WAVES IN WOOD: AXISYMMETRIC GUIDED WAVES ALONG BOREHOLES

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ABSTRACT

Biot showed that elastic waves could be guided by a cylindrical hole in a homogeneous isotropic elastic solid. This problem is generalized to materials with cylindrical orthotropy, this being a plausible model for waves in wood. For time-harmonic motions, the problem is reduced to some coupled ordinary differential equations. Methods for solving these equations are discussed. These include the method of Frobenius (power-series expansions) and the use of asymptotic expansions (valid far from the cylindrical hole).

Keywords : Elastic waves, Cylindrical orthotropy.

1. INTRODUCTION

Rayleigh waves are surface waves. They propagate freely over the traction-free boundary of a homogeneous, isotropic elastic half-space [1:Sect.6.1.4]. Extensions to anisotropic half-spaces have been made by many authors; see, example, Refs. [2,3:Chpt.12,4,5].

The motions in a Rayleigh wave decay exponentially with depth from the free surface. Motions of a similar character exist exterior to a cylindrical cavity: waves can propagate in a direction parallel to the cylinder's generators without attenuation. Such *guided modes* were constructed by Biot [6] for a cavity of circular cross-section; he considered axisymmetric solutions in an isotropic solid. Extensions to non-axisymmetric modes and to non-circular cylinders have been made; see [7~11]. Certain inhomogeneous isotropic solids can also support guided modes [12].

In recent years, we have made several studies of problems involving cylindrical anisotropy, including waves in wooden poles [13] and along wire rope [14]. For this work, we have been guided by Tom Ting's impressive book [3] and his papers on solids with cylindrical anisotropy, such as [15]. Here, we shall describe a preliminary attempt to construct axisymmetric guided modes for a cylindrical cavity of circular cross-section in a solid with cylindrical orthotropy. Thus, assume that the elastic stiffnesses are constants when referred to cylindrical polar coordinates, r, θ, z and that the surface of the cylindrical cavity is $r = a$.

We are interested in the propagation of elastic waves in wooden poles, using the cylindrical-orthotropy model. One application is to understand the use of ultrasonic devices for determining whether wooden telegraph poles have decayed internally [13]. Application of these techniques to live trees has also been made [16]. Further information on waves in wood can be found in the book by Bucur [17].

In this paper, we begin by formulating the problem of wave propagation in a material with cylindrical anisotropy, using matrix notation where possible. We look for time-harmonic solutions, with a prescribed dependence on θ and z, leading to a 3×3 system of coupled ordinary differential equations in the radial direction. To simplify further, we suppose that the material is cylindrically orthotropic and that the motions are axisymmetric. We show that axisymmetric guided modes cannot have a torsional component, so that we are left with a 2×2 system to solve. We review Biot's exact solution for isotropic solids. This motivates our approach to non-isotropic solids. Thus, we first seek expansions in terms of a series in inverse powers of r multiplied by $e^{-\ell r}$. Allowable values for ℓ are found. This leads to asymptotic approximations to the solutions, valid for large r. To make further progress, we discuss alternative expansions; these are needed so that the boundary condition on the cavity can be imposed. One possible expansion comes from the method of Frobenius (expansions in powers of r and log r). This method is described in Sect. 7. Elastic waves in materials with cylindrical orthotropy have been computed using the method of Frobenius by many authors, including Ohnabe and Nowinski [18], Chou and Achenbach [19], Markuš and Mead [20] and Yuan and Hsieh [21].

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However, we are not aware of any attempts to use this method for guided modes.

Another possibility (not explored here) comes from expanding in series of modified Bessel functions, $K_{n+\alpha}(\ell r)$. This latter route seems attractive because it incorporates known facts about the solution (namely, the correct exponential decay), and because the Biot solution involves modified Bessel functions. We have previously used a similar method, using Bessel functions $J_{n+\alpha}(kr)$ (Neumann series), for free vibrations of a wooden pole in a cross-sectional plane (with no dependence on z) [13].

In summary, our purpose is to give a description of an interesting problem in anisotropic elastodynamics, together with a discussion on some semi-analytical techniques for its solution. We hope that this mixture will appeal to Professor Ting on the occasion of his 70th birthday!

