

# Moshinsky's shutter problem: an initial-value problem for the Klein–Gordon equation<sup>+</sup>

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Moshinsky's problem is formulated and solved as a convolution integral. The initial data are discontinuous, giving the possibility of non-uniqueness. Asymptotic properties of the solution are deduced, using variants of the method of stationary phase. Comparisons are made with solutions of analogous problems for the one-dimensional wave equation and the Schrödinger equation.

**Keywords:** Klein–Gordon equation; asymptotic analysis; method of stationary phase; non-uniqueness

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## 1. Introduction

We consider what appear to be 'textbook' examples of initial-value problems. They arise when a wave is incident upon a perfectly absorbing shutter which opens instantaneously at t=0; the problem is to determine the transmitted wave [1]. The wave is governed by one of three equations: the vacuum electromagnetic equation (the ordinary one-dimensional wave equation), the free-particle Schrödinger equation or the free-particle Klein–Gordon equation. The first two of these yield solutions for which numerical results can be obtained readily. However, the Klein–Gordon equation, which is the focus of this study, does not yield such a simple solution.

This Klein–Gordon problem has a long history. Its solution can be written as a convolution integral (using Laplace transforms) [2,3] or expressed as an infinite series (obtained from a Fourier transform inversion) [1]. Chambers' solution [3] is unique in that it is given in terms of wave-equation solutions to the same problem.

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<sup>&</sup>lt;sup>†</sup>This article is dedicated to the memory of Alan Jeffrey. Well-known for his textbooks (including [8]), he is perhaps best known for translating, editing, improving, correcting and promoting several editions of 'Gradshteyn and Ryzhik': generations of mathematicians, scientists and engineers are surely grateful for his efforts.

A numerical approximation for Moshinsky's infinite series solution can be obtained for those spacetime points, beyond the shutter, which can be accessed by the wavegroup generated from the shutter opening. However, the series diverges between this region and the light cone. More recent work on this problem has been reviewed [4].

The difficulty lies not in obtaining the solution in the transform domain but rather in inverting the solution to obtain a numerical result for the parameters of interest physically. Inversion of the Laplace transform can be approximated as a limit of derivatives in the transform domain or as a numerical contour integral [5], but the large value of the speed of light makes such numerical evaluations ineffectual.

Applications of these solutions are to the propagation of electromagnetic waves in waveguides or material with dispersion relations for which the governing equation takes a Klein–Gordon form [2]. In quantum mechanics, free-particle propagation through a shutter is described by the Schrödinger or Klein–Gordon equations, where the latter is appropriate for relativistic particle motion [1]. It is generally accepted that solutions to these two equations should be very similar for particles moving at speeds much less than that of light. However, the solution given by Moshinsky violates that assumption on the light cone, even for Klein–Gordon and Schrödinger particles moving much less than the speed of light [6].

The problem becomes pathological with the imposition of an instantaneous shutter opening. This is manifested mathematically in non-unique solutions, as shown below. Of course, no such instantaneous physical opening can be produced. This raises the question of how the pathology in the initial condition is manifested in the solution. Perhaps the non-zero value of the solution at the light cone is such a manifestation, since one would expect an indication of the shutter opening to travel only at the group velocity. Such issues about the applicability of mathematical models to physically realizable situations remain unresolved in this 'textbook' example. Nevertheless, such a pathological initial condition has been used to determine tunnelling propagation times [7]. Having described the physical motivation behind this study, let us now give mathematical formulations. Moshinsky [1] formulates three initial-value problems for a function  $\psi(x, t)$  with initial conditions

$$\psi(x,0) = f(x)$$
 and  $(\partial \psi/\partial t)(x,0) = g(x), \quad -\infty < x < \infty,$  (1)

one for the wave equation,

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}, \quad -\infty < x < \infty, \quad t > 0,$$
(2)

one for the Klein-Gordon equation,

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} + \mu^2 \psi, \quad -\infty < x < \infty, \quad t > 0,$$
(3)

and one for the Schrödinger equation,

$$\frac{\partial^2 \psi}{\partial x^2} = -2i \frac{\mu}{c} \frac{\partial \psi}{\partial t}, \quad -\infty < x < \infty, \quad t > 0, \tag{4}$$

where c and  $\mu(=mc/\hbar, \text{ using Moshinsky's notation [1]})$  are constants. Equation (4) is to be solved subject to  $\psi(x, 0) = f(x)$ .

We shall mainly be concerned with the first two of these problems, (2) and (3). We shall return to (4) in Section 4.

What makes these problems non-standard is the use of discontinuous initial data, f and g. Specifically, Moshinsky takes

$$g(x) = \beta f(x) \tag{5}$$

with f given by

$$f(x) = \begin{cases} e^{ikx}, & x < 0, \\ 0, & x > 0, \end{cases}$$
(6)

where k and  $\beta$  are constants. Thus, f(x) and g(x) are discontinuous (and undefined) at x = 0.

