# On the diffraction of Poincaré waves 

P. A. Martin*<br>Department of Mathematical and Computer Sciences, Colorado School of Mines, Golden, CO 80401-1887, U.S.A.

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SUMMARY
The diffraction of tidal waves (Poincaré waves) by islands and barriers on water of constant finite depth is governed by the two-dimensional Helmholtz equation. One effect of the Earth's rotation is to complicate the boundary condition on rigid boundaries: a linear combination of the normal and tangential derivatives is prescribed. (This would be an oblique derivative if the coefficients were real.) Corresponding boundary-value problems are treated here using layer potentials, generalizing the usual approach for the standard exterior boundary-value problems of acoustics. Singular integral equations are obtained for islands (scatterers with non-empty interiors) whereas hypersingular integral equations are obtained for thin barriers. Copyright © 2001 John Wiley \& Sons, Ltd.

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## 1. INTRODUCTION

Ocean waves are governed by Laplace's equation in three dimensions. Under linearization and an assumption of constant depth, the governing equation becomes the Helmholtz equation in the horizontal plane. This leads to a widely used model for the diffraction of ocean waves.

When the Earth's rotation is taken into account, one still obtains the two-dimensional Helmholtz equation (Section 2). However, the boundary condition on lateral boundaries is more complicated: on a curve $C$, the boundary condition is

$$
\begin{equation*}
\frac{\partial \zeta}{\partial n}+\mathrm{i} \beta \frac{\partial \zeta}{\partial s}=0 \tag{1}
\end{equation*}
$$

involving a linear combination of the normal and tangential derivatives of the unknown function; the parameter $\beta$ vanishes when there is no rotation. It is the boundary condition (1) that makes the problem interesting from a mathematical point of view.

A familiar method for solving exterior boundary-value problems for the Helmholtz equation is to reduce them to boundary integral equations [1,2]. For the standard boundary conditions,

[^0]namely Neumann $(\beta=0)$ and Dirichlet $(\zeta=0)$, and a scatterer with a non-empty interior and a simple smooth boundary, it can be arranged that these boundary integral equations are Fredholm integral equations of the second kind with continuous kernels (in two dimensions). If the scatterer is thin (empty interior), the boundary-value problems can be reduced to a hypersingular integral equation (Neumann condition) or a Fredholm integral equation of the first kind with a logarithmically singular kernel (Dirichlet condition).

What is the effect of the boundary condition (1)? For Laplace's equation, it is known that complex-variable methods can be used to reduce the boundary-value problem to Cauchysingular integral equations when the scatterer has a non-empty interior. The same result is obtained here. We show that the corresponding water-wave problem can be reduced to a singular integral equation (in fact, two different equations are given), using layer potentials. Irregular frequencies are identified.

For thin scatterers, we derive a hypersingular integral equation. It reduces to the known equation for a thin sound-hard scatterer when $\beta=0$. Indeed, as $\beta$ is often small, one can construct a regular perturbation about the solution for $\beta=0$. Alternatively, the integral equation for $\beta \neq 0$ can be solved numerically, using an expansion-collocation method based on Chebyshev polynomials of the second kind.

## 2. GOVERNING EQUATIONS

Consider an ocean of constant finite depth $h$. The governing equations of motion for long gravity waves, when rotation is taken into account are

$$
\begin{align*}
& \frac{\partial U}{\partial t}-f V+g \frac{\partial Z}{\partial x}=0  \tag{2}\\
& \frac{\partial V}{\partial t}+f U+g \frac{\partial Z}{\partial y}=0  \tag{3}\\
& h \frac{\partial U}{\partial x}+h \frac{\partial V}{\partial y}+\frac{\partial Z}{\partial t}=0 \tag{4}
\end{align*}
$$

where $x$ and $y$ are horizontal Cartesian co-ordinates, $U(x, y, t)$ and $V(x, y, t)$ are the corresponding horizontal velocity components, $Z(x, y, t)$ is the surface elevation, $g$ is the acceleration due to gravity and $f$ is the Coriolis parameter. We have $f=2 \Omega \sin \phi_{0}$, where $\Omega$ is the Earth's angular speed and $\phi_{0}$ is the reference angle of latitude. These equations are given in, for example, References [3, Section 207; 4, Section 116; 5, p. 128]. For background information, see Reference [6].

If we eliminate $U$ and $V$ from (2)-(4), we obtain

$$
\begin{equation*}
\left\{g h \nabla^{2}-f^{2}-\frac{\partial^{2}}{\partial t^{2}}\right\} \frac{\partial Z}{\partial t}=0 \tag{5}
\end{equation*}
$$

a single partial differential equation for $Z(x, y, t)$. Here, $\nabla^{2}$ is the two-dimensional Laplacian. For time-harmonic motions, we can write

$$
U=\operatorname{Re}\left\{u \mathrm{e}^{-\mathrm{i} \omega t}\right\}, \quad V=\operatorname{Re}\left\{v \mathrm{e}^{-\mathrm{i} \omega t}\right\}, \quad Z=\operatorname{Re}\left\{\zeta \mathrm{e}^{-\mathrm{i} \omega t}\right\}
$$

where $\omega$ is the circular frequency. Then, (5) becomes the two-dimensional Helmholtz equation.