2. GOVERNING EQUATIONS

In the absence of body forces, the governing equations of motion are

$$\frac{\partial}{\partial r}(r\boldsymbol{t}_r) + \frac{\partial}{\partial \theta}\boldsymbol{t}_\theta + \mathbf{K}\boldsymbol{t}_\theta + r\frac{\partial}{\partial z}\boldsymbol{t}_z = \rho r \frac{\partial^2}{\partial t^2}\tilde{\boldsymbol{u}} \tag{1}$$

where ρ is the density,

$$\boldsymbol{t}_r = \begin{pmatrix} \tau_{rr} \\ \tau_{r\theta} \\ \tau_{rz} \end{pmatrix}, \quad \boldsymbol{t}_\theta = \begin{pmatrix} \tau_{\theta r} \\ \tau_{\theta\theta} \\ \tau_{\theta z} \end{pmatrix}, \quad \boldsymbol{t}_z = \begin{pmatrix} \tau_{zr} \\ \tau_{z\theta} \\ \tau_{zz} \end{pmatrix} \tag{2}$$

$$\mathbf{K} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{\boldsymbol{u}} = \begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix}$$

is the displacement vector and τ_{ij} are the stress components. In what follows, we generalize the matrix formulation of Ting [15] for static problems. Ting (see Eq. (2.5b) in [15]) gives the following expression for the traction vectors $\mathbf{t}_i$ in terms of $\tilde{\boldsymbol{u}}$:

$$\boldsymbol{t}_r = \mathbf{Q}\frac{\partial}{\partial r}\tilde{\boldsymbol{u}} + \frac{1}{r}\mathbf{R}\left(\frac{\partial}{\partial\theta}\tilde{\boldsymbol{u}} + \mathbf{K}\tilde{\boldsymbol{u}}\right) + \mathbf{P}\frac{\partial}{\partial z}\tilde{\boldsymbol{u}}$$

$$\boldsymbol{t}_\theta = \mathbf{R}^T\frac{\partial}{\partial r}\tilde{\boldsymbol{u}} + \frac{1}{r}\mathbf{T}\left(\frac{\partial}{\partial\theta}\tilde{\boldsymbol{u}} + \mathbf{K}\tilde{\boldsymbol{u}}\right) + \mathbf{S}\frac{\partial}{\partial z}\tilde{\boldsymbol{u}} \tag{3}$$

$$\boldsymbol{t}_z = \mathbf{P}^T\frac{\partial}{\partial r}\tilde{\boldsymbol{u}} + \frac{1}{r}\mathbf{S}^T\left(\frac{\partial}{\partial\theta}\tilde{\boldsymbol{u}} + \mathbf{K}\tilde{\boldsymbol{u}}\right) + \mathbf{M}\frac{\partial}{\partial z}\tilde{\boldsymbol{u}}$$

In these expressions,

$$\mathbf{Q} = \begin{pmatrix} C_{11} & C_{16} & C_{15} \\ C_{16} & C_{66} & C_{56} \\ C_{15} & C_{56} & C_{55} \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} C_{16} & C_{12} & C_{14} \\ C_{66} & C_{26} & C_{46} \\ C_{56} & C_{25} & C_{45} \end{pmatrix}$$

$$\mathbf{T} = \begin{pmatrix} C_{66} & C_{26} & C_{46} \\ C_{26} & C_{22} & C_{24} \\ C_{46} & C_{24} & C_{44} \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} C_{15} & C_{14} & C_{13} \\ C_{56} & C_{46} & C_{36} \\ C_{55} & C_{45} & C_{35} \end{pmatrix}$$

$$\mathbf{M} = \begin{pmatrix} C_{55} & C_{45} & C_{35} \\ C_{45} & C_{44} & C_{34} \\ C_{35} & C_{34} & C_{33} \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} C_{56} & C_{46} & C_{36} \\ C_{25} & C_{24} & C_{23} \\ C_{45} & C_{44} & C_{34} \end{pmatrix}$$

where $\mathbf{R}^T$ is the transpose of $\mathbf{R}$, and we have used the contracted notation $C_{\alpha\beta}$ for the elastic stiffnesses with $(1, 2, 3) = (r, \theta, z)$. Note that $\mathbf{Q}$, $\mathbf{T}$ and $\mathbf{M}$ are symmetric matrices.

We look for time-harmonic solutions of Eq. (1) in the form

$$\tilde{\boldsymbol{u}}(r, \theta, z, t) = \mathrm{Re}_i\{\boldsymbol{u}_m(r)\, e^{jm\theta} e^{i\xi z} e^{-i\omega t}\} \tag{4}$$

where i and j are two non-interacting complex units, m is an integer, ξ is the axial wavenumber, ω is the radian frequency, and Re_i denotes the real part with respect to i. Use of $e^{jm\theta}$ rather than $\cos m\theta$ and $\sin m\theta$ allows us to retain the nice matrix notation in what follows. Thus, we find that $u_m(r)$ solves

$$r\,\mathbf{Q}(r\boldsymbol{u}_m')' + r\mathbf{A}\,\boldsymbol{u}_m' + \mathbf{B}\,\boldsymbol{u}_m = \mathbf{0} \tag{5}$$

where

$$\boldsymbol{A} = \boldsymbol{R}\boldsymbol{K}_m + \boldsymbol{K}_m\boldsymbol{R}^T + i\xi\, r(\boldsymbol{P} + \boldsymbol{P}^T)$$

$$\boldsymbol{B} = \rho\,\omega^2 r^2 \boldsymbol{I} - \xi^2 r^2 \boldsymbol{M} + \boldsymbol{K}_m\boldsymbol{T}\boldsymbol{K}_m + i\,\xi\, r\,(\boldsymbol{P} + \boldsymbol{K}_m\boldsymbol{S} + \boldsymbol{S}^T\boldsymbol{K}_m)$$

