003, −0.07947)	(−0.00003, −0.07902)	(−0.00003, −0.07899)
1	1	(0.00001, 0.01982)	(0.00001, 0.01985)	(0.00001, 0.02001)
1	2	(0.00000, 0.00022)	(0.00000, −0.00007)	(0.00000, 0.00009)
1	3		(0.00000, −0.00019)	(0.00000, −0.00034)
1	4			(0.00000, −0.00017)
2	0	(0.00358, 0.00000)	(0.00356, 0.00000)	(0.00355, 0.00000)
2	1	(−0.00064, 0.00000)	(−0.00063, 0.00000)	(−0.00064, 0.00000)
2	2	(−0.00001, 0.00000)	(0.00001, 0.00000)	(0.00000, 0.00000)
2	3		(0.00001, 0.00000)	(0.00002, 0.00000)
2	4			(0.00001, 0.00000)
3	0		(0.00000, 0.00013)	(0.00000, 0.00013)
3	1		(0.00000, −0.00002)	(0.00000, −0.00002)
3	2		(0.00000, 0.00000)	(0.00000, 0.00000)
3	3		(0.00000, 0.00000)	(0.00000, 0.00000)
3	4			(0.00000, 0.00000)
4	0			(0.00000, 0.00000)
4	1			(0.00000, 0.00000)
4	2			(0.00000, 0.00000)
4	3			(0.00000, 0.00000)
4	4			(0.00000, 0.00000)

On the other hand it should be possible to construct a demonstrably convergent tensor-product type of cubature, using convergent one-dimensional quadrature formulas. This approach seems likely to provide the desired result given the fast convergence demonstrated numerically by the Chebyshev tensor-product collocation earlier in this section.

5. Radiation

The radiation of waves from a heaving horizontal submerged disc was treated by using the expansion–collocation method described in this section and the added mass and damping coefficients were computed. The results obtained recover those of Martin and Farina [20] where a different method was employed. In particular, the occurrence of negative added mass is noticed for small submergences and a critical physical behaviour is observed when both d and K are small. These aspects will also be seen in conjunction with the scattering problem examined next.

6. Scattering

We now turn our attention to the scattering of a regular wavetrain by a submerged horizontal disc. We are especially interested in the scattering cross-section parameters; these are the differential cross-section and the total scattering cross-section. We are not aware of published work on these quantities for a submerged disc, although they have been computed for circular docks; see, for example, Miles [5]. We begin by separating the total potential $\phi_{\rm tot}$ into two components such that

$$\phi_{\rm tot} = \phi_{\rm inc} + \phi \tag{29}$$

where $\phi_{\rm inc}$ corresponds to the unperturbed motion and ϕ corresponds to the scattering of $\phi_{\rm inc}$, by the disc. The incident wave potential is

$$\phi_{\rm inc} = (gA/\omega)\mathrm{e}^{K(z+ix)} \tag{30}$$

where A is the wave amplitude; $\phi_{\rm inc}$ satisfies Eqs. (1) and (2).

The disc is held fixed so that $\partial\phi_{\rm tot}/\partial_n = 0$ on S, whence Eq. (29) gives

$$\frac{\partial\phi}{\partial n} = -\frac{\partial\phi_{\rm inc}}{\partial n} \tag{31}$$

Thus, ϕ must satisfy Eqs. (1)–(4), with

$$V = -A\omega\mathrm{e}^{K(z+ix)} \tag{32}$$

Hence, the scattering potential can be represented by the integral formula, Eq. (6), wherein Eq. (21) is solved with Eq. (8). Note also that $[\phi] = [\phi_{\rm tot}]$ as $[\phi_{\rm inc}] = 0$.

6.1. The scattering amplitude

In this section we examine some parameters of direct physical interest to our problem. They can be computed without great difficulty once we have the scattering potential on the disc.

For deep water problems, the scattered wave satisfies the condition

$$\phi(P) \equiv \phi(r,\theta,z) \sim \frac{1}{\sqrt{Kr}} f(\theta)\mathrm{e}^{Kz}\mathrm{e}^{iKr} \text{ as } r \to \infty \tag{33}$$

where $f(\theta)$ is called the *scattering amplitude* or the *far-field pattern*; $f(\theta)$ describes the angular dependence of the outgoing waves. Let us calculate $f(\theta)$ in terms of $[\phi]$, using Eq. (6).