$$
\begin{equation*}
\left(\nabla^{2}+k^{2}\right) \zeta=0 \tag{6}
\end{equation*}
$$

where

$$
k^{2}=\left(\omega^{2}-f^{2}\right) /(g h)
$$

we have assumed that $\omega^{2}>f^{2}$. It will be convenient to introduce

$$
\beta=f / \omega
$$

a real, dimensionless parameter; we assume that $|\beta|<1$.
If $\zeta(x, y)$ is known, the velocity components can be obtained from (2) and (3):

$$
\begin{aligned}
& u(x, y)=\frac{-\omega}{h k^{2}}\left\{\mathrm{i} \frac{\partial \zeta}{\partial x}-\beta \frac{\partial \zeta}{\partial y}\right\} \\
& v(x, y)=\frac{-\omega}{h k^{2}}\left\{\mathrm{i} \frac{\partial \zeta}{\partial y}+\beta \frac{\partial \zeta}{\partial x}\right\}
\end{aligned}
$$

In terms of a vector $\mathbf{u}=(u, v, 0)$, we can write these formulas concisely as

$$
\mathbf{u}(x, y)=-\omega\left(h k^{2}\right)^{-1}\{\operatorname{igrad} \zeta-\beta \operatorname{curl}(\zeta \mathbf{z})\}
$$

where $\mathbf{z}$ is a unit vector in the vertical direction. Moreover, (6) implies that curl $\mathbf{u}=f(\zeta / h) \mathbf{z}$.
Observe that $u$ and $v$ also satisfy (6). Thus, it is easy to write down some solutions for the velocity components. However, in general, the main difficulty in solving a scattering problem comes from applying the boundary condition on rigid, vertical lateral boundaries. If $C$ is such a boundary, we have

$$
\begin{equation*}
\mathbf{u} \cdot \mathbf{n}=0 \quad \text { on } C \tag{7}
\end{equation*}
$$

where $\mathbf{n}$ is a unit normal vector to $C$. Thus, unless $C$ is parallel to the $x$ - or $y$-axes (7) will involve both $u$ and $v$.

Two elementary problems are solved next. These are plane-wave reflection by a straight coastline and plane-wave scattering by a circular island.

### 2.1. Straight coastline

For a straight coastline of infinite length, given by $y=x \tan \alpha$ (where the water occupies $y>x \tan \alpha$ ), and an incident plane wave

$$
\begin{equation*}
\zeta_{\mathrm{inc}}(x, y)=\exp \left\{\mathrm{i} k\left(x \cos \theta_{\mathrm{i}}-y \sin \theta_{\mathrm{i}}\right)\right\} \tag{8}
\end{equation*}
$$

where $\theta_{\mathrm{i}}$ is the angle of incidence (with $-\alpha<\theta_{\mathrm{i}}<\pi-\alpha$ ), we find that the reflected wave is given by

$$
\zeta_{\mathrm{ref}}(x, y)=R \exp \left\{\mathrm{i} k\left(x \cos \theta_{\mathrm{r}}+y \sin \theta_{\mathrm{r}}\right)\right\}
$$

The boundary condition on $C$ (7) becomes $u \sin \alpha=v \cos \alpha$ on $y=x \tan \alpha$, where $\zeta=\zeta_{\text {inc }}+\zeta_{\text {ref }}$. It follows that $\theta_{\mathrm{r}}=\theta_{\mathrm{i}}+2 \alpha$ and

$$
R=(\sin \Theta+\mathrm{i} \beta \cos \Theta) /(\sin \Theta-\mathrm{i} \beta \cos \Theta)
$$

where $\Theta=\theta_{\mathrm{i}}+\alpha=\theta_{\mathrm{r}}-\alpha$. Note that the complex reflection coefficient $R$ satisfies $|R|=1$ so that $R=\mathrm{e}^{\mathrm{i} \delta}$. Thus, in general, the wave suffers a phase change upon reflection; $\delta=0$ when $\beta=0$.

### 2.2. Circular island

For a circular island, we use plane polar co-ordinates, $r$ and $\theta$, with $x=r \cos \theta$ and $y=r \sin \theta$. Then $C$ is $r=a$, say, and (7) becomes

$$
\begin{equation*}
\mathrm{i} \frac{\partial \zeta}{\partial r}-\frac{\beta}{r} \frac{\partial \zeta}{\partial \theta}=0 \quad \text { on } r=a \tag{9}
\end{equation*}
$$

Take the incident wave (8) as before, whence

$$
\zeta_{\text {inc }}=\sum_{n=-\infty}^{\infty} \mathrm{i}^{n} J_{n}(k r) \mathrm{e}^{\mathrm{i} n\left(\theta+\theta_{\mathrm{i}}\right)}
$$

where $J_{n}$ is a Bessel function. Then, the (outgoing) scattered wave can be written as

$$
\zeta_{\mathrm{sc}}=\sum_{n=-\infty}^{\infty} \mathrm{i}^{n} \zeta_{n} H_{n}(k r) \mathrm{e}^{\mathrm{i} n\left(\theta+\theta_{\mathrm{i}}\right)}
$$

where $H_{n} \equiv H_{n}^{(1)}$ is a Hankel function and the coefficients $\zeta_{n}$ are found by imposing the boundary condition (9):

$$
\zeta_{n}=-\frac{k a J_{n}^{\prime}(k a)-n \beta J_{n}(k a)}{k a H_{n}^{\prime}(k a)-n \beta H_{n}(k a)}
$$

In the absence of rotation $(\beta=0)$, this reduces to the well-known solution for acoustic scattering by a sound-hard circular cylinder. Analogous interior problems (for $r<a$ with $\beta \neq 0$ ) are considered in Reference [3, Section 210]. Chambers [7] has discussed standing-wave solutions (for $r>a$ with $\beta \neq 0$ ). Despite its title, paper [8] is limited to $\beta=0$.