$\mathbf{I}$ is the identify and $\mathbf{K}_m = \mathbf{K} + jm\mathbf{I}$.

Making use of Eqs. (3) and (4), and writing

$$\boldsymbol{t}_r(r, \theta, z, t) = \mathrm{Re}_i\{\boldsymbol{t}_m(r)\, e^{jm\theta} e^{i\xi z} e^{-i\omega t}\}$$

we obtain

$$\boldsymbol{t}_m(r) = \mathbf{Q}\,\boldsymbol{u}_m' + r^{-1}\mathbf{R}\mathbf{K}_m\,\boldsymbol{u}_m + i\xi\mathbf{P}\,\boldsymbol{u}_m \tag{6}$$

For two-dimensional motions independent of z ($\xi = 0$), we recover the equations studied in [13]. If we also put $m = 0$ (axisymmetry) and $\omega = 0$ (static), we obtain the equations solved by Ting [15].

Setting $\boldsymbol{u}_m = (u_m, v_m, w_m)$, Eq. (5) gives three coupled ordinary differential equations for the three components of $\boldsymbol{u}_m$. In general, these equations do not decouple.

2.1 Guided Modes

Consider a circular cylindrical cavity, $r = a$. For a guided mode, we seek solutions of Eq. (5) for $r > a$ that satisfy $\boldsymbol{t}_m(a) = \mathbf{0}$ and that decay rapidly as $r \to \infty$.

To simply the problem, we shall limit attention here to axisymmetric motions ($m = 0$) and to cylindrical orthotropy.

3. CYLINDRICAL ORTHOTROPY

For materials with cylindrical orthotropy, there are nine non-trivial stiffnesses, namely C_{11}, C_{12}, C_{13}, C_{22}, C_{23}, C_{33}, C_{44}, C_{55} and C_{66}. The matrices **Q**, **R**, **T**, **P**, **M** and **S** simplify to

$$\mathbf{Q} = \begin{pmatrix} C_{11} & 0 & 0 \\ 0 & C_{66} & 0 \\ 0 & 0 & C_{55} \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} 0 & C_{12} & 0 \\ C_{66} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{T} = \begin{pmatrix} C_{66} & 0 & 0 \\ 0 & C_{22} & 0 \\ 0 & 0 & C_{44} \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 0 & 0 & C_{13} \\ 0 & 0 & 0 \\ C_{55} & 0 & 0 \end{pmatrix}$$

$$\mathbf{M} = \begin{pmatrix} C_{55} & 0 & 0 \\ 0 & C_{44} & 0 \\ 0 & 0 & C_{33} \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & C_{23} \\ 0 & C_{44} & 0 \end{pmatrix}$$

and the system (5) simplifies accordingly.

Note that isotropy is a special case of cyclindrical orthotropy. For isotropic materials, $C_{11} = C_{22} = C_{33} = \lambda + 2\mu$, $C_{12} = C_{13} = C_{23} = \lambda$ and $C_{44} = C_{55} = C_{66} = \mu$, where λ and μ are the Lamé Moduli. Exact solutions of Eq. (5) are well known for isotropic solids. They are given in textbooks on elastic waves; see, for example, Graff [1:Sect.8.2].

4. AXISYMMETRIC MOTIONS

For axisymmetric motions ($m = 0$) of a cylindrically orthotropic solid, we have $\mathbf{K}_0 = \mathbf{K}$, $\mathbf{RK} = -\mathbf{KR}^T$,

$$\mathbf{A} = i\xi\, r \begin{pmatrix} 0 & 0 & C_{13} + C_{55} \\ 0 & 0 & 0 \\ C_{13} + C_{55} & 0 & 0 \end{pmatrix}$$

and

$$\mathbf{B} = \begin{pmatrix} B_5 r^2 - C_{22} & 0 & i\xi r(C_{13} - C_{23}) \\ 0 & B_4 r^2 - C_{66} & 0 \\ i\xi r(C_{55} + C_{23}) & 0 & B_3 r^2 \end{pmatrix}$$

where

$$B_i = \rho\, \omega^2 - \xi^2 C_{ii} \qquad \text{(no sum)}$$

Similarly, Eq. (6) gives

$$\boldsymbol{t}_0(r) = \begin{pmatrix} C_{11} u_0' + r^{-1} C_{12} u_0 + i\xi C_{13} w_0 \\ C_{66}(v_0' - r^{-1} v_0) \\ C_{55}(w_0' + i\xi u_0) \end{pmatrix} \tag{7}$$

So, the (axisymmetric) torsional components v_0 decouples from the radial and axial components, u_0 and w_0, respectively.

