# Time-domain BEM for 3-D transient elastodynamic problems with interacting rigid movable disc-shaped inclusions 

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Received: 22 October 2013 / Accepted: 6 January 2014 / Published online: 29 January 2014
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#### Abstract

Formulation of time-domain boundary element method for elastodynamic analysis of interaction between rigid massive disc-shaped inclusions subjected to impinging elastic waves is presented. Boundary integral equations (BIEs) with time-retarded kernels are obtained by using the integral representations of displacements in a matrix in terms of interfacial stress jumps across the inhomogeneities and satisfaction of linearity conditions at the inclusion domains. The equations of motion for each inclusion complete the problem formulation. The time-stepping/collocation scheme is implemented for the discretization of the BIEs by taking into account the traveling nature of the generated wave field and local structure of the solution at the inclusion edges. Numerical results concern normal incidence of longitudinal wave onto two coplanar circular inclusions. The inertial effects are revealed by the time dependencies of inclusions' kinematic parameters and dynamic stress intensity factors in the inclusion vicinities for different mass ratios and distances between the interacting obstacles.


Keywords Boundary element method • Time-domain formulation • Elastic matrix - Disc-shaped rigid inclusions • Transient elastic wave • Inclusions interaction

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## 1 Introduction

Many highly demanded elastic materials, in particular advanced nanocomposites, as well as various rock-type natural media contain particles with essential geometric aspect ratio and contrast mechanical properties like rigid disc-shaped inclusions as the filler elements [1-3]. At arbitrary in time loading scenario, presence of such inclusions may lead to other dynamic responses of the structure in comparison with the volumetric inclusions due to the specific wave scattering by the sharp edges of the particles. Most generally these transient processes are exhibited in the three-dimensional case of interacting inclusions, when mixed mode deformation and multiple scattering takes place [4]. Knowledge of the mentioned effects of elastic wave propagation in elastic solids with systems of thin-walled inclusions under impact or shock loading is of great importance for design, optimization and manufacturing of new composite materials with high dynamic strength and stiffness under light weight preservation, fracture and damage analysis, non-destructive material testing, and seismic prospecting of complex geological media.

Application of the time-domain boundary element method (BEM) for the numerical solution of corresponding 3-D elastodynamic problems in the case of infinite matrices is very attractive, because of the reduction by one of space dimensionality in the resulting equations, the automatic satisfaction of Sommerfeld radiation conditions at infinity, and actual parameter histories are obtained directly via step by step procedure. The main advantages of time-domain BEM in comparison to other numerical methods and schemes, for example BEM formulations in the Laplace or Fourier time transform domains, compilation of works on the modeling of an elastodynamic continuum using different modification of time-domain BEM have been presented [5-10]. Involved in
the method elastodynamic fundamental solutions as the functions of time, the receiver and source points are also available in the literature [11]. In particular, within 3D elastodynamic statements, this approach was successfully used for transient analysis of direct and inverse problems for the solids with volumetric inclusions [12,13], contact problems arising in soilstructure interaction analysis [14-16], surface displacement problems on site response analysis of complex topographic structures [17]. Large-scale problems of many interacting (in particular rigid) volumetric inclusions under non-stationary wave incidence are investigated by the fast multipole BEM $[18,19]$. Among other works, there the numerical aspects of the time-domain BEM are described, including proper choice of time and space discretization intervals, appropriate introducing of temporal and spatial shape functions for the unknowns approximation, analytical evaluation of the time-convolution integrals on the considered discretization meshes to provide the convergence and accuracy of the results. Besides, an accounting in the time-domain BEM formulations of movability of rigid massive foundation subjected to transient loadings or waves and joined with an elastic half-space is demonstrated [15].

Concerning sharp-edged or thin-walled stress concentrators, improved techniques are needed to implement the timedomain BEM because the close distances between the opposite surfaces of the object and numerical instability of the classical algorithm. Numerous publications in this direction have been connected with the 3D cracked solids, where the conditions of displacement jumps under preservation of stress continuity across the crack faces should be satisfied. Then, for the simulation numerically of time dependences, the regularized forms of the traction boundary integral equations (BIEs) relative to the crack opening displacements in the time domain and with adoption of convolution quadrature method have been obtained $[8,20]$. Displacement timedomain BEM formulation have been proposed in [21], as well as dual time-domain BEM has been elaborated [22], where displacement and tractions BIEs are used on different crack-surfaces. The acceleration procedures based on the fast multipole method have been also generalized on the timedomain BEM analysis of cracks subjected to impulse wave loading [23]. Described techniques are applied for the estimation of transient elastodynamic fields in 3D bounded cracked solids [21,24], unbounded solids containing crack of complex shape [25], crack interacting with suddenly transformed or expanding zone of eigenstrains [26,27], and multiple interacting cracks [28,29].