## 3. TRAPPING AND UNIQUENESS

A straight coastline can support a trapped wave known as a Kelvin wave [4, Section 133; 5, Section 24]. With the notation of Section 2.1, the Kelvin wave is given by

$$
\zeta=\exp \{\mathrm{i} \lambda(x \cos \alpha+y \sin \alpha)\} \exp \{\lambda \beta(x \sin \alpha-y \cos \alpha)\}
$$

where $\lambda=\omega / \sqrt{g h}$. This expression for $\zeta$ satisfies (6) and $\mathbf{u} \cdot \mathbf{n}=0$ on $y=x \tan \alpha$. It represents a wave travelling along the coastline in the direction of increasing $x$ and $y$, but $\zeta$ decays exponentially away from the coastline. When $\beta=0, \zeta$ reduces to a grazing plane-wave travelling along the coast (without decay).

Can trapped waves exist around islands with vertical coastlines? No. To see this, we modify a standard argument used to prove uniqueness for the exterior boundary-value problems of acoustics [1].

Let $C$ be a simple closed curve representing the island's coastline. Specifically, let us define $C$ in terms of a parametrization,

$$
C=\{(x, y): x=x(\tau), y=y(\tau), 0 \leqslant \tau \leqslant 1\}
$$

where $x(0)=x(1)$ and $y(0)=y(1)$. Then

$$
\mathbf{s}(\tau)=\left(x^{\prime}(\tau), y^{\prime}(\tau)\right) / \Delta
$$

with $\Delta(\tau)=\left[\left(x^{\prime}\right)^{2}+\left(y^{\prime}\right)^{2}\right]^{1 / 2}$, is a unit tangent vector to $C$ and

$$
\mathbf{n}(\tau)=\left(y^{\prime}(\tau),-x^{\prime}(\tau)\right) / \Delta
$$

is a unit normal vector to $C$. If $C$ is traversed anti-clockwise as $\tau$ increases, $\mathbf{n}$ will point into the water. It follows that the boundary condition (7) can be written as

$$
\begin{equation*}
\frac{\partial \zeta}{\partial n}+\mathrm{i} \beta \frac{\partial \zeta}{\partial s}=0 \quad \text { on } C \tag{10}
\end{equation*}
$$

where

$$
\frac{\partial \zeta}{\partial n}=\mathbf{n} \cdot \operatorname{grad} \zeta \quad \text { and } \quad \frac{\partial \zeta}{\partial s}=\mathbf{s} \cdot \operatorname{grad} \zeta
$$

are the normal and tangential derivatives, respectively, of $\zeta$ on $C$.
Solving (6) together with (10) and a radiation condition at infinity is called the Poincaré problem or the oblique-derivative problem: strictly speaking, the Poincaré problem allows an additional term proportional to $\zeta$ on the left-hand side of (10), whereas the oblique-derivative problem has real coefficients on the left-hand side of (10) so that it can be written as the directional derivative of $\zeta$ in a certain direction.

Let us formulate a boundary-value problem.
Poincaré problem: Find $\zeta_{\mathrm{sc}}(x, y)$, where $\zeta_{\text {sc }}$ solves the Helmholtz equation (6) in the unbounded region exterior to $C, B$, together with the boundary condition (10) on $C$, where $\zeta=\zeta_{\text {inc }}+\zeta_{\text {sc }}$ and the incident wave $\zeta_{\text {inc }}$ is given by (8). In addition, the scattered wave $\zeta_{\text {sc }}$ must satisfy the Sommerfeld radiation condition at infinity,

$$
\begin{equation*}
\lim _{r \rightarrow \infty} \sqrt{r}\left(\frac{\partial \zeta_{\mathrm{sc}}}{\partial r}-\mathrm{i} k \zeta_{\mathrm{sc}}\right)=0 \tag{11}
\end{equation*}
$$

where $r=\sqrt{x^{2}+y^{2}}$.
To prove that trapped waves cannot exist, we suppose that $\zeta_{0}$ solves the unforced Poincaré problem. Thus, $\zeta_{0}$ satisfies (6), (10) and (11). Then, we apply Green's first theorem to $\zeta_{0}$ and $\overline{\zeta_{0}}$, the complex conjugate of $\zeta_{0}$, in the region $B_{r} \subset B$, bounded internally by $C$ and externally by a large circle $C_{r}$ of radius $r$. This gives

$$
\begin{aligned}
\int_{B_{r}}\left\{\left|\operatorname{grad} \zeta_{0}\right|^{2}-k^{2}\left|\zeta_{0}\right|^{2}\right\} \mathrm{d} V= & \int_{C_{r}} \overline{\zeta_{0}}\left(\frac{\partial \zeta_{0}}{\partial r}-\mathrm{i} k \zeta_{0}\right) \mathrm{d} s \\
& +\mathrm{i} k \int_{C_{r}}\left|\zeta_{0}\right|^{2} \mathrm{~d} s-\int_{C} \overline{\zeta_{0}} \frac{\partial \zeta_{0}}{\partial n} \mathrm{~d} s
\end{aligned}
$$