It turns out that v_0 cannot yield a guided mode. To see this, we first note that v_0 satisfies

$$r\, C_{66}(r v_0')' + (B_4 r^2 - C_{66})\, v_0 = 0$$

This is Bessel's equation; write it as

$$r(r v_0')' - (\ell_0^2 r^2 + 1)\, v_0 = 0$$

where $\ell_0^2 = \xi^2 - \kappa_0^2$ and $\kappa_0 = \omega\sqrt{\rho / C_{66}}$. We want solutions that decay rapidly as $r \to \infty$; this implies that $|\xi| > \kappa_0$, giving the solution

$$v_0(r) = A\, K_1(\ell_0 r) \tag{8}$$

where $K_n(w)$ is a modified Bessel function and A is an arbitrary constant.

From Eq. (7), we require

$$v_0'(a) - a^{-1} v_0(a) = 0$$

Substitution of Eq. (8) gives $AK_2(\ell_0 a) = 0$. But $K_2(w)$ does not have any real zeros, so that the only possibility is $A = 0$ giving $v_0(r) \equiv 0$.

Having disposed of the torsional component, we consider the pair of equations for u_0 and w_0:

$$r C_{11}(r u_0')' + i\xi\, r^2 (C_{13} + C_{55})\, w_0' + (B_5 r^2 - C_{22})\, u_0 + i\xi\, r(C_{13} - C_{23})\, w_0 = 0 \tag{9}$$

$$r C_{55}(r w_0')' + i\xi\, r^2 (C_{13} + C_{55})\, u_0' + i\xi\, r(C_{55} + C_{23})\, u_0 + B_3 r^2 w_0 = 0 \tag{10}$$

Note that solutions of this system do not depend on the stiffness C_{12}. However, C_{12} does appear in the boundary condition, through the first component of $\boldsymbol{t}_0(r)$; see Eq. (7).

Equations (9) and (10) decouple when $\xi = 0$ [13]. Typical solution pairs are found to be

$$u_0 = 0 \quad \text{and} \quad w_0 = J_0(\kappa\, r)$$

and

$$u_0 = J_r(\kappa_1 r) \quad \text{and} \quad w_0 = 0$$

when $\gamma = \sqrt{C_{22}/C_{11}}$, $\kappa_1 = \omega\sqrt{\rho / C_{11}}$ and $\kappa = \omega\sqrt{\rho / C_{55}}$. Of course, we are really interested in non-zero ξ, as $2\pi/|\xi|$ is the axial wavelength.

5. ISOTROPY: THE BIOT SOLUTION

For isotropic materials, one solution pair for Eqs. (9) and (10) is

$$u_0 = \xi K_1(\ell_s r) \quad \text{and} \quad w_0 = -i\ell_s K_0(\ell_s r) \tag{11}$$

where $\ell_s^2 = \xi^2 - \rho\omega^2/\mu$. Another is

$$u_0 = \ell_p K_1(\ell_p r) \quad \text{and} \quad w_0 = -i\xi\, K_0(\ell_p r) \tag{12}$$

where $\ell_p^2 = \xi^2 - \rho\omega^2/(1 + 2\mu) > \ell_s^2$.

We take a linear combination of these solutions and write

$$u_0 = A\xi\, K_1(\ell_s r) + B\ell_p K_1(\ell_p r)$$

$$w_0 = -i\,[A\ell_s K_0(\ell_s r) + B\xi\, K_0(\ell_p r)]$$

where A and B are arbitrary constants. Next, we impose the boundary condition $\boldsymbol{t}_0(a) = \mathbf{0}$, using Eq. (7). This gives

$$2A\ell_s\, a\, \xi\, K_1'(\ell_s a)$$
$$-[(\xi^2 + \ell_s^2)\, a\, K_0(\ell_p a) + 2\ell_p K_1(\ell_p a)]\, B = 0$$
$$A(\xi^2 + \ell_s^2)\, K_1(\ell_s a) + 2B\xi\ell_p K_1(\ell_p a) = 0$$