The Green's function has the behaviour

$$G \sim 2\pi iK\mathrm{e}^{K(z+\zeta)}H_0^{(1)}(KR) \sim 2\sqrt{\frac{2\pi K}{R}}\mathrm{e}^{K(z+\zeta)+i(KR+\pi/4)} \tag{34}$$

as $R \to \infty$ where we have used the well-known large-argument asymptotic approximation of the Hankel function $H_0^{(1)}$. Then recalling that $x = r\cos\theta$ and $y = r\sin\theta$, we see that

$$R = [(r\cos\theta - \xi)^2 + (r\sin\theta - \eta)^2]^{1/2} \sim r - \xi\cos\theta - \eta\sin\theta$$

as $r \to \infty$.

Hence, from Eqs. (6) and (34),

$$\phi \sim \sqrt{\frac{K}{2\pi r}}\mathrm{e}^{Kz}\mathrm{e}^{iKr+i\pi/4}K\mathrm{e}^{-Kd}\int_S [\phi]\mathrm{e}^{-iK(\xi\cos\theta+\eta\sin\theta)}\,\mathrm{d}S \tag{35}$$

as $r \to \infty$.

6.2. The Kochin function

At this point, it is conventional to introduce a Kochin function, $H(\theta)$. For a closed surface S_B, this can be defined as

$$H(\theta) = K\int_{S_B}\left(\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\right)\mathrm{e}^{K\zeta}\mathrm{e}^{-iK(\xi\cos\theta+\eta\sin\theta)}\,\mathrm{d}S_q$$

where $q \equiv (\xi, \eta, \zeta)$ is a point on S_B, and the normal vector points into the water. This is the definition used by Newman [36]. In particular, for the submerged disc, Eq. (20), we have

$$H(\theta) = -K^2\mathrm{e}^{-Kd}\int_S [\phi]\mathrm{e}^{-iK(\xi\cos\theta+\eta\sin\theta)}\,\mathrm{d}S_q \tag{36}$$

and so Eq. (35) becomes

$$\phi \sim -(2\pi Kr)^{-1/2}H(\theta)\mathrm{e}^{Kz}\mathrm{e}^{iKr+\pi/4} \text{ as } r \to \infty \tag{37}$$

Moreover, comparison with Eq. (33) gives

$$H(\theta) = i(2\pi)^{1/2}f(\theta)\mathrm{e}^{i\pi/4} \tag{38}$$

It is known that H satisfies

$$-\mathrm{Im}H(0)=\frac{\omega}{4\pi gA}\int_0^{2\pi}|H(\theta)|^2\,\mathrm{d}\theta \tag{39}$$

This is known as the "forward-scattering theorem", and is a consequence of energy conservation. See Newman [36] for more information.

Eq. (39) can be used as a test when computing numerical solutions of our problem. After estimating $[\phi]$ numerically, we can compute the Kochin function from Eq. (36) and then see how well Eq. (39) is satisfied.

It is possible to calculate $H(\theta)$ directly in terms of the coefficients a_k^m in Eq. (23). The exponential in the integrand of Eq. (36) is

$$\mathrm{e}^{-iKs\cos(\theta-\alpha)}=\sum_{n=0}^{\infty}\epsilon_n(-i)^nJ_n(Ks)\cos n(\theta-\alpha) \tag{40}$$

where $\epsilon_0=1$ and $\epsilon_n=2$ for $n\geq 1$. Then, after substituting Eq. (23) into Eq. (36), the integration over α is elementary and then the integration over s can be effected using another integral due to Krenk [28], namely

$$\int_0^1 J_m(Ks)P_{m+2n+1}^m(\sqrt{1-s^2})s\,\mathrm{d}s$$
$$=(-1)^nP_{m+2n+1}^{m+1}(0)j_{m+2n+1}(K)$$

where $j_n(w)=(1/2\ \pi/w)^{1/2}\ J_{n+1/2}(\omega)$ is a spherical Bessel function. The result is

$$H(\theta)=2\pi K^2\mathrm{e}^{-Kd}\sum_{k,m}^{N}a_k^m(-i)^m(-1)^{k+1}$$
$$\times P_{m+2k+1}^{m+1}(0)j_{m+2k+1}(K)\cos m\theta$$