The left-hand side is real. The radiation condition ensures that the first integral on the righthand side vanishes as $r \rightarrow \infty$. Hence, taking the imaginary part, we obtain

$$
\begin{aligned}
-k \lim _{r \rightarrow \infty} \int_{C_{r}}\left|\zeta_{0}\right|^{2} \mathrm{~d} s & =-\operatorname{Im} \int_{C} \overline{\zeta_{0}} \frac{\partial \zeta_{0}}{\partial n} \mathrm{~d} s=\frac{\mathrm{i}}{2} \int_{C}\left(\overline{\zeta_{0}} \frac{\partial \zeta_{0}}{\partial n}-\zeta_{0} \frac{\partial \overline{\zeta_{0}}}{\partial n}\right) \mathrm{d} s \\
& =\frac{\beta}{2} \int_{C}\left(\overline{\zeta_{0}} \frac{\partial \zeta_{0}}{\partial s}+\zeta_{0} \frac{\partial \overline{\zeta_{0}}}{\partial s}\right) \mathrm{d} s=\frac{\beta}{2} \int_{C} \frac{\partial}{\partial s}\left|\zeta_{0}\right|^{2} \mathrm{~d} s \\
& =\frac{\beta}{2} \int_{0}^{1} \frac{\mathrm{~d}}{\mathrm{~d} \tau}\left|\zeta_{0}\right|^{2} \mathrm{~d} \tau=0
\end{aligned}
$$

Rellich's lemma [1, Lemma 3.11] then implies that $\zeta_{0} \equiv 0$, as required.
Note that Longuet-Higgins [9] has shown that trapped waves do exist for circular islands when $\beta^{2}>1\left(\omega^{2}<f^{2}\right)$; in this case, (6) should be replaced by the modified Helmholtz equation.

It is well known that the Poincare problem for Laplace's equation can be reduced to a singular integral equation on $C$; see, for example References [10, Section 74, 11, p. 185]. Extensions to other elliptic partial differential equations, including the Helmholtz equation, can be made [10, Section 76]. See also the recent book by Paneah [12]. The problem with an open $C$ (several thin rigid barriers) has been discussed recently by Krutitskii [13]. Exact solutions for a thin, straight semi-infinite barrier have been given by Crease [14], Chambers [15], Kapoulitsas [16] and Haines [17], using the Wiener-Hopf technique.

## 4. POTENTIAL THEORY

Introduce a fundamental solution, $G$, defined by $G(P, Q)=-\frac{1}{2} \mathrm{i} H_{0}^{(1)}(k R)$, where $R$ is the distance between the two points $P$ and $Q$. Using $G$, we define single- and double-layer potentials by

$$
(S \mu)(P)=\int_{C} \mu(q) G(P, q) \mathrm{d} s_{q}
$$

and

$$
(D v)(P)=\int_{C} v(q) \frac{\partial}{\partial n_{q}} G(P, q) \mathrm{d} s_{q}
$$

respectively, where $P \notin C$. $(S \mu)(P)$ is continuous in $P$ as $P$ crosses $C$, whereas both $D v$ and the normal derivative of $S \mu$ exhibit jumps given by

$$
D v=(\mp I+K) v
$$

and

$$
\begin{equation*}
\frac{\partial}{\partial n_{p}} S \mu=\left( \pm I+K^{\prime}\right) \mu \tag{12}
\end{equation*}
$$

respectively, where, in each case, the upper (lower) sign corresponds to $P \rightarrow p \in C$ from the exterior (interior) of $C$. (Recall that $\mathbf{n}$ points into the exterior $B$.) Here, $K$ and $K^{\prime}$ are boundary integral operators defined by

$$
\begin{aligned}
K v & =\int_{C} v(q) \frac{\partial}{\partial n_{q}} G(p, q) \mathrm{d} s_{q} \\
K^{\prime} \mu & =\int_{C} \mu(q) \frac{\partial}{\partial n_{p}} G(p, q) \mathrm{d} s_{q}
\end{aligned}
$$

and $p \in C$. All these formulas hold if $C$ is twice-differentiable and the densities, $\mu$ and $v$, are continuous. Moreover, for such curves $C$, the kernels of $K$ and $K^{\prime}$ are continuous.

The boundary condition in the Poincaré problem (10) suggests that we will require the normal derivative of $D v$ and tangential derivatives of $S \mu$ and $D v$. Sufficient conditions for these to exist are that $\mu$ is Hölder continuous and that $v$ has a Hölder-continuous tangential derivative on $C$. Then

$$
\begin{equation*}
N v=\frac{\partial}{\partial n_{p}}(D v) \quad \text { and } \quad L^{\prime} \mu=\frac{\partial}{\partial s_{p}}(S \mu) \tag{13}
\end{equation*}
$$

are well defined (no jumps). The operator $N$ is hypersingular; it may be represented as an integral operator involving a finite-part integral [18]. The operator $L^{\prime}$ is a singular integral operator, it can be written as

$$
\begin{equation*}
\left(L^{\prime} \mu\right)(p)=f_{C} \mu(q) \frac{\partial}{\partial s_{q}} G(p, q) \mathrm{d} s_{q} \tag{14}
\end{equation*}
$$

where the integral must be interpreted as a Cauchy principal-value integral; see References [1, Theorem 2.17, 19, Theorem 7.27] for the analogous results for the two-dimensional Laplace equation and the three-dimensional Helmholtz equation, respectively. It is also natural to introduce an operator $L$ defined by

$$
\begin{aligned}
(L v)(p) & =\int_{C} v(q) \frac{\partial}{\partial s_{q}} G(p, q) \mathrm{d} s_{q} \\
& =-\int_{C} v^{\prime}(q) G(p, q) \mathrm{d} s_{q}=-\left(S v^{\prime}\right)
\end{aligned}
$$

after an integration by parts, where $v^{\prime}(q)$ is the tangential derivative of $v(q)$ at $q \in C$.
For the gradient of $D v$, we follow Kress [19, Section 7.5] and deduce that

$$
\operatorname{grad}(D v)=k^{2} \int_{C} G(P, q) v(q) \mathbf{n}(q) \mathrm{d} s_{q}+\left(-\frac{\partial W}{\partial y}, \frac{\partial W}{\partial x}\right)
$$

where $P$ is at $(x, y)$ and

$$
W(P)=\int_{C} v(q) \frac{\partial}{\partial s_{q}} G(P, q) \mathrm{d} s_{q}=-\left(S v^{\prime}\right)(P)
$$