For the wave equation, the textbook solution of the initial-value problem is given by d'Alembert's formula,

$$\psi(x,t) = \frac{1}{2} \left\{ f(x-ct) + f(x+ct) \right\} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y) \mathrm{d}y, \tag{7}$$

see, for example, [8, p. 134] or [9, p. 41]. In particular, this formula gives

$$\lim_{t \to 0} \psi(0, t) = \frac{1}{2} \left\{ f(0+) + f(0-) \right\}$$
(8)

and

$$\lim_{t \to 0} \frac{\partial \psi}{\partial t}(0, t) = \frac{1}{2}c \left\{ f'(0+) - f'(0-) \right\} + \frac{1}{2}c \left\{ g(0+) + g(0-) \right\}.$$
(9)

Also, if we insert (5) and (6) in (7), and choose  $\beta = -ikc$  [1, Equation (17)], we obtain

$$\psi(x,t) = \begin{cases} e^{ik(x-ct)}, & x < -ct, \\ e^{ik(x-ct)} - \frac{1}{2}, & -ct < x < ct, \\ 0, & x > ct. \end{cases}$$
(10)

See Figure 1 for a sketch of the three regions in the *xt*-plane. Moshinsky gave the solution (10) for x > 0 [1, Equation (18)], and he commented on the discontinuity



Figure 1. The *xt*-plane. The lines  $x = \pm ct$  are characteristics.

across x = ct. However, he did not notice that the problem as posed exhibits nonuniqueness. To see this, return to the basic d'Alembert solution of (2),

$$\psi(x,t) = L(x+ct) + R(x-ct),$$
(11)

where L and R are functions of one variable. The formula (11) generates a solution of (2) provided L and R are twice differentiable. Then, applying the initial conditions, (1), gives

$$f(x) = L(x) + R(x), \quad g(x) = cL'(x) - cR'(x).$$
 (12)

Hence cf' + g = 2cL', cf' - g = 2cR',

$$L(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{a}^{x} g(y)dy + A,$$
(13)

$$R(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{a}^{x} g(y)dy - A,$$
(14)

where a and A are arbitrary. Substitution in (11) gives the d'Alembert formula (7).

For Moshinsky's problem, f(x) = g(x) = 0 for x > 0. Then, solving (12) (or by taking a = 0 in (13) and (14)) gives

$$L(x) = A, \quad R(x) = -A, \quad x > 0$$

We also have  $f(x) = e^{ikx}$  and g(x) = -ikcf(x) for x < 0. Then, solving (12) gives

$$L(x) = B$$
,  $R(x) = e^{ikx} - B$ ,  $x < 0$ ,

where B is another arbitrary constant. Hence, using (11),

$$\psi(x,t) = \begin{cases} e^{ik(x-ct)}, & x < -ct, \\ e^{ik(x-ct)} + C, & -ct < x < ct, \\ 0, & x > ct, \end{cases}$$
(15)

where C = A - B is an arbitrary constant; cf (10). In general,  $\psi(x, t)$  is discontinuous across both characteristic lines x = ct and x = -ct.

The influence of the discontinuity in the initial data at x = 0 is the presence of the constant *C*. If we insist on having

$$f(x) = \begin{cases} e^{ikx}, & x < 0, \\ 0, & x \ge 0, \end{cases}$$
(16)

we obtain C = -1, giving continuity across x = ct. Instead, we may require that

$$f(x) = \begin{cases} e^{ikx}, & x \le 0, \\ 0, & x > 0, \end{cases}$$
(17)

the result is C = 0 and continuity across x = -ct. The d'Alembert formula gives (10), corresponding to  $C = -\frac{1}{2}$ : it picks out a particular solution, with f(0) and g(0) given by (8) and (9). This solution is discontinuous across both x = ct and x = -ct.

In this article, we make a similar investigation for the Klein–Gordon equation. We begin with a formal solution of the initial-value problem, analogous to the d'Alembert formula for the wave equation. Asymptotic approximations are then obtained, mainly using variants of the method of stationary phase. Some comparisons with the Schrödinger equation are given in Section 4. Further comments on non-uniqueness are made in Section 5.

## 2. Moshinsky's relativistic shutter problem

The governing equation for  $\psi(x, t)$  is the Klein–Gordon equation, (3), with initial conditions given by (1), (5) and (6). The parameter  $\beta$  in (5) is given by

$$\beta = -iEc, \quad \text{where } E = \sqrt{k^2 + \mu^2}, \tag{18}$$

c, k and  $\mu$  are constants. Again, we note that f(x) and g(x) are discontinuous and undefined at x = 0, so we expect non-uniqueness.

We start by deriving a formula for solving Moshinsky's initial-value problem. Natural tools to use are Laplace transforms with respect to t or Fourier transforms with respect to x. We choose the latter, and define

$$\Psi(\xi, t) = \int_{-\infty}^{\infty} \psi(x, t) \mathrm{e}^{-\mathrm{i}\xi x} \,\mathrm{d}x.$$

Transforming (3) gives  $\partial^2 \Psi / \partial t^2 = -\epsilon^2 c^2 \Psi$ , where  $\epsilon(\xi) = \sqrt{\xi^2 + \mu^2}$ . Hence,

 $\Psi(\xi, t) = F(\xi)\cos\left(\epsilon ct\right) + (\epsilon c)^{-1}G(\xi)\sin\left(\epsilon ct\right),$ 

where *F* and *G* are the Fourier transforms of *f* and *g*, respectively; from (5) and (18), we obtain G = -iEcF.