Dynamic response of disc-shaped inclusions on the incident waves in 3D acoustic and elastic media was investigated in details under the assumption of single scatterer. Corresponding frequency-domain solutions for the rigid disc are given in [30-32], time-domain BEM solutions are constructed both for rigid [32-34] and compliant [35] disc-
shaped inclusions. In particular, in [34] the BEM formulations for the model of rigid disc-shaped inclusion in an elastic matrix with so-called "anticrack" conditions of displacement continuity and stress jumps across the inclusion faces are accompanied by the equation of inclusion motion to show its inertial properties. Time-dependent solutions for single inclusion due to the retarded or traveling character of waves describe also the situation with multiple inclusions until time necessary the longitudinal wave induced by closest particles to arrive at the actual point. However, after that moment cooperative influence of inclusions onto transient process should be considered. It should be mentioned, that the formulations of wave propagation problems for multiple thin-walled scatterers are complicated not only by superposition relations in general form, but also by the considering the special behavior of solutions at the inclusions edges. To the best of the authors' knowledge, numerical simulation of interacting rigid disc-shaped inclusions in 3-D matrix under transient elastic wave loading was yet not reported in the literature. It is the subject of current work under the assumptions of infinite elastic isotropic matrix, arbitrary locations and movability of inclusions as the thin-walled rigid units of given masses.

The paper content is organized as the following. In Sect. 2 the corresponding problem is reduced in time-domain to the weakly-singular BIEs relative to the interfacial stress jumps (ISJs) across the inclusion surfaces; the unknown kinematic parameters of inclusions, namely their translations and rotations, are presented in the BIEs as the free terms. The dynamic interaction between arbitrarily located inclusions is described in the BIEs by the regular kernels in an explicit form. Completeness of the mathematical model is achieved by the joining of BIEs with the ordinary differential equations of motion for each inclusion. In Sect. 3 BIEs obtained are adapted to the effective solution by the time-domain BEM on the example of antisymmetric problem for coplanar circular inclusions. It consists in the transformation of BIEs to non-singular form with new smooth functions as the unknowns. To this end the singularity subtraction technique is used for the proper interpretation of weakly-singular kernels at source point, and a mapping of the circular integration domains into the rectangular domains is adopted to avoid the peculiarities at the contours of integration domains. They arise due to the exact accounting of "square-root" behavior of solutions near the edges of multiple inclusions. The space collocation approach in conjunction with the marching in time approach to construct the discrete analogue of the problem as the recurrent systems of linear algebraic equations is described in this section also. It involves the analytical over the time and both analytical and numerical over the space integration schemes for the calculation of influence matrix coefficients. The subelement technique is used to more correct satisfaction of causality conditions for the introduced space and temporal meshes.

Several examples of the method implementation are considered to demonstrate its robustness and efficiency in Sect. 4. Calculations are carried out for a pair of coplanar circular disc-shaped inclusions of equal size and different masses under normal incidence of a plane longitudinal elastic wave with the step-like and hill-like time profiles. Unlike of single inclusion, the general motion of inclusions including their translations as well as rotations is realized in the considered configuration. The histories of inclusions displacements and dynamic stress intensity factors in the inclusions vicinities are assessed from the point of view of the shielding and amplification effects of inclusions neighborhood. The conclusions are listed in Sect. 5.

## 2 Basic formulations

Let an infinite isotropic elastic matrix contain $N$ arbitrarily located disc-shaped rigid inclusions of masses $M_{n}(n=$ $1,2, \ldots, N)$. Their mid-surfaces occupy the plane regions $S_{n}(n=1,2, \ldots, N)$ as shown in Fig. 1. The thicknesses of the inclusions are much smaller than the diameters of corresponding domains $S_{n}$. Perfect contact between the matrix and the inclusions is assumed, which implies that the displacements and the stresses are continuous across the matrixinclusion interfaces. Transient deformation process in the solid is generated by an incident elastic wave with the given distribution in the space $\mathbf{x}$ and the time $t$ of the displacement vector $\mathbf{u}^{\text {in }}$.

To describe the problem geometry, $N$ Cartesian coordinate systems are introduced, so that the coordinate system $O^{(n)} x_{1}^{(n)} x_{2}^{(n)} x_{3}^{(n)}$ is connected with the $n$-th inclusion by the coincidence of its center $O^{(n)}$ with the center of mass of the inhomogeneity, the axis $O^{(n)} x_{3}^{(n)}$ is perpendicular to the surface $S_{n}$ (Fig. 1). The mutual locations of the $n$ -


Fig. 1 Problem geometry
th and $k$-th inclusions are fixed by the vectors $\mathbf{O}^{(n)} \mathbf{O}^{(k)}=$ $-\mathbf{O}^{(k)} \mathbf{O}^{(n)}$ and the direction cosines $l_{j}^{(n k)}, m_{j}^{(n k)}, p_{j}^{(n k)}(j=$ $1,2,3$ ) of the unit vector $\mathbf{e}^{(n)}$ belonging to the axis $O^{(n)} x_{j}^{(n)}$ in the $k$-th coordinate system. Then denoting the point by the position vector $\mathbf{x}^{(n)}\left(x_{1}^{(n)}, x_{2}^{(n)}, x_{3}^{(n)}\right)$ in the coordinate system $O^{(n)} x_{1}^{(n)} x_{2}^{(n)} x_{3}^{(n)}$ yields the position vector $\mathbf{x}^{(k n)}\left(x_{1}^{(k n)}, x_{2}^{(k n)}, x_{3}^{(k n)}\right)$ of the same point in the coordinate system $O^{(k)} x_{1}^{(k)} x_{2}^{(k)} x_{3}^{(k)}$ as $\mathbf{x}^{(k n)}=\mathbf{O}^{(k)} \mathbf{O}^{(n)}+\mathbf{x}^{(n)}$.