Hence, making use of (12) and the relation between $\mathbf{s}=\left(s_{1}, s_{2}\right)$ and $\mathbf{n}=\left(s_{2},-s_{1}\right)$, we obtain

$$
\begin{align*}
\frac{\partial}{\partial s_{p}}(D v) & =\frac{\partial W}{\partial n_{p}}+k^{2} \mathbf{s} \cdot(S\{v \mathbf{n}\}) \\
& =\left(\mp I-K^{\prime}\right) v^{\prime}+k^{2} \mathbf{s} \cdot(S\{v \mathbf{n}\}) \tag{15}
\end{align*}
$$

for $p \in C$. Similarly, we use (14) and obtain

$$
\begin{align*}
\frac{\partial}{\partial n_{p}}(D v) & =-\frac{\partial W}{\partial s_{p}}+k^{2} \mathbf{n} \cdot(S\{v \mathbf{n}\}) \\
& =L^{\prime} v^{\prime}+k^{2} \mathbf{n} \cdot(S\{v \mathbf{n}\}) \tag{16}
\end{align*}
$$

explicitly, we have Maue's formula [18, p. 343],

$$
\begin{aligned}
\frac{\partial}{\partial n_{p}} \int_{C} v(q) \frac{\partial}{\partial n_{q}} G(p, q) \mathrm{d} s_{q}= & -\frac{\partial}{\partial s_{p}} \int_{C} v(q) \frac{\partial}{\partial s_{p}} G(p, q) \mathrm{d} s_{q} \\
& +k^{2} \int_{C} v(q)\{\mathbf{n}(p) \cdot \mathbf{n}(q)\} G(p, q) \mathrm{d} s_{q}
\end{aligned}
$$

We shall also make use of representations based on Green's theorem. Thus, if we apply Green's theorem in $B$ to $\zeta_{\text {sc }}$ and $G$, we obtain

$$
\begin{equation*}
2 \zeta_{\mathrm{sc}}(P)=\int_{C}\left\{G(P, q) \frac{\partial \zeta_{\mathrm{sc}}}{\partial n_{q}}-\zeta_{\mathrm{sc}}(q) \frac{\partial}{\partial n_{q}} G(P, q)\right\} \mathrm{d} s_{q}, \quad P \in B \tag{17}
\end{equation*}
$$

Similarly, applying Green's theorem in the bounded interior of $C, B_{i}$, to $\zeta_{\text {inc }}$ and $G$, and adding the result to (17), we obtain

$$
\begin{equation*}
2 \zeta_{\mathrm{sc}}(P)=\int_{C}\left\{G(P, q) \frac{\partial \zeta}{\partial n_{q}}-\zeta(q) \frac{\partial}{\partial n_{q}} G(P, q)\right\} \mathrm{d} s_{q}, \quad P \in B \tag{18}
\end{equation*}
$$

where $\zeta=\zeta_{\text {inc }}+\zeta_{\text {sc }}$ is the total field.
Apart from this last formula, all of the results in this section are valid when $C$ is a simple smooth open arc, provided $p$ is not taken at an end point, and provided $v(q)=0$ when $q$ is at an end point (otherwise the use of integration by parts will generate end-point contributions).

## 5. DIFFRACTION BY AN ISLAND

Here, we suppose that $C$ is a simple smooth closed curve with a non-empty interior $B_{i}$. Then, we can use (18) to represent the scattered field $\zeta_{\text {sc }}$. Making use of the boundary condition (10), we obtain

$$
\begin{align*}
2 \zeta_{\mathrm{sc}}(P) & =\int_{C}\left\{G(P, q)\left(-\mathrm{i} \beta \frac{\partial \zeta}{\partial s_{q}}\right)-\zeta(q) \frac{\partial}{\partial n_{q}} G(P, q)\right\} \mathrm{d} s_{q} \\
& =\int_{C}\left\{\mathrm{i} \beta \frac{\partial \zeta}{\partial s_{q}} G(P, q)-\frac{\partial}{\partial n_{q}} G(P, q)\right\} \zeta(q) \mathrm{d} s_{q} \tag{19}
\end{align*}
$$

for $P \in B$, which is a formula for $\zeta_{\mathrm{sc}}(P)$ in terms of the boundary values of $\zeta$. Hence, letting $P \rightarrow p \in C$, we obtain

$$
\begin{equation*}
(I-\mathrm{i} \beta L+K) \zeta=2 \zeta_{\mathrm{inc}} \tag{20}
\end{equation*}
$$

which is a singular integral equation for $\zeta(q)$.
As an alternative, we may seek a solution in the form of a single-layer potential,

$$
\begin{equation*}
\zeta_{\mathrm{sc}}(P)=\int_{C} \mu(q) G(P, q) \mathrm{d} s_{q}, \quad P \in B \tag{21}
\end{equation*}
$$

where the density $\mu$ is to be found. We apply the boundary condition (10), using (12) and $(13)_{2}$, giving