Let  $S(\xi, t) = \epsilon^{-1} \sin(\epsilon c t)$ . Inverting  $\Psi$  gives

$$\psi(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{F(\xi)\cos\left(\epsilon ct\right) - iEF(\xi)S(\xi,t)\} e^{i\xi x} d\xi = \frac{1}{c}\frac{\partial\phi}{\partial t} - iE\phi,$$
(19)

say, where

$$\phi(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\xi) S(\xi,t) \mathrm{e}^{\mathrm{i}\xi x} \,\mathrm{d}\xi.$$

This is a Fourier convolution because S is a Fourier transform. For, if

$$s(x,t) = \begin{cases} \frac{1}{2} J_0 \left( \mu \sqrt{c^2 t^2 - x^2} \right), & |x| < ct, \\ 0, & |x| > ct, \end{cases}$$
(20)

where  $J_n$  is a Bessel function, then

$$\int_{-\infty}^{\infty} s(x,t) \mathrm{e}^{-\mathrm{i}\xi x} \,\mathrm{d}x = \int_{0}^{ct} J_0\left(\mu\sqrt{c^2t^2 - x^2}\right) \cos\xi x \,\mathrm{d}x = S(\xi,t),\tag{21}$$

using [10, 6.677 (6)]. Hence,

$$\phi(x,t) = \int_{-\infty}^{\infty} f(x-y)s(y,t)dy = \frac{1}{2} \int_{-ct}^{ct} f(x-y)J_0\left(\mu\sqrt{c^2t^2-y^2}\right)dy.$$
 (22)

This is our solution for  $\phi(x, t)$ , with  $-\infty < x < \infty$  and t > 0. Then,  $\psi(x, t)$  is given by (19); with  $\chi(y, t) = \mu \sqrt{c^2 t^2 - y^2}$ ,

$$\psi(x,t) = \frac{1}{2} \left\{ f(x-ct) + f(x+ct) \right\} - \frac{1}{2} \int_{-ct}^{ct} \left\{ \frac{\mu^2 ct}{\chi} J_1(\chi) + iE J_0(\chi) \right\} f(x-y) dy.$$
(23)

Formulae of this kind are known; see, for example, [11, Equation (69), p. 255] or [2, Equation (17)].

When  $\mu = 0$ , E = k, (3) becomes the wave equation, and (23) reduces to the d'Alembert formula.

For Moshinsky's specific problem, f is given by (6). Then, when evaluating (22) or (23), there are three cases.

Solution for x > ct. In this case, x > y in (22), so f(x-y) = 0 and, therefore,  $\phi(x, t) = \psi(x, t) = 0$  for x > ct.

Solution for x < -ct. In this case, x < y in (22), so  $f(x - y) = e^{ik(x - y)}$  and

$$\phi(x,t) = \frac{1}{2} e^{ikx} \int_{-ct}^{ct} J_0 \left( \mu \sqrt{c^2 t^2 - y^2} \right) e^{-iky} dy$$
  
=  $e^{ikx} \int_0^{ct} J_0 \left( \mu \sqrt{c^2 t^2 - y^2} \right) \cos ky \, dy = e^{ikx} S(k,t) = e^{ikx} \frac{\sin (Ect)}{E}$ 

using (21) and  $\epsilon(k) = E$ . Hence, from (19),

$$\psi(x,t) = \frac{1}{c} \frac{\partial \phi}{\partial t} - iE\phi = e^{i(kx - Ect)}, \quad x < -ct.$$

This wave solution has frequency  $\omega(k) = Ec = c\sqrt{k^2 + \mu^2}$ , phase speed  $c_p = Ec/k$  and group velocity

$$c_{\rm g} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{ck}{E} = \frac{k^2 c_{\rm p}}{k^2 + \mu^2}.$$

Thus,  $c_g < c < c_p$ .

Solution for -ct < x < ct. In this case, with  $\chi(y,t) = \mu \sqrt{c^2 t^2 - y^2}$ ,

$$\phi(x,t) = \frac{1}{2} e^{ikx} \int_x^{ct} J_0(\chi) e^{-iky} \, \mathrm{d}y$$

and, using  $J_0(0) = 1$  and  $J'_0 = -J_1$ ,

$$\psi(x,t) = \frac{1}{2} e^{ik(x-ct)} - \frac{1}{2} e^{ikx} \int_{x}^{ct} \left\{ \frac{\mu^2 ct}{\chi} J_1(\chi) + iE J_0(\chi) \right\} e^{-iky} \, \mathrm{d}y.$$
(24)

The  $J_1$  term could be simplified using  $(2/\chi)J_1(\chi) = J_0(\chi) + J_2(\chi)$ . As a check, suppose that  $\mu = 0$ . Then, as E = k, (24) reduces to

$$\psi(x,t) = \frac{1}{2} e^{ik(x-ct)} - \frac{ik}{2} e^{ikx} \int_x^{ct} e^{-iky} \, \mathrm{d}y = e^{ik(x-ct)} - \frac{1}{2}.$$

Thus, we find agreement with the d'Alembert solution, (10).