For non-trivial solutions of this system, we set the determinant equal to zero. This gives

$$\begin{aligned} &4\xi^2\, \ell_s\, \ell_p\, a\, K_1(\ell_p a)\, K_1'(\ell_s a) \\ &+ (\xi^2 + \ell_s^2)[(\xi^2 + \ell_s^2)\, a\, K_0(\ell_p a) + 2\ell_p K_1(\ell_p a)] \\ &\cdot K_1(\ell_s a) = 0 \end{aligned} \tag{13}$$

This equation, sometimes called the *secular equation*, can be thought of as giving the axial wavenumber ξ in terms of the frequency and the elastic properties. More precisely, it gives the dimensionless wavenumber ξa in terms of Poisson's ratio and the dimensionless quantity $a\omega\sqrt{\rho/\mu}$. Equation (13) was obtained by Biot [6]; see also [22:Sect.6.6]. For a detailed discussion (and extension to non-axisymmetric modes), see [8].

There is a cut-off frequency, below which guided modes do not exist. As we want ℓ_p and ℓ_s to be real, cut-off corresponds to $\ell_s = 0$, which means that the axial wavenumber must satisfy $|\xi| \geq \omega\sqrt{\rho/\mu}$, the shear-wave speed.

Notice that the solutions (11) and (12) decay exponentially with r; specifically,

$$K_\nu(w) \sim \sqrt{\frac{\pi}{2w}} e^{-w} \left\{ 1 + \frac{4\nu^2 - 1}{8w} + \frac{(4\nu^2 - 1)(4\nu^2 - 9)}{128 w^2} + \cdots \right\} \tag{14}$$

as $w \to \infty$ [23:Eq.9.7.2].

6. THE COUPLED SYSTEM: ASYMPTOTIC APPROXIMATIONS

For axisymmetric motions of a cylindrically orthotropic material, we have to solve the coupled system, Eqs. (9) and (10). Let us begin with a slight simplification of notation, defining dimensionless stiffnesses by

$$c_1 = \frac{C_{11}}{C_{55}}, \quad c_2 = \frac{C_{22}}{C_{55}}, \quad c_3 = \frac{C_{33}}{C_{55}}$$
$$c_{12} = \frac{C_{12}}{C_{55}}, \quad c_{13} = \frac{C_{13}}{C_{55}} \quad \text{and} \quad c_{23} = \frac{C_{23}}{C_{55}} \tag{15}$$

For example, the data in [13] for Scots pine gives

$$c_1 \simeq 1.31, \quad c_2 \simeq 0.67, \quad c_3 \simeq 14.9$$
$$c_{12} \simeq 0.47, \quad c_{13} \simeq 0.96 \quad \text{and} \quad c_{23} \simeq 0.56 \tag{16}$$

Making use of Eq. (15), Eqs. (9) and (10) become

$$\begin{aligned} &rc_1(ru_0')' + i\xi\, r^2(1 + c_{13})w_0' \\ &+ [(\kappa^2 - \xi^2)r^2 - c_2]\, u_0 + i\xi\, r(c_{13} - c_{23})\, w_0 = 0 \end{aligned} \tag{17}$$

$$\begin{aligned} &r(rw_0')' + i\xi\, r^2(1 + c_{13})u_0' \\ &+ i\xi\, r(1 + c_{23})\, u_0 + (\kappa^2 - c_3\xi^2)\, r^2\, w_0 = 0 \end{aligned} \tag{18}$$

where $\kappa^2 = \rho\omega^2/\, C_{55}$. We want to solve this system for $r > a$, and we are looking for solutions that decay exponentially as $r \to \infty$. Thus, motivated by the isotropic results in Sect. 5, we try

$$u_0(r) = e^{-\ell r} \sum_{n=0} U_n\, r^{\sigma - n} \tag{19}$$

$$w_0(r) = i e^{-\ell r} \sum_{n=0} W_n\, r^{\sigma - n} \tag{20}$$

where ℓ, σ, U_n and W_n are to be found. Differentiating, we have, for example,

$$r\, u_0' = e^{-\ell r} \sum_{n=-1} \{(\sigma - n)\, U_n - \ell\, U_{n+1}\}\, r^{\sigma - n}$$

and

$$r(ru_0')' = e^{-\ell r} \sum_{n=-2} \{(\sigma - n)^2\, U_n - \ell(2\sigma - 2n - 1)\, U_{n+1} + \ell^2\, U_{n+2}\}\, r^{\sigma - n}$$

where we define $U_{-1} = U_{-2} = 0$. Substitution in Eqs. (17) and (18) gives

$$\begin{aligned} &A_{11}\, U_{n+2} + A_{12}\, W_{n+2} - c_1\, \ell(2\sigma - 2n - 1)U_{n+1} \\ &- \xi\, [(1 + c_{13})(\sigma - n) - 1 - c_{23}]\, W_{n+1} \\ &+ [c_1(\sigma - n)^2 - c_2]\, U_n = 0 \end{aligned} \tag{21}$$