$$
\begin{equation*}
\left(I+K^{\prime}+\mathrm{i} \beta L^{\prime}\right) \mu=-f_{\text {inc }} \tag{22}
\end{equation*}
$$

where

$$
\begin{equation*}
f_{\mathrm{inc}}(p)=\left(\frac{\partial}{\partial n_{p}}+\mathrm{i} \beta \frac{\partial}{\partial s_{p}}\right) \zeta_{\mathrm{inc}} \tag{23}
\end{equation*}
$$

is known. Equation (22) is another singular boundary integral equation.
Equations (20) and (22) are so-called quasi-Fredholm integral equations [10], provided $\beta^{2} \neq 1$. (Recall that we assumed that $\beta^{2}<1$.) They have an index of zero, which means that the usual Fredholm structure is obtained. In particular, existence follows from uniqueness. Also, (20) and (22) are Hermitian adjoints with respect to the $L^{2}$ inner product, and so we can limit our analysis to (22).

To justify these claims, we write (22) as

$$
\left(A_{0}+A_{1}\right) \mu=-f_{\mathrm{inc}}
$$

where $A_{0}$ is the dominant part of the operator on the left-hand side of (22), and $A_{1}$ is less singular. Approximating the kernel of $L^{\prime}$ for small $R$ shows that

$$
A_{0}=I+\beta A
$$

where we have noted that $G(P, Q) \sim \pi^{-1} \log R$ as $R \rightarrow 0$ and

$$
(A \varphi)(z)=\frac{1}{\pi \mathrm{i}} \int_{C} \frac{\varphi(w)}{w-z} \mathrm{~d} w
$$

is the Cauchy integral operator. Explicitly, we find that the dominant part of $\left(L^{\prime} \mu\right)(p)$ is

$$
\frac{1}{\pi} \int_{0}^{1} \frac{\mu(q(\tau))}{\sigma-\tau} \mathrm{d} \tau \simeq \frac{1}{\pi} \int_{0}^{1} \frac{\mu(q(w))}{z-w} \mathrm{~d} w
$$

where $p$ is at $z=x(\sigma)+\mathrm{i} y(\sigma), q$ is at $w=x(\tau)+\mathrm{i} y(\tau)$ and we have used the parametrization described in Section 3. As $A^{2}=I$, we deduce that $A^{-1}=I-\beta A$, provided that $\beta^{2} \neq 1$. The remaining arguments are standard; see, for example, References [10, 11 or 19, Chapter 7].

To proceed, we suppose that $\mu_{0}$ is a non-trivial solution of the homogeneous form of (22),

$$
\begin{equation*}
\left(I+K^{\prime}+\mathrm{i} \beta L^{\prime}\right) \mu_{0}=0 \tag{24}
\end{equation*}
$$

Construct

$$
\zeta_{0}(P)=\int_{C} \mu_{0}(q) G(P, q) \mathrm{d} s_{q}
$$

For $P \in B$, we see that $\zeta_{0}$ solves the homogeneous Poincaré problem, whence $\zeta_{0} \equiv 0$ in $B$. In particular, $\zeta_{0}=0$ on $C$. Then, as $\left(S \mu_{0}\right)(P)$ is continuous across $C$, we see that $\zeta_{0}(P)$ solves the interior Dirichlet problem for $B_{i}$. So, if $k^{2}$ is not an eigenvalue of this problem, we obtain $\zeta_{0} \equiv 0$ in $B_{i}$. In particular, $\partial \zeta_{0} / \partial n+\mathrm{i} \beta \partial \zeta_{0} / \partial s=0$ when computed from the interior, which gives

$$
\left(-I+K^{\prime}+\mathrm{i} \beta L^{\prime}\right) \mu_{0}=0
$$

When this is combined with (24), we deduce that $\mu_{0} \equiv 0$.
Conversely, suppose that $k^{2}$ is an eigenvalue of the interior Dirichlet problem. Hence, there exists an interior field $\zeta_{1}(P) \not \equiv 0$, with $\zeta_{1}=0$ on $C$. An application of Green's theorem in $B_{i}$ to $\zeta_{1}$ and $G$ gives (cf. (17))

$$
-2 \zeta_{1}(P)=\int_{C} G(P, q) \frac{\partial \zeta_{1}}{\partial n_{q}} \mathrm{~d} s_{q}, \quad P \in B_{i}
$$

Hence, setting $\mu_{0}=\partial \zeta_{1} / \partial n_{q}$, we deduce that

$$
S \mu_{0}=0, \quad L^{\prime} \mu_{0}=0 \quad \text { and } \quad\left(I+K^{\prime}\right) \mu_{0}=0
$$

It follows that $\mu_{0}$ also solves (24).
Summarising, unless $k^{2}$ is an eigenvalue of the interior Dirichlet problem, the Poincaré problem can be solved using the single-layer representation (21) where the density $\mu$ solves the singular integral equation (22). Alternatively, one can use the representation based on Green's theorem, (19), together with the singular integral equation (20). Moreover, one can remove the irregular values of $k^{2}$ by modifying the fundamental solution in a standard manner; see, for example Reference [1, Section 3.6].

## 6. DIFFRACTION BY A THIN BARRIER

Let $\Gamma$ be a thin barrier, with two sides, $\Gamma^{+}$and $\Gamma^{-}$. We suppose that the given incident wave $\zeta_{\text {inc }}$ is scattered by $\Gamma$. Then, the scattered field $\zeta_{\mathrm{sc}}$ solves the Poincaré problem; we require, in addition, that $\zeta_{\mathrm{sc}}$ is bounded in the water $B$, including at the two ends of $\Gamma$.