The substructure concept is applied here for the statement of kinematic interaction in the system "infinite matrixmultiple inclusions". According to this, the governing equations of each element of the system are derived independently and then the compatibility conditions are imposed at the contact surface between the substructures to achieve the formulation completeness.

Within the considered transient problem, the input equation for the displacement vector $\mathbf{u}\left(\mathbf{x}^{(n)}, t\right)$ in the matrix is the equation of motion [7],
$c_{1}^{2} \nabla(\nabla \cdot \mathbf{u})-c_{2}^{2} \nabla \times(\nabla \times \mathbf{u})-\frac{\partial^{2} \mathbf{u}}{\partial t^{2}}=0$,
where $\nabla\left(\partial / \partial x_{1}, \partial / \partial x_{2}, \partial / \partial x_{3}\right)$ is the three-dimensional nabla-vector, $c_{2}=\sqrt{G / \rho}$ and $c_{1}=\sqrt{2(1-\nu) /(1-2 \nu)} c_{2}$ are the transverse and longitudinal waves velocities, $G$ is the shear modulus of matrix material, $\rho$ is its density, and $\nu$ is the Poisson's ratio.

Concerning the moving inclusions, reaction of a matrix on their presence is defined by the time-dependent forces with the principal vectors $\mathbf{P}^{(n)}(n=1,2, \ldots, N)$ acting on each inhomogeneity, and by the moments $\mathbf{Z}^{(n)}(n=1,2, \ldots, N)$ of these forces relative to the corresponding centers of masses. Interpretation of the inclusions as the absolute rigid units enables the modeling of their motions by the wellknown differential equations:

$$
\begin{align*}
M_{n} \frac{\mathrm{~d}^{2} \mathbf{U}^{(n)}}{\mathrm{d} t^{2}} & =\mathbf{P}^{(n)}, \quad M_{n}\left(\mathbf{r}^{(n)} \cdot \mathbf{e}^{(n)}\right)^{2} \frac{\mathrm{~d}^{2} \Omega^{(n)}}{\mathrm{d} t^{2}}=\mathbf{Z}^{(n)} \\
n & =1,2, \ldots, N \tag{2}
\end{align*}
$$

Here $\mathbf{U}^{(n)}$ and $\Omega^{(n)}$ are translation and rotation angle vectors of $n$-th inclusion, respectively, $\mathbf{r}^{(n)}$ is its vector radius of inertia relative to the coordinate axes.

With accepted restrictions on the kinetics of thin-walled inclusions, which allow translations and rotations, the boundary conditions in their domains take the form
$\mathbf{u}=\mathbf{U}^{(n)}+\Omega^{(n)} \times \mathbf{x}^{(n)}, \quad \mathbf{x}^{(n)} \in S_{n}, n=1,2, \ldots, N$.
Due to perfect contact between the matrix and the inclusions, the compatibility conditions mean that for the matrix points the displacements (3) take place also, besides force influence on it is characterized by the opposite values of parameters $\mathbf{P}^{(n)}$ and $\mathbf{Z}^{(n)}$.

In accordance with the superposition principle and for the correct statement of initial conditions, the total nonstationary displacement $\mathbf{u}$ in the matrix with multiple inclusions or scatterers is presented as a sum, namely
$\mathbf{u}=\mathbf{u}^{i n}+\sum_{k=1}^{N} \mathbf{u}^{(k)}$,
where the term $\mathbf{u}^{(k)}$ describes the contribution to the scattered wave field of $k$-th inclusion. Then by fixing of the process from the moment $t=0$ when incident wave front hits the closest inclusion, the initial conditions on the scattered displacements $\mathbf{u}^{(n)}$, the kinematic parameters $\mathbf{U}^{(n)}$ and $\Omega^{(n)}$ of inclusions can be written as

$$
\begin{align*}
\left.\mathbf{u}^{(n)}\right|_{t=0} & =\left.\frac{\partial \mathbf{u}^{(n)}}{\partial t}\right|_{t=0}=0 \\
\left.\mathbf{U}^{(n)}\right|_{t=0} & =\left.\frac{\mathrm{d} \mathbf{U}^{(n)}}{\mathrm{d} t}\right|_{t=0}=0,\left.\quad \Omega^{(n)}\right|_{t=0}=\left.\frac{\mathrm{d} \Omega^{(n)}}{\mathrm{d} t}\right|_{t=0}=0 \\
n & =1,2, \ldots, N \tag{5}
\end{align*}
$$