$$\begin{aligned} &-A_{12}\, U_{n+2} + A_{22}\, W_{n+2} - \xi\, [(1 + c_{13})(\sigma - n) - c_{13} + c_{23}]\, U_{n+1} \\ &- \ell(2\sigma - 2n - 1)\, W_{n+1} + (\sigma - n)^2 W_n = 0 \end{aligned} \tag{22}$$

for $n = -2, -1, 0, 1, \ldots$, where

$$A_{11} = c_1\, \ell^2 + \kappa^2 - \xi^2, \quad A_{12} = \xi\ell\, (1 + c_{13}) \quad \text{and}$$
$$A_{22} = \ell^2 + \kappa^2 - c_3\, \xi^2$$

Solution for $n = -2$

When $n = -2$, Eqs. (21) and (22) reduce to

$$A_{11}\, U_0 + A_{12}\, W_0 = 0 \tag{23}$$

$$-A_{12}\, U_0 + A_{22}\, W_0 = 0 \tag{24}$$

This homogeneous system will have non-trivial solutions provided that

$$A_{11}A_{22} + A_{12}^2 = 0 \tag{25}$$

This condition yields a quadratic for ℓ^2:

$$c_1\ell^4 + \ell^2\{\xi^2[(1+c_{13})^2 - 1 - c_1c_3] + (1+c_1)\kappa^2\} + (\xi^2 - \kappa^2)(c_3\xi^2 - \kappa^2) = 0 \tag{26}$$

For an isotropic solid, Eq. (26) reduces to

$$(\ell^2 - \ell_p^2)(\ell^2 - \ell_s^2) = 0 \tag{27}$$

as expected, where ℓ_p and ℓ_s are defined in Sect. 5. At the cut-off frequency, $\ell_s = 0$ and then $\ell = 0$ is a solution of Eq. (27).

Similarly, Eq. (26) shows that cut-off occurs when $\xi^2 = \kappa^2$ or when $c_3\xi^2 = \kappa^2$. When $\xi^2 = \kappa^2$, Eq. (26) reduces to

$$\ell^2 = \kappa^2\{c_3 - 1 - (1+c_{13})^2/c_1\}$$

whereas when $\xi^2 = \kappa^2/c_3$, Eq. (26) reduces to

$$\ell^2 = -\kappa^2\{c_{13}(2+c_{13}) + c_3\}/(c_1c_3)$$

The right-hand sides of these expressions must be positive at cut-off. For Scots pine, the data (16) give

$$\ell^2 \simeq 11\kappa^2 \quad \text{and} \quad \ell^2 \simeq -0.9\kappa^2$$

respectively, so that cut-off occurs at $\xi^2 = \kappa^2$ (for this particular material).

Solution for $n = -1$

When $n = -1$, Eqs. (21) and (22) reduce to

$$A_{11}U_1 + A_{12}W_1 = F_1 \tag{28}$$

$$-A_{12}U_1 + A_{22}W_1 = G_1 \tag{29}$$

where

$$F_1 = c_1\ell(2\sigma+1)U_0 + \xi\{(1+c_{13})\sigma + c_{13} - c_{23}\}W_0$$

$$G_1 = \xi\{(1+c_{13})(\sigma+1) - c_{13} + c_{23}\}U_0 - \ell(2\sigma+1)W_0$$

The system (28) and (29) cannot be uniquely solvable. In order to have any solutions, a consistency condition must be satisfied. This is obtained from the condition (25): $F_1A_{22} - G_1A_{12} = 0$. As $A_{22}W_0 = A_{12}U_0$ and $A_{12}W_0 = -A_{11}U_0$, we find that

$$(2\sigma+1)U_0\{(A_{11} + c_1A_{22})\ell + (1+c_{13})\xi A_{12}\} = 0$$

for consistency; this can be ensured by taking $\sigma = -1/2$ (as might have been expected by inspection of Eq. (14)). Hence, U_1 and W_1 must satisfy either one of Eqs. (28) and (29); taking the first gives

$$A_{11}U_1 + A_{12}W_1 = \xi\left\{-\frac{1}{2}(1+c_{13}) + c_{13} - c_{23}\right\}W_0 \tag{30}$$

We obtain a second equation by going to the next order in n.