If we surround $\Gamma$ by a closed curve $C$ which we then allow to shrink onto $\Gamma$, we obtain an integral representation from (17), namely

$$
\begin{equation*}
2 \zeta_{\mathrm{sc}}(P)=\int_{\Gamma^{+} \cup \Gamma^{-}}\left\{G(P, q) \frac{\partial \zeta_{\mathrm{sc}}}{\partial n_{q}}-\zeta_{\mathrm{sc}}(q) \frac{\partial}{\partial n_{q}} G(P, q)\right\} \mathrm{d} s_{q}, \quad P \in B \tag{25}
\end{equation*}
$$

We have

$$
\begin{equation*}
\int_{\Gamma^{+} \mathrm{U}-} \zeta_{\mathrm{sc}}(q) \frac{\partial}{\partial n_{q}} G(P, q) \mathrm{d} s_{q}=\int_{\Gamma}\left[\zeta_{\mathrm{sc}}(q)\right] \frac{\partial}{\partial n_{q}} G(P, q) \mathrm{d} s_{q} \tag{26}
\end{equation*}
$$

where

$$
\left[\zeta_{\mathrm{sc}}(q)\right]=\lim _{Q \rightarrow q^{+}} \zeta_{\mathrm{sc}}(Q)-\lim _{Q \rightarrow q^{-}} \zeta_{\mathrm{sc}}(Q)
$$

and $q^{+}$and $q^{-}$are corresponding points on $\Gamma^{+}$and $\Gamma^{-}$, respectively: thus, square brackets denote the discontinuity in a quantity across $\Gamma$. Also, we define $\mathbf{n}(q)$ for $q \in \Gamma$ to be the unit normal vector to $\Gamma^{+}, \mathbf{n}\left(q^{+}\right)$.

Next, write (10) explicitly as

$$
\begin{equation*}
\frac{\partial \zeta_{\mathrm{sc}}}{\partial n}+\mathrm{i} \beta \frac{\partial \zeta_{\mathrm{sc}}}{\partial s}=-\frac{\partial \zeta_{\mathrm{inc}}}{\partial n}-\mathrm{i} \beta \frac{\partial \zeta_{\mathrm{inc}}}{\partial s} \tag{27}
\end{equation*}
$$

We have $\mathbf{n}\left(q^{+}\right)=-\mathbf{n}\left(q^{-}\right)$and $\mathbf{s}\left(q^{+}\right)=-\mathbf{s}\left(q^{-}\right)$. As $\zeta_{\text {inc }}$ is continuous across $\Gamma$, we obtain

$$
\begin{aligned}
0 & =\frac{\partial \zeta_{\mathrm{sc}}}{\partial n_{q}}\left(q^{+}\right)+\mathrm{i} \beta \frac{\partial \zeta_{\mathrm{sc}}}{\partial s_{q}}\left(q^{+}\right)+\frac{\partial \zeta_{\mathrm{sc}}}{\partial n_{q}}\left(q^{-}\right)+\mathrm{i} \beta \frac{\partial \zeta_{\mathrm{sc}}}{\partial s_{q}}\left(q^{-}\right) \\
& =\frac{\partial \zeta_{\mathrm{sc}}}{\partial n_{q}}\left(q^{+}\right)+\frac{\partial \zeta_{\mathrm{sc}}}{\partial n_{q}}\left(q^{-}\right)+\mathrm{i} \beta \frac{\partial}{\partial s_{q}}\left[\zeta_{\mathrm{sc}}(q)\right]
\end{aligned}
$$

where we define $\mathbf{s}(q)$ for $q \in \Gamma$ to be the unit tangent vector to $\Gamma^{+}, \mathbf{s}\left(q^{+}\right)$. Hence

$$
\int_{\Gamma^{+} \cup \Gamma^{-}} G(P, q) \frac{\partial \zeta_{\mathrm{sc}}}{\partial n_{q}} \mathrm{~d} s_{q}=-\mathrm{i} \beta \int_{\Gamma} G(P, q) \frac{\partial}{\partial s_{q}}\left[\zeta_{\mathrm{sc}}(q)\right] \mathrm{d} s_{q} .
$$

If we integrate by parts, noting that $\left[\zeta_{\mathrm{sc}}\right]=0$ at the two ends of $\Gamma$, and combine with (26), we see that (25) becomes

$$
\begin{equation*}
2 \zeta_{\mathrm{sc}}(P)=-\int_{\Gamma}\left\{\frac{\partial}{\partial n_{q}} G(P, q)-\mathrm{i} \beta \frac{\partial}{\partial s_{q}} G(P, q)\right\}[\zeta(q)] \mathrm{d} s_{q}, \quad P \in B \tag{28}
\end{equation*}
$$

here, we have used $\left[\zeta_{\text {sc }}\right]=[\zeta]$. Observe that (28) is similar to (19).
Equation (28) shows that the waves scattered by a thin rigid barrier can be represented in terms of [弓]; the integrand is composed of a certain linear combination of normal and tangential dipoles. In the absence of rotation $(\beta=0)$, $(28)$ reduces to the well-known fact that normal dipoles suffice [18, Equation (1.5)].