From the above definition of the displacement vector $\mathbf{u}^{(n)}$ it follows that the integral representation of its components $u_{j}^{(n)}(j=1,2,3)$ is the same as that for a nonstationary disturbed single inclusion in an infinite matrix. As a result of applying the Betti-Rayleigh reciprocity theorem and using the properties of 3-D time-domain elastodynamic fundamental solutions, the displacement components $u_{j}^{(n)}(j=1,2,3)$ can be given through the combinations of retarded potentials by [34]

$$
\begin{align*}
u_{j}^{(n)}\left(\mathbf{x}^{(n)}, t\right)= & \frac{1}{4 \pi G}\left\{\iint_{S_{n}} \frac{1}{\left|\mathbf{x}^{(n)}-\mathbf{y}^{(n)}\right|}\right. \\
& {\left[-\int_{\gamma}^{1} \Delta \sigma_{j}^{(n)}\left(\mathbf{y}^{(n)}, t-\tau\left|\mathbf{x}^{(n)}-\mathbf{y}^{(n)}\right| / c_{2}\right) \tau d \tau\right.} \\
& \left.+\Delta \sigma_{j}^{(n)}\left(\mathbf{y}^{(n)}, t-\left|\mathbf{x}^{(n)}-\mathbf{y}^{(n)}\right| / c_{2}\right)\right] d S_{\mathbf{y}} \\
& +\sum_{i=1}^{3} \iint_{S_{n}} \frac{\left(x_{j}^{(n)}-y_{j}^{(n)}\right)\left(x_{i}^{(n)}-y_{i}^{(n)}\right)}{\left|\mathbf{x}^{(n)}-\mathbf{y}^{(n)}\right|^{3}} \\
& \times\left[3 \int_{\gamma}^{1} \Delta \sigma_{i}^{(n)}\left(\mathbf{y}^{(n)}, t-\tau\left|\mathbf{x}^{(n)}-\mathbf{y}^{(n)}\right| / c_{2}\right) \tau d \tau\right. \\
& -\Delta \sigma_{i}^{(n)}\left(\mathbf{y}^{(n)}, t-\left|\mathbf{x}^{(n)}-\mathbf{y}^{(n)}\right| / c_{2}\right) \\
& \left.\left.+\gamma^{2} \Delta \sigma_{i}^{(n)}\left(\mathbf{y}^{(n)}, t-\gamma\left|\mathbf{x}^{(n)}-\mathbf{y}^{(n)}\right| / c_{2}\right)\right] d S_{\mathbf{y}}\right\} \\
& j=1,2,3, n=1,2, \ldots, N . \tag{6}
\end{align*}
$$

Here the potential densities $\Delta \sigma_{j}^{(n)}(j=1,2,3)$ are the unknown ISJs across the $n$-th inclusion, which are defined
by means of the stress components $\sigma_{j 3}$ as

$$
\begin{align*}
& \Delta \sigma_{j}^{(n)}\left(\mathbf{x}^{(n)}, t\right)= \sigma_{j 3}^{+}\left(\mathbf{x}^{(n)}, t\right)-\sigma_{j 3}^{-}\left(\mathbf{x}^{(n)}, t\right), \\
& j=1,2,3, \mathbf{x}^{(n)} \in S_{n}, \\
& \sigma_{j 3}^{ \pm}\left(\mathbf{x}^{(n)}, t\right)=\lim _{x_{3}^{(n)} \rightarrow \pm 0} \sigma_{j 3}\left(\mathbf{x}^{(n)}, t\right), \tag{7}
\end{align*}
$$

in accordance with the causality principle $\Delta \sigma_{j}^{(n)}\left(\mathbf{x}^{(n)}, t\right)=$ 0 , when $t \leq 0, \gamma=c_{2} / c_{1},\left|\mathbf{x}^{(n)}-\mathbf{y}^{(n)}\right|$ is the distance between the receiver $\mathbf{x}^{(n)}\left(x_{1}^{(n)}, x_{2}^{(n)}, x_{3}^{(n)}\right)$ and the source $\mathbf{y}^{(n)}\left(y_{1}^{(n)}, y_{2}^{(n)}, 0\right)$.