6.3. Scattering cross-sections

We are interested in how effective the disc is at scattering energy. The average flux of energy through a fixed control surface S is

$$\frac{1}{2}\rho\omega\mathrm{Im}\int_S\varphi^*\frac{\partial\varphi}{\partial n}\,\mathrm{d}S$$

where φ is a velocity potential, n is in the direction of the flux and the asterisk denotes complex conjugation. Thus, the average flux of energy in the incident wave Eq. (30) passing through a fixed vertical plane parallel to the wave crests is

$$\frac{1}{4}\rho g^2A^2/\omega$$

per unit width; this is P_w, the mean power per unit crest length, familiar in the theory of wave-power devices [37].

The average flux of energy in the scattered waves, away from the disc, is

$$P=\frac{1}{2}\rho\omega\mathrm{Im}\int_{S_\infty}\phi^*\frac{\partial\phi}{\partial n}\,\mathrm{d}S$$

where S_∞ is a large vertical cylindrical surface enclosing the disc; P will be evaluated by inserting the far-field asymptotic behaviour of ϕ. Then, the (dimensionless) *total scattering cross-section* σ is usually defined by

$$\sigma=P/(2aP_w)$$

where $2aP_w$ is the mean incident power in a channel of width $2a$, which is the diameter of the disc. We find it convenient to work with a scaled version of σ, and define

$$Q=2(Ka)^2\sigma$$

We find that

$$P=\frac{1}{2}\rho\omega K\int_{-\infty}^{0}\int_0^{2\pi}|\phi(r,\theta,z)|^2r\mathrm{d}\theta\mathrm{d}z=\frac{\rho w}{8\pi K}\int_0^{2\pi}|H(\theta)|^2\,\mathrm{d}\theta$$

using $\partial\phi/\partial r\sim iK\phi$ and Eq. (37). Hence

$$Q=\frac{Ka}{2\pi}\int_0^{2\pi}|\tilde{H}(\theta)|^2\mathrm{d}\theta$$

where

$$\tilde{H}(\theta)=[\omega/(gA)]H(\theta)$$

is a dimensionless Kochin function. From Eq. (39), we have

$$-\mathrm{Im}\tilde{H}(0)=\frac{1}{4\pi}\int_0^{2\pi}|\tilde{H}(\theta)|^2\,\mathrm{d}\theta \tag{41}$$

whence

$$Q=-2Ka\mathrm{Im}\tilde{H}(0)$$

(Note that Miles [5] uses a slightly different definition of Q.)

Q gives a global measure of the energy scattered by the disc. In scattering theory, it is common to introduce the *differential cross-section* $\mathcal{D}(\theta)$, defined by

$$\mathcal{D}(\theta)=(2\pi)^{-1}Ka|\tilde{H}(\theta)|^2$$

so that

$$Q=\int_0^{2\pi}\mathcal{D}(\theta)\,\mathrm{d}\theta$$

Thus, $\mathcal{D}$ gives an indication of the directional distribution of the scattered energy.

6.4. Results

We have used the expansion–collocation method to compute the total scattering cross-section Q as a function of the dimensionless wavenumber, K, for values of the submergence depth d varying between 0.8 and 0.04. (Recall that,