We cannot evaluate (24) in general but we can find  $\psi$  on the characteristics,  $x = \pm ct$  (see Figure 1). Trivially, we obtain

$$\psi(ct,t) = \frac{1}{2},\tag{25}$$

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which agrees with [1, Equation (32)]. We also find that

$$\psi(-ct, t) = e^{-i(k+E)ct} - \frac{1}{2}.$$
 (26)

To see this, note that (24) gives

$$2e^{ikct}\psi(-ct,t) = e^{-ikct} - 2\int_0^{ct} \left\{ \frac{\mu^2 ct}{\chi} J_1(\chi) + iEJ_0(\chi) \right\} \cos ky \, dy$$
  
=  $e^{-ikct} - 2i\sin(Ect) - 2\mu ct \int_0^{\pi/2} J_1(\mu ct\sin z) \cos(kct\cos z) \, dz.$ 

From [10, 6.688 (1)], the remaining integral is

$$\frac{\pi}{2}J_{1/2}(ct[E+k]/2)J_{1/2}(ct[E-k]/2) = \frac{2}{\mu ct}\sin\left(ct[E+k]/2\right)\sin\left(ct[E-k]/2\right),$$

whence

$$2e^{ikct}\psi(-ct,t) = e^{-ikct} - 2i\sin(Ect) - 2\{\cos(kct) - \cos(Ect)\} = 2e^{-iEct} - e^{ikct}.$$

This reduces to (26).

We observe that  $\psi(x, t)$  is discontinuous across both x = ct and x = -ct. Also, from (25) or (26), we see that the formula (24) implies that

$$\psi(0,0) = \frac{1}{2}.$$
(27)

Let us mention some previous work on solving related problems for the Klein–Gordon equation. Reiss [12, Appendix] considered (3) for  $x \ge 0$  with f=g=0 and  $\psi(0, t) = h(t)$ , see also [13, Appendix] and [14, Section 2]. Bleistein and Handelsman [15, Section 7.5] considered an inhomogeneous form of (3), forced by  $\delta(x)e^{i\omega t}$ , with f=g=0. A similar problem but with  $e^{i\omega t}$  replaced by  $\psi(0, t)$  and with Moshinsky's initial conditions is discussed in [6]. Babich [16] solved (3) subject to g(x) = 0, f(x) = 1 for  $x \ge 0$  and f(x) = 0 for x < 0.

# 3. Asymptotic approximations when |x| < ct

As we cannot evaluate (24) in general, we resort to asymptotic approximations. We begin by introducing dimensionless quantities, using  $E^{-1}$  as a length scale:

$$X = Ex, \quad T = Ect, \quad M = \mu/E \quad \text{and} \quad \varepsilon = k/E,$$
 (28)

in applications,  $\varepsilon \ll 1$ . Then, (24) becomes

$$\psi(x,t) = \frac{1}{2} e^{ikx} \{ e^{-ikct} - \Phi(X,T) \},$$
(29)



Figure 2. The lines  $x = \pm ct$  are characteristics. Asymptotic approximations are given in the region above these two lines. The dashed line is used for fixed x and large t. Estimates along the sloping line,  $x = ct \cos \alpha$  ( $0 < \alpha < \pi$ ), as  $t \to \infty$  are also given. Region I is bounded by  $x = c_g t$  and x = ct. Region II is bounded by x = -ct and  $x = c_g t$ . The estimates are different in these two regions.

where

$$\Phi(X,T) = \int_X^T \left\{ \frac{MT}{\sqrt{T^2 - Y^2}} J_1 \left( M \sqrt{T^2 - Y^2} \right) + i J_0 \left( M \sqrt{T^2 - Y^2} \right) \right\} e^{-i\varepsilon Y} dY.$$

We consider two kinds of asymptotic approximations. First, we fix x and then let  $t \rightarrow \infty$  (along a vertical line in Figure 2). Second, as an alternative, we move away from the origin in Figure 2, along a sloping line defined by  $x = ct \cos \alpha$ .

We start by estimating  $\Phi(X, T)$  as  $T \to \infty$ , with X fixed. Assuming that M > 0, the integrand is small except when Y is near T, so we write

$$\Phi(X, T) = \Phi(0, T) - \{\Phi(0, T) - \Phi(X, T)\}$$

and then

$$\Phi(0,T) - \Phi(X,T) = \int_0^X \{\cdots\} e^{-i\varepsilon Y} dY \sim X\{MJ_1(MT) + iJ_0(MT)\},\$$

which is  $O(1/\sqrt{T})$  as  $T \to \infty$ . Thus, we conclude that, when M > 0,  $\Phi(X, T) \sim \Phi(0, T)$  as  $T \to \infty$ , with no dependence on X at leading order. On the other hand, when M = 0,  $\Phi(X, T) = i \int_X^T e^{-iY} dY = e^{-iX} - e^{-iT}$  (in agreement with d'Alembert's particular solution, (10)). Hence, the limits  $M \to 0$  and  $T \to \infty$  do not commute.

