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PERTURBED CRACKS IN TWO DIMENSIONS: A REPRISE

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Abstract. A nominally straight crack of finite length is subjected to plane-strain loadings. A perturbation method is developed for calculating the stress-intensity factors, based on an asymptotic analysis of the governing hypersingular boundary integral equation for the crack-opening displacement. Comparisons with a recent paper by Ballarini and Villaggio are made.

Keywords: curved cracks, perturbation methods, integral equations.

1. Introduction. Consider a nominally straight crack under plane-strain loading. The problem is to calculate the stress-intensity factors, correct to first order in ε, where the perturbed crack is defined by

$$y = \varepsilon f(x), \qquad -1 \leq x \leq 1; \tag{1}$$

here, x and y are Cartesian coordinates, f is a given function and ε is a small parameter.

In a previous paper (Martin, 2000; henceforth, we denote this paper by M), we solved the plane-strain problem for a slightly curved crack, using integral-equation methods. We began by reformulating the boundary-value problem as a boundary integral equation; we chose to use a hypersingular integral equation for the crack-opening displacement (COD). Next, we parametrised the curve defining the crack, leading to a one-dimensional hypersingular integral equation on a finite interval. Then, we introduced (1), leading to a sequence of hypersingular integral equations for each term in the regular expansion of the COD in powers of ε. Each integral equation can be solved exactly. Each is of the form $\mathcal{H}u = b$, where $b(x)$ is known and the operator $\mathcal{H}$ is defined by

$$(\mathcal{H}u)(x_0) = \frac{1}{\pi} \fint_{-1}^{1} \frac{u(x)}{(x - x_0)^2}\, \mathrm{d}x, \qquad -1 < x_0 < 1. \tag{2}$$

Each has to be solved for $u(x)$, $-1 < x < 1$, subject to $u(1) = u(-1) = 0$. The easiest way to do this is to use Chebyshev polynomials of the second kind, $U_n(x)$, defined by $U_n(\cos\theta) = [\sin(n+1)\theta]/\sin\theta$; for example, $U_0(x) = 1$, $U_1(x) = 2x$ and $U_2(x) = 4x^2 - 1$. Then, if

$$b(x) = \sum_{n=0} b_n\, U_n(x), \tag{3}$$

the unique solution of $\mathcal{H}u = b$ (with $u(1) = u(-1) = 0$) is given by

$$u(x) = -\sqrt{1 - x^2} \sum_{n=0} (n+1)^{-1} b_n\, U_n(x). \tag{4}$$

This approach is especially convenient when $b(x)$ is a polynomial.

In M, we gave results for quadratic cracks (defined by $f(x) = a_0 + a_1 x + a_2 x^2$) correct to second order in ε, and we showed agreement with the known exact solution for a circular-arc crack under constant loads. The basic method is systematic, it permits non-uniform loadings, and it has been exended to three-dimensional problems for "wrinkled penny-shaped cracks" (Martin, 2001).

In a recent paper, Ballarini and Villaggio (2006; henceforth, we denote this paper by BV) have also considered the plane-strain problem for a slightly curved crack. They assume uniform loading and give results to first order only. They also begin with an integral-equation formulation, but choose Cauchy-singular integral equations with dislocation components as unknowns; these are tangential derivatives of the COD components. To solve their equations, they use power series multiplied by an appropriate square-root factor. (This is analogous to replacing $U_n(x)$ in (4) by x^n.) Many singular integrals are evaluated analytically (see p. 63 of BV), and then the integral equations yield systems of linear algebraic equations for the coefficients in the power-series expansions. These are solved and expressions for the stress-intensity factors are obtained; we shall see that these contain errors. Notice that the method in M avoids linear systems: this is a major benefit of using orthogonal polynomials. The method in M also provides higher-order approximations in a systematic manner.

In the next section, we give results when f is a cubic polynomial. Comparisons with BV are made in Section 3.