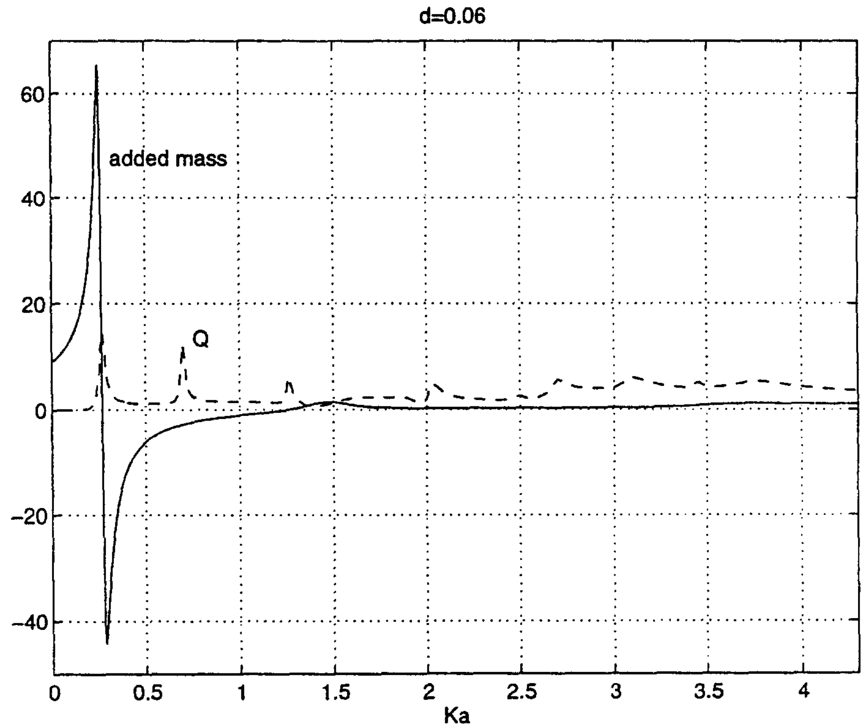

Fig. 1. The added mass (solid line) and the total scattering cross-section, Q (the dotted line), as functions of K, for submergence $d = 0.06$.

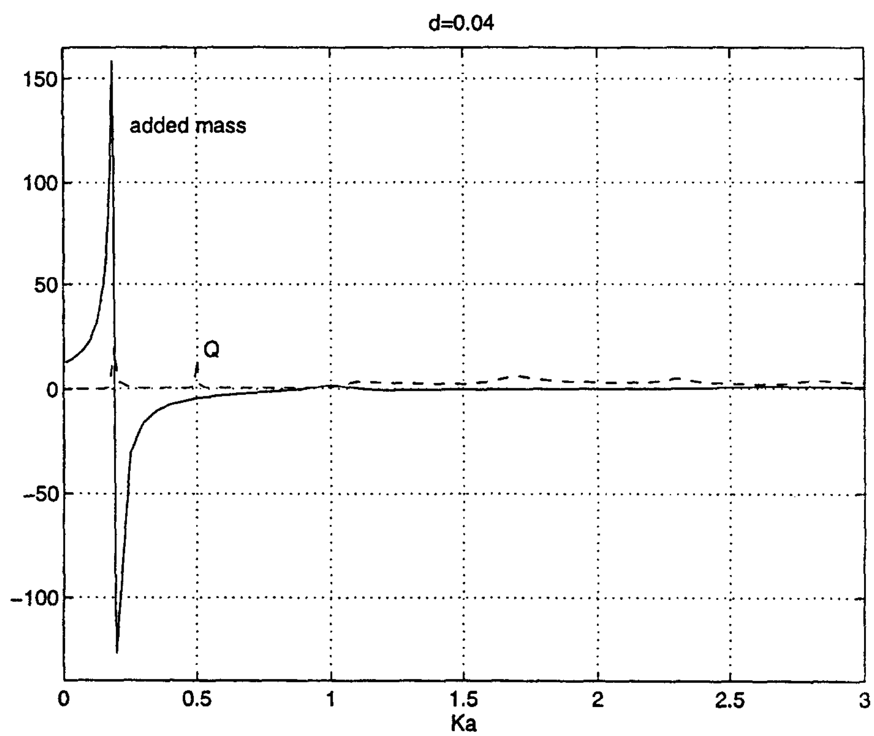

Fig. 2. The added mass (solid line) and the total scattering cross-section, Q (dotted line), as functions of K, for submergence $d = 0.04$.