Physical meaning of the functions $\Delta \sigma_{j}^{(n)}(j=1,2,3)$ in the integral representations (6) yields the following relations between them and the components $P_{j}^{(n)}(j=1,2,3)$ of principal vector $\mathbf{P}^{(n)}$ of the forces to be transferred from the matrix to the $n$-th inclusion and the components $Z_{j}^{(n)}(j=$ $1,2,3$ ) of moments $\mathbf{Z}^{(n)}$ of these forces:

$$
\begin{align*}
& P_{j}^{(n)}(t)=-\iint_{S_{n}} \Delta \sigma_{j}^{(n)}\left(\mathbf{y}^{(n)}, t\right) d S_{\mathbf{y}}, \quad j=1,2,3, \\
& Z_{j}^{(n)}(t)=(-1)^{j} \iint_{S_{n}} y_{3-j}^{(n)} \Delta \sigma_{3}^{(n)}\left(\mathbf{y}^{(n)}, t\right) d S_{\mathbf{y}}, \quad j=1,2, \\
& Z_{3}(t)=\iiint_{S_{n}}\left[y_{2}^{(n)} \Delta \sigma_{1}^{(n)}\left(\mathbf{y}^{(n)}, t\right)-y_{1}^{(n)} \Delta \sigma_{2}^{(n)}\left(\mathbf{y}^{(n)}, t\right)\right] d S_{\mathbf{y}} . \tag{8}
\end{align*}
$$

Thus, by the relations (4) and (6) the displacements in the matrix and by the relations (8) the force influences on the inclusions are expressed in terms of the ISJs. To determine these functions the boundary conditions (3) should be applied, taking into account that the components of total displacement (4) in the $n$-th coordinate system have the form

$$
\begin{align*}
& u_{j}\left(\mathbf{x}^{(n)}, t\right)=u_{j}^{i n}\left(\mathbf{x}^{(n)}, t\right) \\
& \quad+\sum_{k=1}^{N}\left[u_{1}^{(k)}\left(\mathbf{x}^{(k n)}, t\right)\left(l_{1}^{(k n)} \delta_{1 j}+m_{1}^{(k n)} \delta_{2 j}+p_{1}^{(k n)} \delta_{3 j}\right)\right. \\
& \quad+u_{2}^{(k)}\left(\mathbf{x}^{(k n)}, t\right)\left(l_{2}^{(k n)} \delta_{1 j}+m_{2}^{(k n)} \delta_{2 j}+p_{2}^{(k n)} \delta_{3 j}\right) \\
& \left.\quad+u_{3}^{(k)}\left(\mathbf{x}^{(k n)}, t\right)\left(l_{3}^{(k n)} \delta_{1 j}+m_{3}^{(k n)} \delta_{2 j}+p_{3}^{(k n)} \delta_{3 j}\right)\right], \tag{9}
\end{align*}
$$

where $\delta_{i j}$ is the Kronecker symbol.
Substituting the relations (9) with the integral representations (6) in the boundary conditions (3) we arrive to such system of BIEs, where the ISJs as the integral densities, and the translations and rotations of inclusions as the free terms are theunknowns:

$$
\begin{align*}
& \iint_{S_{n}} \frac{\mathbf{B}_{1}^{\left|\mathbf{x}^{(n)}-\mathbf{y}^{(n)}\right|}\left[\Delta \sigma_{3}^{(n)}\left(\mathbf{y}^{(n)}, t\right)\right]}{\left|\mathbf{x}^{(n)}-\mathbf{y}^{(n)}\right|} d S_{\mathbf{y}} \\
& +\sum_{k=1}^{N}\left(1-\delta_{k n}\right) \sum_{i=1}^{3} \iint_{S_{k}} \frac{\mathbf{K}_{3 i}^{\mathbf{x}^{(k n)}, \mathbf{y}^{(k)}}\left[\Delta \sigma_{i}^{(k)}\left(\mathbf{y}^{(k)}, t\right)\right]}{\left|\mathbf{x}^{(k n)}-\mathbf{y}^{(k)}\right|^{3}} d S_{\mathbf{y}} \\
& =4 \pi G\left[-u_{3}^{i n}\left(\mathbf{x}^{(n)}, t\right)+U_{3}^{(n)}(t)-\Omega_{2}^{(n)}(t) x_{1}^{(n)}\right. \\
& \left.+\Omega_{1}^{(n)}(t) x_{2}^{(n)}\right], \quad \mathbf{x}^{(n)} \in S_{n}, n=1,2, \ldots, N, \\
& \iint_{S_{n}} \frac{1}{\left|\mathbf{x}^{(n)}-\mathbf{y}^{(n)}\right|}\left\{\mathbf{B}_{1}^{\left|\mathbf{x}^{(n)}-\mathbf{y}^{(n)}\right|}\left[\Delta \sigma_{j}^{(n)}\left(\mathbf{y}^{(n)}, t\right)\right]\right. \\
& +\frac{\left(x_{1}^{(n)}-y_{1}^{(n)}\right)\left(x_{2}^{(n)}-y_{2}^{(n)}\right)}{\left|\mathbf{x}^{(n)}-\mathbf{y}^{(n)}\right|^{2}} \mathbf{B}_{2}^{\left|\mathbf{x}^{(n)}-\mathbf{y}^{(n)}\right|}\left[\Delta \sigma_{3-j}^{(n)}\left(\mathbf{y}^{(n)}, t\right)\right] \\
& \left.+\frac{\left(x_{j}^{(n)}-y_{j}^{(n)}\right)^{2}}{\left|\mathbf{x}^{(n)}-\mathbf{y}^{(n)}\right|^{2}} \mathbf{B}_{2}^{\left|\mathbf{x}^{(n)}-\mathbf{y}^{(n)}\right|}\left[\Delta \sigma_{j}^{(n)}\left(\mathbf{y}^{(n)}, t\right)\right]\right\} d S_{\mathbf{y}} \\
& +\sum_{k=1}^{N}\left(1-\delta_{k n}\right) \sum_{i=1}^{3} \iint_{S_{k}} \frac{\mathbf{K}_{j i}^{\mathbf{x}^{(k n)}, \mathbf{y}^{(k)}}\left[\Delta \sigma_{i}^{(k)}\left(\mathbf{y}^{(k)}, t\right)\right]}{\left|\mathbf{x}^{(k n)}-\mathbf{y}^{(k)}\right|^{3}} d S_{\mathbf{y}} \\
& =4 \pi G\left[-u_{j}^{i n}\left(\mathbf{x}^{(n)}, t\right)+U_{j}^{(n)}(t)+(-1)^{j} \Omega_{3}^{(n)}(t) x_{3-j}^{(n)}\right] \text {, } \\
& \mathbf{x}^{(n)} \in S_{n}, j=1,2, n=1,2, \ldots, N . \tag{10}
\end{align*}
$$