Now, in the integral defining  $\Phi(0, T)$ , put  $Y = T\cos \varphi$ , giving

$$\Phi(0,T) = T \int_0^{\pi/2} \left\{ M J_1(MT\sin\varphi) + i J_0(MT\sin\varphi)\sin\varphi \right\} e^{-iT\varepsilon\cos\varphi} \,\mathrm{d}\varphi.$$

We shall estimate this integral using the method of stationary phase. Before doing that, suppose that, instead of fixing X, we move away from the origin along a line  $x = ct \cos \alpha$ , that is, along  $X = T \cos \alpha$  with  $0 < \alpha < \pi$  (because |x| < ct). We find that

$$\Phi(T\cos\alpha, T) = T \int_0^\alpha \left\{ M J_1(MT\sin\varphi) + i J_0(MT\sin\varphi)\sin\varphi \right\} e^{-iT\varepsilon\cos\varphi} \,\mathrm{d}\varphi, \qquad (30)$$

which reduces to  $\Phi(0, T)$  when  $\alpha = \pi/2$ .

To estimate  $\Phi(T \cos \alpha, T)$  as  $T \to \infty$ , we first write  $\int_0^{\alpha} = \int_0^{\delta} + \int_{\delta}^{\alpha}$ , where  $0 < \delta < \alpha$  and  $\delta \ll 1$ . The last integral will be estimated by the method of stationary phase but the first integral will be estimated directly. (We shall explain later why we cannot work directly with the original integral.) Thus,

$$T \int_{0}^{\delta} \{ MJ_{1}(MT\sin\varphi) + iJ_{0}(MT\sin\varphi)\sin\varphi \} e^{-iT\varepsilon\cos\varphi} d\varphi$$
  

$$\sim T \int_{0}^{\delta} \{ MJ_{1}(MT\varphi) + iJ_{0}(MT\varphi)\varphi \} e^{-iT\varepsilon} d\varphi$$
  

$$= e^{-iT\varepsilon} \int_{0}^{MT\delta} \{ J_{1}(\sigma) + \frac{i\sigma}{M^{2}T} J_{0}(\sigma) \} d\sigma = e^{-iT\varepsilon} \Big[ -J_{0}(\sigma) + \frac{i\sigma}{M^{2}T} J_{1}(\sigma) \Big]_{0}^{MT\delta}$$
  

$$= e^{-iT\varepsilon} \{ 1 - J_{0}(MT\delta) + i(\delta/M) J_{1}(MT\delta) \} \sim e^{-iT\varepsilon} = e^{-ikct} \text{ as } T \to \infty.$$

This exponential cancels the exponential in (29), whence

$$\psi(x,t) \sim -\frac{1}{2} e^{ikx} \Phi_{\pi/2}(T)$$
, as  $T \to \infty$  with fixed x,

and

$$\psi(ct\cos\alpha, t) \sim -\frac{1}{2} e^{ikx} \Phi_{\alpha}(T), \text{ as } T \to \infty,$$

where

$$\Phi_{\alpha}(T) = T \int_{\delta}^{\alpha} \left\{ M J_1(MT \sin \varphi) + i J_0(MT \sin \varphi) \sin \varphi \right\} e^{-iT\varepsilon \cos \varphi} \, \mathrm{d}\varphi.$$

To estimate  $\Phi_{\alpha}(T)$  as  $T \to \infty$ , we replace the Bessel functions by integral representations,

$$J_0(z) = \frac{1}{2\pi} \int e^{iz\cos\theta} d\theta, \quad J_1(z) = \frac{1}{2\pi i} \int e^{iz\cos\theta} \cos\theta d\theta,$$

where the integration is over any interval of length  $2\pi$ ; we choose  $-\pi/4 < \theta < 7\pi/4$  so that  $\theta = 0$  and  $\theta = \pi$  are interior points. Thus,

$$\Phi_{\alpha}(T) = \frac{T}{2\pi i} \int_{\delta}^{\alpha} \int_{-\pi/4}^{7\pi/4} K(\theta, \varphi) e^{iTH(\theta, \varphi)} \, \mathrm{d}\theta \, \mathrm{d}\varphi, \tag{31}$$

where

$$K(\theta, \varphi) = M \cos \theta - \sin \varphi, \quad H(\theta, \varphi) = M \cos \theta \sin \varphi - \varepsilon \cos \varphi.$$

Next, we use the two-dimensional method of stationary phase. We have

$$\frac{\partial H}{\partial \theta} = -M \sin \theta \sin \varphi$$
 and  $\frac{\partial H}{\partial \varphi} = M \cos \theta \cos \varphi + \varepsilon \sin \varphi$ .