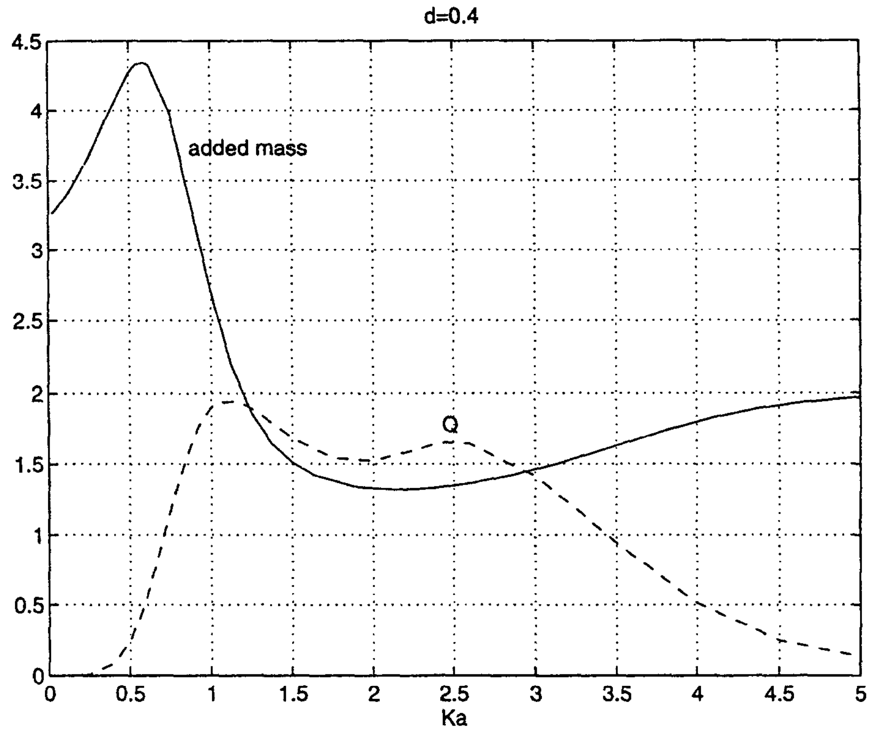

Fig. 3. The added mass (solid line) and the total scattering cross-section, Q (dotted line), as functions of K, for submergence $d = 0.4$.

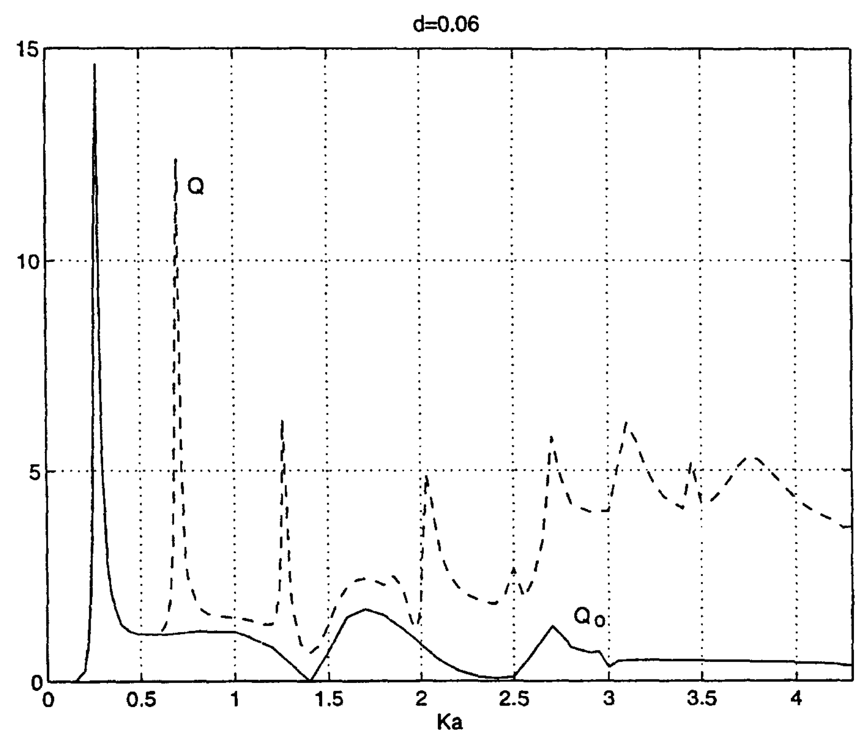

Fig. 4. The total scattering cross-section, Q, and the component Q_0, as functions of K, for submergence $d = 0.06$.