Here $U_{j}^{(n)}(j=1,2,3)$ and $\Omega_{j}^{(n)}(j=1,2,3)$ are the components of translation and rotation angle vectors $\mathbf{U}^{(n)}$ and $\Omega^{(n)}$ of the $n$-th inclusion, respectively, the operators $\mathbf{B}_{j}^{|\mathbf{x}-\mathbf{y}|}$ in the weakly-singular integrals are the same as in the BIEs for single inclusion [34], and act on the function by the law of time retardation, namely:

$$
\begin{align*}
& \mathbf{B}_{j}^{\left|\mathbf{x}^{(n)}-\mathbf{y}^{(n)}\right|}\left[\Delta \sigma_{i}^{(n)}\left(\mathbf{y}^{(n)}, t\right)\right]=b_{1 j} \int_{\gamma}^{1} \Delta \sigma_{i}^{(n)}\left(\mathbf{y}^{(n)},\right. \\
& \left.t-\tau\left|\mathbf{x}^{(n)}-\mathbf{y}^{(n)}\right| / c_{2}\right) \tau d \tau+b_{2 j} \Delta \sigma_{i}^{(n)}\left(\mathbf{y}^{(n)},\right. \\
& \left.t-\left|\mathbf{x}^{(n)}-\mathbf{y}^{(n)}\right| / c_{2}\right)+b_{3 j} \Delta \sigma_{i}^{(n)}\left(\mathbf{y}^{(n)},\right. \\
& \left.t-\gamma\left|\mathbf{x}^{(n)}-\mathbf{y}^{(n)}\right| / c_{2}\right), j=1,2, i=1,2,3 \\
& b_{11}=-1, b_{21}=1, b_{31}=0, \quad b_{12}=3, b_{22}=-1, \\
& b_{32}=\gamma^{2} \tag{11}
\end{align*}
$$

Operators $\mathbf{K}_{j i}^{\mathbf{x}, \mathbf{y}}$ in the remaining integrals are regular (because $\mathbf{x}^{(k n)} \neq \mathbf{y}^{(k)}$ ), they describe the nonstationary interaction of scatterers, and they are determined by the formulas

$$
\begin{aligned}
& \mathbf{K}_{j i}^{\left.\mathbf{x}^{(k n)}\right) \mathbf{y}^{(k)}}\left[\Delta \sigma_{i}^{(k)}\left(\mathbf{y}^{(k)}, t\right)\right]=K_{1 j i}^{(k n)}\left(\mathbf{x}^{(k n)}, \mathbf{y}^{(k)}\right) \int_{\gamma}^{1} \Delta \sigma_{i}^{(k)}\left(\mathbf{y}^{(k)},\right. \\
& \left.\quad t-\tau\left|\mathbf{x}^{(k n)}-\mathbf{y}^{(k)}\right| / c_{2}\right) \tau d \tau
\end{aligned}
$$