Setting these to zero, we see that there are stationary-phase points at  $(\theta, \varphi) = (\pi, \varphi_g)$  and  $(0, \pi - \varphi_g)$ , where

$$\sin \varphi_{\rm g} = M \quad \text{and} \quad \cos \varphi_{\rm g} = \varepsilon.$$
 (32)

Denote these points by  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , respectively. For these points to be relevant, they have to be inside the domain of integration, and this depends on  $\alpha$ . However, when  $\mathcal{P}_2$  is relevant (which means when  $\alpha > \pi - \varphi_g$ ), its contribution is negligible because  $K(0, \pi - \varphi_g) = 0$ . Thus, there are two cases:

Region I. When  $0 < \alpha < \varphi_g$ , there are no relevant points. Region II. When  $\varphi_g < \alpha < \pi$ ,  $\mathcal{P}_1$  is relevant.

To interpret these regions, we split the region |x| < ct into two. As  $\cos \varphi_g = \varepsilon = k/E = c_g/c$ , we see that Region I is bounded by the lines  $x = c_g t$  and x = ct (Figure 2). As there are no relevant stationary-phase points,  $\Phi_{\alpha}(T)$  is negligible, and hence, in Region I,

$$\psi(x, t) = o(1)$$
 as  $t \to \infty$ , where  $x = ct \cos \alpha$  and  $0 < \alpha < \varphi_{g}$ . (33)

We show in Section 3.1 that o(1) in (33) is actually  $O(T^{-1/2})$ .

For Region II, we must evaluate the contribution from the stationary-phase point,  $\mathcal{P}_1$ . At this point,  $\theta = \pi$ ,  $\varphi = \varphi_g$ , K = -2M and H = -1. Also, at  $\mathcal{P}_1$ , the matrix

$$A = \begin{pmatrix} H_{\theta\theta} & H_{\theta\varphi} \\ H_{\theta\varphi} & H_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} M^2 & 0 \\ 0 & 1 \end{pmatrix}$$

has two positive eigenvalues with det  $A = M^2$ . Then, using a known formula from the book by Bleistein and Handelsman [15, Equation (8.4.44)],<sup>1</sup>

$$\Phi_{\alpha}(T) \sim \frac{T}{2\pi i} \frac{2\pi}{T} \frac{(-2M)}{\sqrt{M^2}} \exp\{-iT + i\pi/2\} = -2e^{-iT} \text{ as } T \to \infty, \ \varphi_{g} < \alpha < \pi$$

Hence, in Region II (between x = -ct and  $x = +c_g t$ , see Figure 2)

$$\psi(x, t) \sim e^{i(kx - Ect)}$$
 as  $t \to \infty$ , where  $x = ct \cos \alpha$  and  $\varphi_g < \alpha < \pi$ . (34)

Also,

$$\psi(x,t) \sim e^{i(kx - Ect)}$$
 as  $t \to \infty$ , with x fixed. (35)

Let us conclude with some remarks on the introduction of  $\delta$ . If the range of integration had been  $0 < \varphi < \alpha$ , we would have found boundary stationary-phase points at  $(\theta, \varphi) = (\pi/2, 0)$  and  $(3\pi/2, 0)$ . At these points, K=0 (which suggests a negligible contribution) but  $H_{\theta\theta} = 0$  (giving det A = 0). Rather than modify the method of stationary phase, we gave a direct treatment; it turns out that there is a non-negligible contribution.

# 3.1. Region I: more detail

Return to (31) in which  $0 < \alpha < \varphi_g$ . Let  $\mathcal{D}$  denote the rectangular integration domain  $(-\pi/4 < \theta < 7\pi/4, \delta < \varphi < \alpha)$ ; we shall let  $\delta \rightarrow 0$  later. By assumption,  $|\text{grad } H| \neq 0$  in  $\mathcal{D}$  or around the boundary,  $\partial \mathcal{D}$ . Therefore, we can define [15, Section 8.2]

$$\mathbf{h} = K \frac{\text{grad } H}{|\text{grad } H|^2},$$

we have  $\operatorname{div}(e^{iTH}\mathbf{h}) = iTKe^{iTH} + e^{iTH}\operatorname{div}\mathbf{h}$ . Hence, substituting in (31) and then using the divergence theorem, we obtain

$$\Phi_{\alpha}(T) = \frac{1}{2\pi\mathcal{D}} e^{iTH} \operatorname{div} \mathbf{h} \, \mathrm{d}\theta \, \mathrm{d}\varphi - \frac{1}{2\pi} \int_{\partial \mathcal{D}} e^{iTH} \mathbf{h} \cdot \mathbf{n} \, \mathrm{d}s, \tag{36}$$

where **n** is the unit normal vector on  $\partial D$  pointing out of D.

The first integral on the right-hand side of (36) is similar to (31): it is a factor of  $T^{-1}$  smaller than  $\Phi_{\alpha}(T)$ , and so we neglect it.

For the integral around the rectangle  $\partial D$ , we find that the contributions from the sides at  $\theta = -\pi/4$  and  $\theta = 7\pi/4$  cancel. Next, consider the piece of  $\partial D$  at  $\varphi = \delta$ , and let  $\delta \to 0$ . We have  $H = -\varepsilon$ ,  $\partial H/\partial \theta = 0$ , and  $K = \partial H/\partial \varphi = M \cos \theta$ , whence  $\mathbf{h} \cdot \mathbf{n} = -1$  and

$$\left. \frac{1}{2\pi} \int_{-\pi/4}^{7\pi/4} \left( \mathrm{e}^{\mathrm{i}TH} \, \mathbf{h} \cdot \mathbf{n} \right) \right|_{\varphi=0} \mathrm{d}\theta = \mathrm{e}^{-\mathrm{i}T\varepsilon}$$

This cancels the term  $e^{-ikct}$  in (29), as expected.