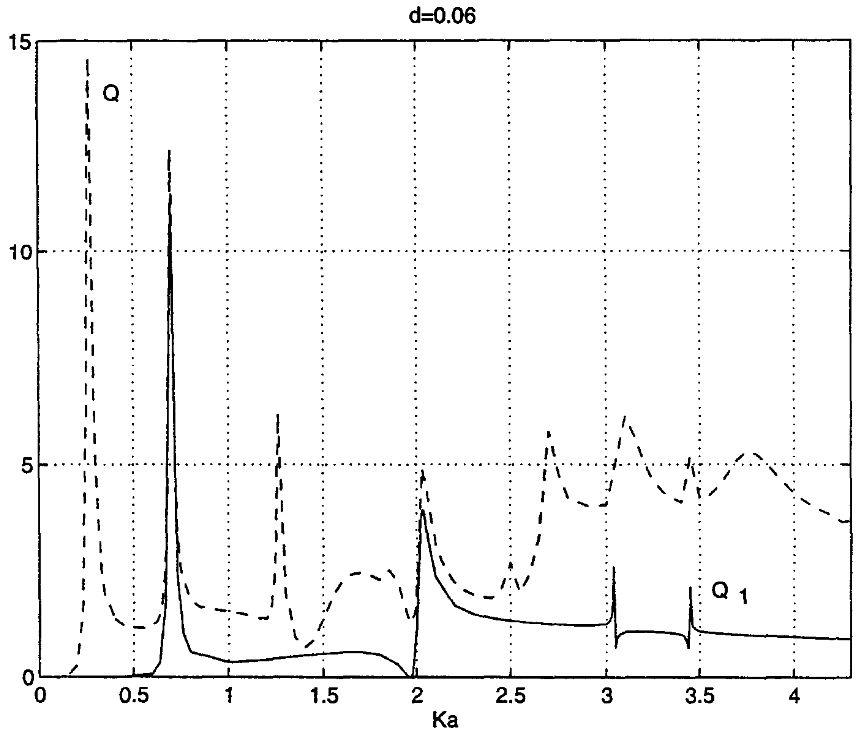

Fig. 5. The total scattering cross-section, Q, and the component Q_1, as functions of K, for submergence $d = 0.06$.

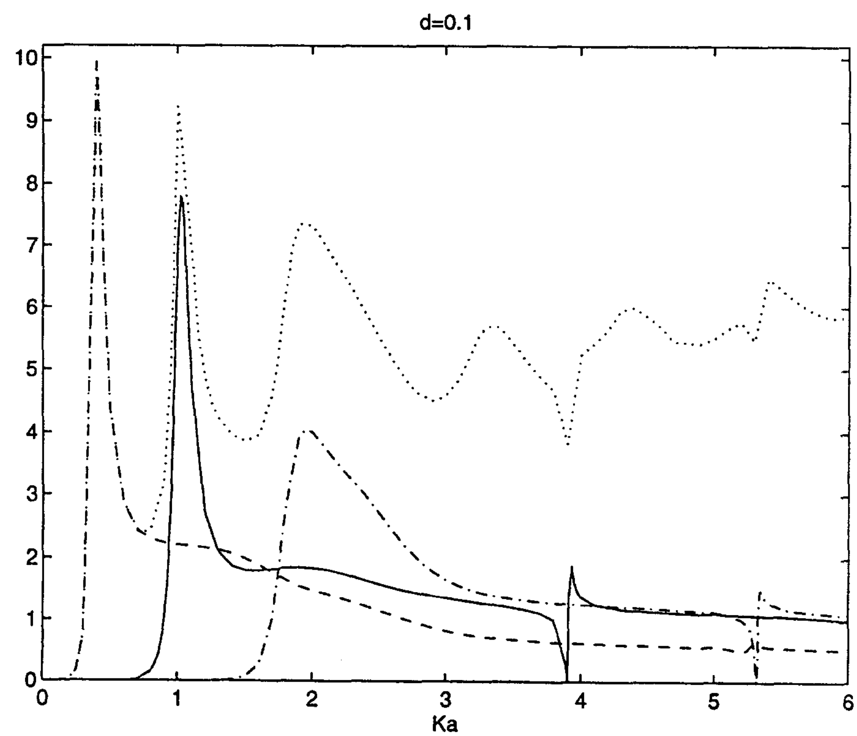

Fig. 6. The total scattering cross-section $Q(\cdots)$, and the components Q_0(dashed line), Q_1 (solid line) and Q_2(dash–dotted line), as functions of K, for submergence $d = 0.1$.