Solution for $n = 0, 1, 2, \ldots$

For $n \geq 0$, Eqs. (21) and (22) reduce to

$$A_{11}U_{n+2} + A_{12}W_{n+2} = F_{n+2} \tag{31}$$

$$-A_{12}U_{n+2} + A_{22}W_{n+2} = G_{n+2} \tag{32}$$

where

$$F_{n+2} = -2c_1\ell(n+1)U_{n+1} - \xi\left[(1+c_{13})\left(n+\frac{1}{2}\right) + 1 + c_{23}\right]W_{n+1} - \left[c_1\left(n+\frac{1}{2}\right)^2 - c_2\right]U_n$$

$$G_{n+2} = \xi\left[(1+c_{13})\left(n+\frac{1}{2}\right) + c_{13} - c_{23}\right]U_{n+1} - 2\ell(n+1)W_{n+1} - \left(n+\frac{1}{2}\right)^2 W_n$$

We also have the consistency condition,

$$A_{22}F_{n+2} - A_{12}G_{n+2} = 0 \tag{33}$$

We now have enough equations to determine all the unknowns in the expansions (19) and (20). Thus, we first determine ℓ from Eq. (26). Then, choose U_0 arbitrarily (non-zero) and calculate W_0 from either Eq. (23) or (24). Next, calculate U_1 and W_1 by solving Eqs. (30) and (33) (with $n = 0$). Subsequent coefficients are determined recursively, using Eqs. (31) and (33) (with n replaced by $n + 1$ in the latter).

7. THE COUPLED SYSTEM: CONVERGENT EXPANSIONS

A standard technique for solving ordinary differential equations is the method of Frobenius, in which one looks for solutions in the form of power series. The method proceeds by writing

$$u_0(r) = \sum_{n=0}^{\infty} a_n(\kappa r)^{n+\alpha} \quad \text{and} \quad w_0(r) = i\sum_{n=0}^{\infty} b_n(\kappa r)^{n+\alpha}$$

where the coefficients a_n and b_n, and the exponent α are to be determined by substitution in Eqs. (17) and (18). If we define σ by

$$\xi = \sigma\kappa$$

we find that

$$\{c_1(n+\alpha)^2 - c_2\}a_n - \sigma\{(1+c_{13})(n+\alpha-1) + c_{13} - c_{23}\}b_{n-1} + (1-\sigma^2)a_{n-2} = 0$$

and

$$(n+\alpha)^2\, b_n + \sigma\{(1+c_{13})(n+\alpha-1)+1+c_{23}\}\, a_{n-1} + (1-c_3\sigma^2)\, b_{n-2} = 0$$

for $n = 0, 1, 2, \ldots$, where, by definition, $a_{-2} = a_{-1} = b_{-2} = b_{-1} = 0$. When $n = 0$, these equations reduce to

$$(c_1\alpha^2 - c_2)\, a_0 = 0 \quad \text{and} \quad \alpha^2 b_0 = 0$$

Thus we can take $\alpha = \pm\sqrt{c_2/c_1}$, $b_0 = 0$ and $a_0 \neq 0$, leading to two solutions. A third solution comes by taking $\alpha = 0$, $a_0 = 0$ and $b_0 \neq 0$. The fact that $\alpha = 0$ is a double root implies that the fourth solution will involve $\log r$; it may be constructed in a standard manner.

Having selected α and starting values for a_0 and b_0, all the remaining coefficients can be calculated efficiently by recursion.

We are interested in constructing solutions that decay exponentially as $r \to \infty$. These solutions must be singular at $r = 0$, which means that we should take the solution constructed with $\alpha = -\sqrt{c_2/c_1}$ and the logarithmic solution corresponding to the double root at $\alpha = 0$. Notice that isotropy is a degenerate case, because it has $c_2/c_1 = 1$. (The method of Frobenius must be modified when some of the roots α are equal or differ by an integer.) We now consider the construction of these two solutions in detail.

Denote the solution obtained with $\alpha = -\sqrt{c_2/c_1}$ and $b_0 = 0$ with a superscript 1, so that

$$u_0^{(1)}(r) = \sum_{n=0}^{\infty} a_n^{(1)} (\kappa r)^{2n+\alpha} \quad \text{and} \quad w_0^{(1)}(r) = i\sum_{n=0}^{\infty} b_n^{(1)} (\kappa r)^{2n+1+\alpha}$$

where $a_0^{(1)}$ is arbitrary,

$$(2n+1+\alpha)^2\, b_n^{(1)} = -\sigma\{(1+c_{13})(2n+\alpha)+1+c_{23}\}\, a_n^{(1)} + (c_3\sigma^2 - 1)\, b_{n-1}^{(1)}$$

for $n = 0, 1, 2, \ldots$ with $b_{-1}^{(1)} = 0$, and

$$\{c_1(2n+\alpha)^2 - c_2\}\, a_n^{(1)} = \sigma\{(1+c_{13})(2n+\alpha-1)+c_{13}-c_{23}\}\, b_{n-1}^{(1)} + (\sigma^2-1)\, a_{n-1}^{(1)}$$

for $n = 1, 2, \ldots$ (Here, we have removed a_{2n+1} and b_{2n} as they all vanish.)