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So, at this stage, we have

$$\Phi_{\alpha}(T) = \mathrm{e}^{-\mathrm{i}T\varepsilon} - \frac{1}{2\pi} \mathrm{e}^{-\mathrm{i}T\varepsilon\cos\alpha} \int_{-\pi/4}^{7\pi/4} K_{1}(\theta) \mathrm{e}^{\mathrm{i}TM\sin\alpha\cos\theta} \,\mathrm{d}\theta + O(T^{-1})$$

as  $T \to \infty$ , where the integral comes from the piece of  $\partial D$  at  $\varphi = \alpha$  and

$$K_1(\theta) = \frac{(M\cos\theta - \sin\alpha)(M\cos\theta\cos\alpha + \varepsilon\sin\alpha)}{(M\sin\theta\sin\alpha)^2 + (M\cos\theta\cos\alpha + \varepsilon\sin\alpha)^2}.$$

The integral can be estimated (as  $TM \sin \alpha \to \infty$ ) by the one-dimensional method of stationary phase [15, Section 6.1]. There are stationary-phase points at  $\theta = 0$  and  $\theta = \pi$ . Evaluating their contributions gives

$$\Phi_{\alpha}(T) \sim e^{-iT\varepsilon} - \frac{e^{-iT\varepsilon\cos\alpha}}{\sqrt{2\pi TM\sin\alpha}} \{K_1(0)e^{iTM\sin\alpha}e^{-i\pi/4} + K_1(\pi)e^{-iTM\sin\alpha}e^{i\pi/4}\},\$$

where, using  $M = \sin \varphi_{\rm g}$  and  $\varepsilon = \cos \varphi_{\rm g}$ ,

$$K_1(0) = \frac{\sin([\varphi_g - \alpha]/2)}{\sin([\varphi_g + \alpha]/2)}$$
 and  $K_1(\pi) = \frac{1}{K_1(0)}$ .

Hence, in Region I,

$$\psi(ct\cos\alpha, t) \sim \frac{1}{2\sqrt{2\pi TM\sin\alpha}} \{ K_1(0) \mathrm{e}^{\mathrm{i}TM\sin\alpha} \mathrm{e}^{-\mathrm{i}\pi/4} + K_1(\pi) \mathrm{e}^{-\mathrm{i}TM\sin\alpha} \mathrm{e}^{\mathrm{i}\pi/4} \}.$$
(37)

As we might expect, the estimate (37) breaks down on the boundaries of Region I, at  $\alpha = \pi$  and  $\alpha = \varphi_g$  (where  $K_1(\pi)$  is singular). When  $\alpha \simeq \varphi_g$ , (37) simplifies to

$$\psi(ct\cos\alpha, t) \sim \sqrt{\frac{\sin\varphi_{\rm g}}{2\pi\mu ct}} \frac{e^{i\pi/4}}{(\varphi_{\rm g} - \alpha)} e^{-i\mu ct\sin\alpha}.$$
(38)

Later, this estimate will be compared with an estimate coming from the Schrödinger equation, as discussed next.

# 4. Schrödinger's equation

It is of interest to compare our results with Moshinsky's solution of the analogous initial-value problem for the Schrödinger equation, (4), with  $\psi(x, 0) = f(x)$  and f given by (6). Before recalling Moshinsky's solution, we observe that  $\psi = e^{i(kx - \omega t)}$  solves (4) provided  $\omega = \frac{1}{2}k^2c/\mu = \omega_s$ , say; the corresponding group velocity is

$$v = ck/\mu = c_{\rm g}\sqrt{1 + (k/\mu)^2}.$$

Thus,  $v > c_g$  although, in applications,  $\mu \gg k$  so that  $v \simeq c_g$ . Under the same approximation, we have  $e^{i(kx-Ect)} \simeq e^{-i\mu ct}e^{i(kx-\omega_s t)}$ ; we will see the same factor of  $e^{-i\mu ct}$  when we compare our Klein–Gordon results with Moshinsky's solution of the Schrödinger problem. His solution [1, Equations (3) and (6)] can be written as

$$\psi(x,t) = \frac{1}{2} e^{y^2} \exp\left(\frac{i\mu x^2}{2ct}\right) \operatorname{erfc}(y), \qquad (39)$$

where erfc is the complementary error function [10] and

$$y = \mathrm{e}^{-\mathrm{i}\pi/4} \left( x - vt \right) \sqrt{\frac{\mu}{2ct}}.$$

The solution is defined for all x and for all t > 0: there are no discontinuities.

Clearly, the sign of (x - vt) affects the qualitative properties of the solution. Note that the line x = vt is in Region I (Figure 2), between  $x = c_g t$  and x = ct, but typically very close to  $x = c_g t$ . On x = vt, y = 0 and  $\psi(vt, t) = \frac{1}{2}e^{i\omega_s t}$ , which is  $\frac{1}{2}e^{i(kx-\omega t)}$  with x = vt and  $\omega = \omega_s$ . In particular,  $\psi(0, 0) = \frac{1}{2}$ .