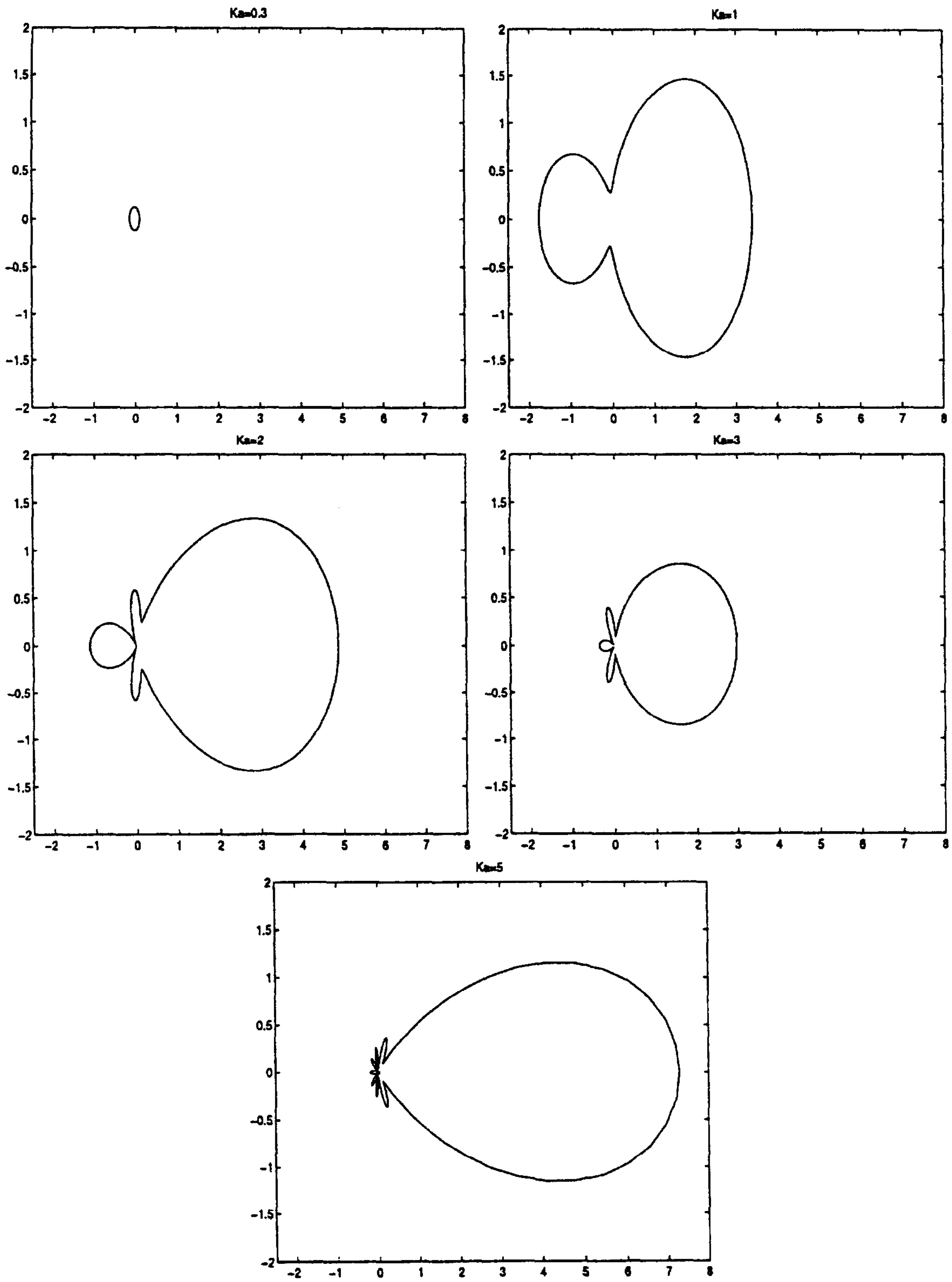

Fig. 7. The differential cross-section $D(\theta)$. The vertical and horizontal axes are the y and x axis, respectively, and (0, 0) indicates the centre of the disc. The submergence depth is given by $d = 0.1$.