2. An example: cubic cracks. Consider a cubic crack, defined by

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3, \tag{5}$$

under uniform loading; here, a_0, a_1, a_2 and a_3 are constants. The traction components in M(14) are $t_i^0 = B\tau_{i2}$ and $t_i^1(x_0) = -B\tau_{i1} f'(x_0)$, where $i = 1, 2$, $-\tau_{ij}$ are the components of the constant stress field at infinity, $B = 2(1-\nu)/\mu$, ν is Poisson's ratio and μ is the shear modulus.

Our goal here is to calculate the stress-intensity factors, correct to first order in ε. As in M, we expand the crack-opening displacement $(\widetilde{u}_1, \widetilde{u}_2)$, writing $\widetilde{u}_i = u_i^0 + \varepsilon u_i^1 + \cdots$. We have $\mathcal{H}u_i^0 = t_i^0$ so that $u_i^0(x) = -B\tau_{i2}\sqrt{1-x^2}$, the solution for a straight crack ($\varepsilon = 0$). Then, $\mathcal{H}u_i^1 = b_i^1$, where b_1^1 and b_2^1 are given by the two formulas at the bottom of p. 323 of M, involving the function S_{12}^1. We have

$$S_{12}^1 = 2\frac{f(x) - f(x_0)}{x - x_0} - f'(x) - f'(x_0) = -a_3(x - x_0)^2,$$

using (5), giving

$$\begin{aligned}
b_1^1(x_0) &= -B\left\{3a_3\tau_{11}x_0^2 + 2a_2\tau_{11}x_0 + a_1\tau_{11} + \tfrac{1}{2}a_3\tau_{22}\right\} \\
&= -B\left\{\tfrac{3}{4}a_3\tau_{11}U_2(x_0) + a_2\tau_{11}U_1(x_0)\right. \\
&\qquad\qquad \left. + \left[\left(a_1 + \tfrac{3}{4}a_3\right)\tau_{11} + \tfrac{1}{2}a_3\tau_{22}\right]U_0(x_0)\right\} \\
b_2^1(x_0) &= -B\tau_{12}\left\{3a_3x_0^2 + 2a_2x_0 + a_1 + \tfrac{1}{2}a_3\right\} \\
&= -B\tau_{12}\left\{\tfrac{3}{4}a_3U_2(x_0) + a_2U_1(x_0) + \left(a_1 + \tfrac{5}{4}a_3\right)U_0(x_0)\right\}.
\end{aligned}$$

Use of (3) and (4) then gives

$$\begin{aligned}
u_1^1(x) &= B\left\{(a_3x^2 + a_2x + a_1)\tau_{11} + \tfrac{1}{2}a_3(\tau_{11} + \tau_{22})\right\}\sqrt{1-x^2}, \\
u_2^1(x) &= B\tau_{12}\left(a_3x^2 + a_2x + a_1 + a_3\right)\sqrt{1-x^2}.
\end{aligned}$$

The stress-intensity factors, K_1 and K_2, are defined by M(16) as

$$u_n(x) \sim -BK_1\sqrt{2\rho} \quad \text{and} \quad u_t(x) \sim -BK_2\sqrt{2\rho} \quad \text{as } \rho \to 0, \tag{6}$$

where $\rho = \sqrt{(1-x)^2 + \varepsilon^2[f(1) - f(x)]^2}$ is distance from the edge at $x = 1$, and u_n and u_t are the normal and tangential components, respectively, of the COD. Near $x = 1$, as on p. 325 of M, we have

$$u_n \sim u_2^0 + \varepsilon\{u_2^1 - u_1^0 f'(1)\} \sim -B\sqrt{2(1-x)}\{\tau_{22} - \varepsilon(2a_1 + 3a_2 + 5a_3)\tau_{12}\}$$

and

$$\begin{aligned} u_t &\sim u_1^0 + \varepsilon\{u_1^1 + u_2^0 f'(1)\} \\ &\sim -B\sqrt{2(1-x)}\{\tau_{12} + \varepsilon[(a_1 + 2a_2 + \tfrac{5}{2}a_3)\tau_{22} - (a_1 + a_2 + \tfrac{3}{2}a_3)\tau_{11}]\}. \end{aligned}$$