$$
\begin{align*}
& \quad+K_{2 j i}^{(k n)}\left(\mathbf{x}^{(k n)}, \mathbf{y}^{(k)}\right) \Delta \sigma_{i}^{(k)}\left(\mathbf{y}^{(k)}, t-\left|\mathbf{x}^{(k n)}-\mathbf{y}^{(k)}\right| / c_{2}\right) \\
& \quad+K_{3 j i}^{(k n)}\left(\mathbf{x}^{(k n)}, \mathbf{y}^{(k)}\right) \Delta \sigma_{i}^{(k)}\left(\mathbf{y}^{(k)}, t-\gamma \mid \mathbf{x}^{(k n)}-\mathbf{y}^{\left.(k) \mid / c_{2}\right)}\right. \\
& K_{1 j i}^{(k n)}\left(\mathbf{x}^{(k n)}, \mathbf{y}^{(k)}\right)=-\left(l_{i}^{(k n)} \delta_{1 j}+m_{i}^{(k n)} \delta_{2 j}\right. \\
& \left.\quad+p_{i}^{(k n)} \delta 3 j\right) \mid \mathbf{x}^{(k n)}-\mathbf{y}^{\left.(k)\right|^{2}+3 \tilde{K}_{j i}^{(k n)}\left(\mathbf{x}^{(k n)}, \mathbf{y}^{(k)}\right),} \\
& K_{2 j i}^{(k n)}\left(\mathbf{x}^{(k n)}, \mathbf{y}^{(k)}\right)=\left(l_{i}^{(k n)} \delta_{1 j}+m_{i}^{(k n)} \delta_{2 j}\right. \\
& \left.\quad+p_{i}^{(k n)} \delta_{3 j}\right) \mid \mathbf{x}^{(k n)}-\mathbf{y}^{\left.(k)\right|^{2}-\tilde{K}_{j i}^{(k n)}\left(\mathbf{x}^{(k n)}, \mathbf{y}^{(k)}\right),} \\
& K_{3 j i}^{(k n)}\left(\mathbf{x}^{(k n)}, \mathbf{y}^{(k)}\right)=\gamma^{2} \tilde{K}_{j i}^{(k n)}\left(\mathbf{x}^{(k n)}, \mathbf{y}^{(k)}\right), \\
& \tilde{K}_{j i}^{(k n)}\left(\mathbf{x}^{(k n)}, \mathbf{y}^{(k)}\right)=\left[\left(x_{1}^{(k n)}-y_{1}^{(k)}\right) \delta_{1 i}+\left(x_{2}^{(k n)}-y_{2}^{(k)}\right) \delta_{2 i}\right. \\
& \left.\quad+x_{3}^{(k n)} \delta_{3 i}\right]\left[( l _ { 1 } ^ { ( k n ) } \delta _ { 1 j } + m _ { 1 } ^ { ( k n ) } \delta _ { 2 j } + p _ { 1 } ^ { ( k n ) } \delta _ { 3 j } ) \left(x_{1}^{(k n)}\right.\right. \\
& \left.\quad-y_{1}^{(k)}\right)+\left(l_{2}^{(k n)} \delta_{1 j}+m_{2}^{(k n)} \delta_{2 j}+p_{2}^{(k n)} \delta_{3 j}\right)\left(x_{2}^{(k n)}-y_{2}^{(k)}\right) \\
& \left.\quad+\left(l_{3}^{(k n)} \delta_{1 j}+m_{3}^{(k n)} \delta_{2 j}+p_{3}^{(k n)} \delta_{3 j}\right) x_{3}^{(k n)}\right] . \tag{12}
\end{align*}
$$

For completeness, BIEs (10) should be accompanied by the differential Eq. (2), which in scalar form and using Eqs. (8), are transformed to $\left(r_{j}^{(n)}\right.$ is the radius of inertia of the $n$-th inclusion relative to the axis $O^{(n)} x_{j}^{(n)}$ ):

$$
\begin{align*}
& M_{n} \frac{\mathrm{~d}^{2} U_{3}^{(n)}(t)}{\mathrm{d} t^{2}}=-\iint_{S_{n}} \Delta \sigma_{3}^{(n)}\left(\mathbf{y}^{(n)}, t\right) d S_{\mathbf{y}}, \\
& n=1,2, \ldots, N, \\
& M_{n}\left(r_{j}^{(n)}\right)^{2} \frac{\mathrm{~d}^{2} \Omega_{j}^{(n)}(t)}{\mathrm{d} t^{2}}=(-1)^{j} \iint_{S_{n}} y_{3-j}^{(n)} \Delta \sigma_{3}^{(n)}\left(\mathbf{y}^{(n)}, t\right) d S_{\mathbf{y}}, \\
& \quad j=1,2, n=1,2, \ldots, N, \\
& \begin{aligned}
M_{n} \frac{\mathrm{~d}^{2} U_{j}^{(n)}(t)}{\mathrm{d} t^{2}}=-\iint_{S_{n}} \Delta \sigma_{j}^{(n)}\left(\mathbf{y}^{(n)}, t\right) d S_{\mathbf{y}}, \\
j=1,2, n=1,2, \ldots, N
\end{aligned} \\
& \begin{array}{r}
M_{n}\left(r_{3}^{(n)}\right)^{2} \frac{\mathrm{~d}^{2} \Omega_{3}^{(n)}(t)}{\mathrm{d} t^{2}}=\iint_{S_{n}}\left[y_{2}^{(n)} \Delta \sigma_{1}^{(n)}\left(\mathbf{y}^{(n)}, t\right)\right. \\
\\
\left.\quad-y_{1}^{(n)} \Delta \sigma_{2}^{(n)}\left(\mathbf{y}^{(n)}, t\right)\right] d S_{\mathbf{y}}, n=1,2, \ldots, N .
\end{array}
\end{align*}
$$

Therefore, the initial problem is reduced to the system of $9 N$ connected scalar equations consisting of $3 N$ BIEs (10) and $6 N$ ordinary differential Eq. (13) for the functions $\Delta \sigma_{j}^{(n)}, U_{j}^{(n)}$ and $\Omega_{j}^{(n)}(j=1,2,3 ; n=1,2, \ldots, N)$. For Eqs. (13) the zero initial conditions (5) on the components of inclusion translations and rotations are added also.