for, dimensionless results, we scaled with respect to the disc's radius, and so set $a = 1$.) We notice the existence of peaks in the graphs of Q against K. These peaks are sharper and more pronounced when K is small and d goes to zero. A similar striking behaviour was observed before in our computations of the hydrodynamic coefficients (added mass and damping) for a heaving submerged horizontal disc [20].

As the frequency is increased from zero, for a given small depth of submergence, the first peak of Q is found to occur

Table 3
Relative error when evaluating the Kochin function relationship. The depth of submergence is $d = 0.1$ and 49 collocation points were used

K	Relative error
0.1	0.00000
0.5	0.00003
1.0	0.00005
1.5	0.00001
2.0	0.00020
2.5	0.00014
3.0	0.00007
3.5	0.00003
4.0	0.00091
4.5	0.00028
5.0	0.00144
5.5	0.01693
6.0	0.05710

Table 4
Relative error when evaluating the Kochin function relationship. The depth of submergence is $d = 0.06$ and 64 collocation points were used

K	Relative error
0.1	0.00005
0.9	0.00000
1.0	0.00002
1.5	0.00010
2.0	0.00111
2.5	0.01560
3.0	0.00977
3.5	0.00127
4.0	0.00305
4.3	0.00201

at the same frequency as the first peak of the added mass. See Figs 1 and 2.

This relation is less clear for $d \geq 0.2$; see Fig. 3.

In fact, this maximum in Q has its origin in the axisymmetric mode present in the incident potential, Eq. (30). To show this, we decompose the incident potential into its Fourier components as

$$\phi_{\text{inc}}(r,\theta,z) = \sum_{k=0}^{\infty} \phi_{\text{inc}}^{k}(r,\theta,z)$$

where

$$\phi_{\text{inc}}^{k}(r,\theta,z) = (gA/\omega)e^{Kz}\epsilon_k i^k J_k(Kr)\cos k\theta$$

and we have used Eq. (40). Let Q_k denote the total scattering cross-section obtained by solving the scattering problem with

$$\phi_{\text{inc}} = \phi_{\text{inc}}^{k}$$

Fig. 4 shows numerical results which indicate that the first peak in Q has its origin in Q_0; indeed, the two curves are indistinguishable for K less than about 0.6.

Furthermore we can see (Figs 5 and 6) that the subsequent extrema are related to different modes ϕ_{inc}^{k}. The calculations also confirm that $Q = \Sigma_k Q_k$, which can be deduced from the linearity of the problem. The spikes in the physical parameters occur at frequencies called *resonant frequencies*. Determining the location of these frequencies in the present problems has been studied by Farina [8].

In Fig. 7, the differential cross-section $\mathcal{D}(\theta)$ is plotted for five values of K and $d = 0.1$. It is interesting to note the varying angular dependence as the frequency increases and the areas defined by the closed curves which give the corresponding total scattering cross-sections, described previously.

Table 3 and Table 4 give the relative error found in Eq. (41) for two values of d; this error is defined by $|\text{LHS} - \text{RHS}|/|\text{LHS}|$. The small relative errors suggest the good accuracy and stability of the method even for small submergences, such as $d = 0.06$. Moreover only a modest number of collocation points is needed, which means a low computational cost.

7. Conclusions

An expansion–collocation method has been developed for solving the two-dimensional hypersingular integral equations that arise in the radiation and scattering problems for a submerged circular plate. This is an extension of a well-known method for solving one-dimensional hypersingular integral equations. The results for the radiation problem recover those of Martin and Farina [20]. The results for the scattering problem reveal a strong dependence on the frequency, especially when the plate is close to the free surface. Similar results were found for the heaving disc. The relationships between the scattering cross-section and the peaks in the added mass have been explored. The connections between those peaks and so-called *resonant frequencies* are currently being investigated.

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