We now compare these expressions with (6), noting that $\sqrt{\rho} \sim \sqrt{1-x}$, giving

$$K_1 = \tau_{22} - \varepsilon(2a_1 + 3a_2 + 5a_3)\tau_{12}, \tag{7}$$
$$K_2 = \tau_{12} + \varepsilon\left\{(a_1 + 2a_2 + \tfrac{5}{2}a_3)\tau_{22} - (a_1 + a_2 + \tfrac{3}{2}a_3)\tau_{11}\right\}, \tag{8}$$

correct to first order in ε.

3. Comparison with Ballarini and Villaggio. In BV, the authors define their crack by

$$y(X) = A_1 X + A_2 X^2 + A_3 X^3, \quad 0 < X < 1,$$

and then calculate the stress-intensity factors near the edge at $X = 0$. We introduce a linear change of variables, mapping $X = 0$ to $x = 1$ and $X = 1$ to $x = -1$; thus $X = (1-x)/2$, $x = 1 - 2X$ and

$$y = \tfrac{1}{2}A_1 + \tfrac{1}{4}A_2 + \tfrac{1}{8}A_3 - \left(\tfrac{1}{2}A_1 + \tfrac{1}{2}A_2 + \tfrac{3}{8}A_3\right)x + \left(\tfrac{1}{4}A_2 + \tfrac{3}{8}A_3\right)x^2 - \tfrac{1}{8}A_3 x^3.$$

In this formula, the coefficient of x^n equals εa_n in (5). Substituting in (7) and (8) gives

$$K_1 = \tau_{22} + \tfrac{1}{2}\tau_{12}\left(2A_1 + \tfrac{1}{2}A_2 + \tfrac{1}{2}A_3\right), \tag{9}$$
$$K_2 = \tau_{12} + \tfrac{1}{2}\tau_{11}\left(A_1 + \tfrac{1}{2}A_2 + \tfrac{3}{8}A_3\right) + \tfrac{1}{2}\tau_{22}\left(-A_1 + \tfrac{1}{8}A_3\right). \tag{10}$$

These should be compared with equations (5.4a,b) of BV, namely

$$K_1^{\mathrm{BV}} = \tau_{22} - \tau_{12}\left(2A_1 + \tfrac{1}{2}A_2 + \tfrac{1}{8}A_3\right), \tag{11}$$
$$K_2^{\mathrm{BV}} = \tau_{12} - \tau_{11}\left(A_1 + \tfrac{1}{2}A_2\right) - \tau_{22}\left(-A_1 + \tfrac{1}{8}A_3\right). \tag{12}$$

For quadratic cracks, $A_3 = 0$, and then these formulas agree apart from a consistent factor of $-\frac{1}{2}$ in all the first-order terms. This is an error in (11) and (12). We know this because (9) and (10) give the correct results for shallow circular-arc cracks; see Section 6 of M, where agreement to second order is also shown. Ballarini and Villaggio also compare with the circular-arc crack but their comparison contains errors. First, they used $y(X) = 2\alpha X(1-X)$ whereas they should have used $y(X) = -2\alpha X(1-X)$ when comparing with Cotterell and Rice (1980). Second, the parameter α is related to the geometry. Thus, the crack subtends an angle of 2α at the centre of the circle, and $y(\frac{1}{2}) = -\frac{1}{2}\alpha$, so that Figure 2a in BV is incorrect.

The terms involving A_3 in (9)–(12) show some differences. We have not checked these because the calculations in BV are complicated: our use of orthogonal polynomials means that we do not have to solve any systems of linear algebraic equations.

In conclusion, our method (Martin, 2000) is preferable to the method of Ballarini and Villaggio (2006) because it is systematic, it does not require the solution of systems of linear algebraic equations, it can give higher-order approximations, it can be used for non-uniform loadings, and it can be extended to certain three-dimensional problems (Martin, 2001).

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