Denote the (regular) solution obtained with $\alpha = 0$ and $a_0 = 0$ with a superscript 2, so that

$$u_0^{(2)}(r) = \sum_{n=0}^{\infty} a_n^{(2)} (\kappa\, r)^{2n+1} \quad \text{and} \quad w_0^{(2)}(r) = i\sum_{n=0}^{\infty} b_n^{(2)} (\kappa\, r)^{2n}$$

where $b_0^{(2)}$ is arbitrary,

$$\{(2n+1)^2 c_1 - c_2\}\, a_n^{(2)} = \sigma\,\{2n(1+c_{13})+c_{13}-c_{23}\}\, b_n^{(2)} + (\sigma^2-1)\, a_{n-1}^{(2)}$$

for $n = 0, 1, 2, \ldots$ with $a_{-1}^{(2)} = 0$, and

$$4nb_n^{(2)} = -\sigma\{(2n-1)(1+c_{13})+1+c_{23}\}\, a_{n-1}^{(2)} + (c_3\sigma^2-1)\, b_{n-1}^{(2)}$$

for $n = 1,2, \ldots$ (Here, we have removed a_{2n} and b_{2n+1} as they all vanish.) However, we want the other solution corresponding to $\alpha = 0$. To construct it, we write

$$u_0^{(3)}(r) = u_0^{(2)}(r)\log r + \widetilde{u}(r) \quad \text{and}$$
$$w_0^{(3)}(r) = w_0^{(2)}(r)\log r + \widetilde{w}(r)$$

Substitution in Eqs. (17) and (18) gives

$$rc_1(r\widetilde{u}')' + i\xi\, r^2(1+c_{13})\,\widetilde{w}' + [(\kappa^2-\xi^2)\, r^2 - c_2]\,\widetilde{u} + i\xi\, r(c_{13}-c_{23})\,\widetilde{w} = \widetilde{f}(r)$$

$$r(r\widetilde{w}')' + i\xi\, r^2(1+c_{13})\,\widetilde{u}' + i\xi\, r(1+c_{23})\,\widetilde{u} + (\kappa^2 - c_3\xi^2)\, r^2\,\widetilde{w} = \widetilde{g}(r)$$

where

$$\widetilde{f}(r) = -2c_1\, r u_0^{(2)\prime} - i\xi\,(1+c_{13})\, r w_0^{(2)} = \sum_{n=0}^{\infty} \widetilde{f}_n (\kappa r)^{2n+1}$$

$$\widetilde{g}(r) = -2r w_0^{(2)\prime} - i\xi\,(1+c_{13})\, r u_0^{(2)} = i\sum_{n=1}^{\infty} \widetilde{g}_n (\kappa r)^{2n}$$

$$\widetilde{f}_n = -2(2n+1)\, c_1\, a_n^{(2)} + \sigma(1+c_{13})\, b_n^{(2)}$$

$$\widetilde{g}_n = -4n\, b_n^{(2)} - \sigma(1+c_{13})\, a_{n-1}^{(2)}$$

Hence, if we write

$$\widetilde{u}(r) = \sum_{n=0}^{\infty} \widetilde{a}_n (\kappa r)^{2n+1} \quad \text{and} \quad \widetilde{w}(r) = i\sum_{n=1}^{\infty} \widetilde{b}_n (\kappa r)^{2n}$$

we find that

$$\{(2n+1)^2 c_1 - c_2\}\,\widetilde{a}_n = \sigma\,\{2n(1+c_{13})+c_{13}-c_{23}\}\,\widetilde{b}_n + (\sigma^2-1)\,\widetilde{a}_{n-1} + \widetilde{f}_n$$

for $n = 0, 1, 2, \ldots$ with $\widetilde{a}_{-1} = \widetilde{b}_0 = 0$, and

$$4n\,\widetilde{b}_n = -\sigma\,\{(2n-1)(1+c_{13})+1+c_{23}\}\,\widetilde{a}_{n-1} + (c_3\sigma^2-1)\,\widetilde{b}_{n-1} + \widetilde{g}_n$$

for $n = 1, 2, \ldots$ This completes the construction of $u_0^{(3)}$ and $w_0^{(3)}$.

For the guided-wave problem, we set

$$u_0(r) = A\, u_0^{(1)}(r) + B\, u_0^{(3)}(r)$$
$$w_0(r) = A\, w_0^{(1)}(r) + B\, w_0^{(3)}(r)$$

where A and B are arbitrary constants, and then impose the boundary condition $\boldsymbol{t}_0(a) = \mathbf{0}$, using Eq. (7). In principle, this procedure will lead to a secular equation, in just the same way as Biot obtained his secular equation, Eq. (13). Numerical implementation of this procedure will be described elsewhere.

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