Suppose, now, that  $x = ct \cos \alpha$  and t is large. Then, we can use large-argument estimates of erfc(y) [1, Equation (5)] in (39) to estimate  $\psi(ct \cos \alpha, t)$  as  $t \to \infty$ . Suppose first that  $\cos \alpha < k/\mu$ ; this is the region above x = vt with t > 0, including the line  $x = c_g t$ . We find that

$$\psi(ct\cos\alpha, t) \sim \exp\left\{i(kct\cos\alpha - \omega_{\rm s}t)\right\}$$
 as  $t \to \infty$ .

This is the expected wave solution, as noted by Moshinsky [1, Equation (7)].

Suppose instead that  $\cos \alpha > k/\mu$ ; this is the region below x = vt with t > 0. Then, use of [1, Equation (5)] in (39) gives  $\psi(ct \cos \alpha, t) \to 0$  as  $t \to \infty$ . In more detail, using [17, Equation 7.1.23],

$$\psi(ct\cos\alpha, t) \sim (2\pi\mu ct)^{-1/2}(\cos\alpha - k/\mu)^{-1} e^{i\pi/4} \exp\{i(\mu/2)ct\cos^2\alpha\},$$
 (40)

as  $t \to \infty$ , so that  $\psi \to 0$  as  $t^{-1/2}$ . This rate of decay is what we found in Region I for the Klein–Gordon equation. In fact, we can compare with (38) when  $\mu \gg k$ . In that case,  $\varepsilon \simeq k/\mu \ll 1$  and  $\varphi_g \simeq \frac{1}{2}\pi - \varepsilon$  (see (32)). Then, if  $\alpha = \frac{1}{2}\pi - \alpha'$ , where  $0 < \alpha' \ll 1$ (so that the three lines,  $x = c_g t$ , x = vt and  $x = ct \cos \alpha$  are all almost vertical),  $\sin \alpha \simeq 1 - {\alpha'}^2/2$ ,  $\cos \alpha \simeq \alpha'$  and  $\varphi_g - \alpha \simeq \alpha' - \varepsilon$ . It follows that (38) and (40) agree precisely (under the assumptions made), apart from the expected multiplicative factor of  $e^{-i\mu ct}$ .

## 5. Discussion

Let us summarize our results for Moshinsky's problem. We have  $\psi(x, t) = e^{i(kx-Ect)}$  for x < -ct. This solution is also approached if we fix x and then let  $t \to \infty$  (along the dashed line in Figure 2). We also have  $\psi(x, t) = 0$  for x > ct.

The behaviour of  $\psi(x, t)$  for  $-ct \le x \le ct$  is more complicated. If we examine  $\psi(ct \cos \alpha, t)$  as  $t \to \infty$  (sloping line in Figure 2), we either approach  $e^{i(kx-Ect)}$  (Region II) or zero (Region I) as  $t \to \infty$ , depending on the angle  $\alpha$ , with  $0 < \alpha < \pi$ . In fact, in Region I,  $\psi$  goes to zero as  $t^{-1/2}$ ; see (37). The dividing line between Regions I and II,  $x = c_g t$ , involves the group velocity. The exact solution,  $\psi$ , cannot be discontinuous along this line, it can only be discontinuous across the characteristics,  $x = \pm ct$ . The apparent discontinuity across  $x = c_g t$  could be smoothed out with a uniform asymptotic expansion: it is similar to the behaviour across a shadow boundary [14].

We also have two exact results, corresponding to  $\alpha = 0$  and  $\alpha = \pi$ :  $\psi(ct, t) = \frac{1}{2}$  and  $\psi(-ct, t) = e^{-i(k+E)ct} - \frac{1}{2}$ . Thus, the formula for  $\psi$ , (24), exhibits unexpected behaviour along  $x = \pm ct$ . However, by analogy with the wave equation, we can exploit the inherent non-uniqueness in the problem to eliminate one of the discontinuities: we can add any multiple of s(x, t), defined by (20), without violating the initial conditions. For example, if we replace  $\psi$  by

$$\psi(x,t) + s(x,t) = \psi_1(x,t),$$

say, we obtain a solution of Moshinsky's problem with no discontinuity along x = -ct; note that  $s(\pm ct, t) = \frac{1}{2}$ . The function  $\psi_1$  solves Moshinsky's problem with f given by (17) instead of (6). The presence of s in  $\psi_1$  alters the behaviour along x = ct but it does not affect the leading order asymptotics in Region II because the Bessel function decays. In Region I, we find that the behaviour of  $\psi_1$  is given by (37) but with 1 added to both  $K_1(0)$  and  $K_1(\pi)$ .

As noted below (28), the parameter  $\varepsilon$  is small in physical applications. This fact has not been exploited in the asymptotic results given here (except for the comparisons made at the end of Section 4). Therefore, there is scope for further analysis in future.

#### Note

1. This formula is given incorrectly in the 1975 edition of [15].

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