To determine [弓], we apply the boundary condition on $\Gamma^{+}$. Write (28) as

$$
2 \zeta_{\mathrm{sc}}(P)=-(D v)(P)-\mathrm{i} \beta\left(S v^{\prime}\right)(P)
$$

where $v \equiv[\zeta]$. Then, (13) $)_{1}$ and (12) give

$$
2 \frac{\partial}{\partial n_{p}} \zeta_{\mathrm{sc}}=-N v-\mathrm{i} \beta\left(I+K^{\prime}\right) v^{\prime}
$$

Similarly, (15) and (13) $)_{2}$ give

$$
2 \frac{\partial}{\partial s_{p}} \zeta_{\mathrm{sc}}=\left(I+K^{\prime}\right) v^{\prime}-k^{2} \mathbf{s} \cdot(S\{v \mathbf{n}\})-\mathrm{i} \beta L^{\prime} v^{\prime}
$$

But (16) gives $L^{\prime} v^{\prime}=N v-k^{2} \mathbf{n} \cdot(S\{v \mathbf{n}\})$, whence

$$
2\left\{\frac{\partial}{\partial n_{p}}+\mathrm{i} \beta \frac{\partial}{\partial s_{p}}\right\} \zeta_{\mathrm{sc}} \equiv-\left(1-\beta^{2}\right) N v-\mathrm{i} \beta k^{2}\{\mathbf{s}+\mathrm{i} \beta \mathbf{n}\} \cdot(S\{v \mathbf{n}\})
$$

Hence, the boundary condition (27) gives

$$
\begin{equation*}
\left(1-\beta^{2}\right) N v+\mathrm{i} \beta k^{2}\{\mathbf{s}+\mathrm{i} \beta \mathbf{n}\} \cdot(S\{v \mathbf{n}\})=2 f_{\mathrm{inc}} \tag{29}
\end{equation*}
$$

where $f_{\text {inc }}$ is defined by (23).
Equation (29) is a hypersingular integral equation for $v(q) \equiv[\zeta(q)]$, the discontinuity in $\zeta$ across $\Gamma$. It is to be solved subject to $v=0$ at the two ends of $\Gamma$. When $\beta=0$, (29) reduces to

$$
\begin{equation*}
N v=2(\partial / \partial n) \zeta_{\mathrm{inc}} \tag{30}
\end{equation*}
$$

which is a well-studied integral equation. Very effective numerical methods for its solution have been developed, in which $\Gamma$ is parametrized and the strong singularity in $N$ is extracted, leading to an integral equation of the form

$$
\int_{-1}^{1} \frac{v(t)}{(x-t)^{2}} \mathrm{~d} t+\int_{-1}^{1} v(t) \mathscr{K}(x, t) \mathrm{d} t=f(x), \quad-1<x<1
$$

Here, the first integral is a finite-part integral and $\mathscr{K}$ is a known kernel with a weak singularity. Then, an approximation

$$
v(t) \simeq \sqrt{1-t^{2}} \sum_{n=0}^{N} a_{n} U_{n}(t)
$$

can be sought in terms of Chebyshev polynomials of the second kind, $U_{n}$, defined by $U_{n}(\cos \theta)=\{\sin (n+1) \theta\} / \sin \theta$. The coefficients $a_{n}$ are determined by collocation. The method is effective because the known square-root zeros of $v(t)$ at the two end-points are incorporated, and because the finite-part integral can be evaluated explicitly:

$$
\int_{-1}^{1} \frac{\sqrt{1-t^{2}}}{(x-t)^{2}} U_{n}(t) \mathrm{d} t=-\pi(n+1) U_{n}(x)
$$

For more information on this method, see, for example, References [18] or [20].
The effect of rotation is merely to alter the weakly singular kernel $\mathscr{K}$; the dominant part of the integral equation is the operator $N$, and this is treated effectively by the expansioncollocation method using Chebyshev polynomials.

Moreover, if $\beta$ is small (recall that $|\beta|<1$ ), a simple perturbation method can be developed. Thus, put

$$
v=v_{0}+\mathrm{i} \beta v_{1}+\cdots
$$

Then, (29) gives

$$
\begin{align*}
& N v_{0}=2(\partial / \partial n) \zeta_{\text {inc }}  \tag{31}\\
& N v_{1}=2(\partial / \partial s) \zeta_{\text {inc }}-k^{2} \mathbf{s} \cdot\left(S\left\{v_{0} \mathbf{n}\right\}\right) \tag{32}
\end{align*}
$$

and so on. Equation (31) gives the leading-order term $v_{0}$; it is the same as (30) and must be solved subject to $v_{0}=0$ at the two ends of $\Gamma$. The first correction for small $\beta, v_{1}$, is obtained by solving (32), which is the same hypersingular integral equation as (31), but with a different right-hand side.

## 7. DISCUSSION

It is interesting to compare our method with that described by Krutitskii [13] for thin barriers. He uses a combination of single-layer and 'angular' potentials, the latter being defined by

$$
\int_{\Gamma} \mu(q) V(P, q) \mathrm{d} s_{q}
$$

where

$$
V\left(P, q^{\prime}\right)=\int_{q_{0}}^{q^{\prime}} \frac{\partial}{\partial n_{q}} G(P, q) \mathrm{d} s_{q}
$$

and $q_{0}$ is one end of $\Gamma$. This approach leads to a singular integral equation on $\Gamma$. The integral equation involves a constant that has to be determined using a side condition. Its presence is not surprising, because transforming from a hypersingular to a singular integral equation requires one integration. In fact, Krutitskii considers $n$ barriers and so he must determine $n$ constants; our formulation does not change if the number of barriers is increased.

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[^0]:    * Correspondence to: P. A. Martin, Department of Mathematical and Computer Sciences, Colorado School of Mines, Golden, CO-80401-1887, U.S.A.