It should be mentioned that the system of BIEs is considerably simplified and divided into two independent subsystems for coplanar inclusions, for example, located in the plane $x_{3}^{(n)}=0(n=1,2, \ldots, N)$. Then, by putting in the expressions (12) $l_{j}^{(k n)}=\delta_{1 j}, m_{j}^{(k n)}=\delta_{2 j}, p_{j}^{(k n)}=\delta_{3 j}, x_{3}^{(k n)}=0$,
from Eqs. (10) we obtain the following system of $N$ BIEs corresponding to the antisymmetric problem

$$
\begin{align*}
& \sum_{k=1}^{N} \iint_{S_{k}} \frac{\mathbf{B}_{1}^{\left|\mathbf{x}^{(k n)}-\mathbf{y}^{(k)}\right|}\left[\Delta \sigma_{3}^{(k)}\left(\mathbf{y}^{(k)}, t\right)\right]}{\left|\mathbf{x}^{(k n)}-\mathbf{y}^{(k)}\right|} d S_{\mathbf{y}} \\
& =4 \pi G\left[-u_{3}^{i n}\left(\mathbf{x}^{(n)}, t\right)+U_{3}^{(n)}(t)\right. \\
& \left.\quad-\Omega_{2}^{(n)}(t) x_{1}^{(n)}+\Omega_{1}^{(n)}(t) x_{2}^{(n)}\right], \\
& \mathbf{x}^{(n)} \in S_{n}, n=1,2, \ldots, N \tag{14}
\end{align*}
$$

and the system of $2 N$ BIEs corresponding to the symmetric problem

$$
\begin{align*}
& \sum_{k=1}^{N} \iint_{S_{k}} \frac{1}{\left|\mathbf{x}^{(k n)}-\mathbf{y}^{(k)}\right|}\left\{\mathbf{B}_{1}^{\left|\mathbf{x}^{(k n)}-\mathbf{y}^{(k)}\right|}\left[\Delta \sigma_{j}^{(k)}\left(\mathbf{y}^{(k)}, t\right)\right]\right. \\
& \quad+\frac{\left(x_{1}^{(k n)}-y_{1}^{(k)}\right)\left(x_{2}^{(k n)}-y_{2}^{(k)}\right)}{\left|\mathbf{x}^{(k n)}-\mathbf{y}^{(k)}\right|^{2}} \mathbf{B}_{2}^{\left|\mathbf{x}^{(k n)}-\mathbf{y}^{(k)}\right|}\left[\Delta \sigma_{3-j}^{(k)}\left(\mathbf{y}^{(k)}, t\right)\right] \\
& \left.\quad+\frac{\left(x_{j}^{(k n)}-y_{j}^{(k)}\right)^{2}}{\left|\mathbf{x}^{(k n)}-\mathbf{y}^{(k)}\right|^{2}} \mathbf{B}_{2}^{\left|\mathbf{x}^{(k n)}-\mathbf{y}^{(k) \mid}\right|}\left[\Delta \sigma_{j}^{(k)}\left(\mathbf{y}^{(k)}, t\right)\right]\right\} d S_{\mathbf{y}} \\
& =4 \pi G\left[-u_{j}^{i n}\left(\mathbf{x}^{(n)}, t\right)+U_{j}^{(n)}(t)+(-1)^{j} \Omega_{3}^{(n)}(t) x_{3-j}^{(n)}\right], \\
& \quad \mathbf{x}^{(n)} \in S_{n}, j=1,2, n=1,2, \ldots, N . \tag{15}
\end{align*}
$$

Then the complete formulation of antisymmetric problem includes a system of $4 N$ equations ( $N$ BIEs (14) are joined with the first $3 N$ differential Eq. (13)) for the jumps of normal stresses $\Delta \sigma_{3}^{(n)}(n=1,2, \ldots, N)$ across the inclusions and the parameters $U_{3}^{(n)}, \Omega_{1}^{(n)}, \Omega_{2}^{(n)}(n=1,2, \ldots, N)$ of their transverse motion. The symmetric problem is reduced to a system of $5 N$ equations ( $2 N$ BIEs (15) are joined with the last $3 N$ differential Eq. (13)) for the jumps of tangential stresses $\Delta \sigma_{1}^{(n)}, \Delta \sigma_{2}^{(n)}(n=1,2, \ldots, N)$ across the inclusions and the parameters $U_{1}^{(n)}, U_{2}^{(n)}, \Omega_{3}^{(n)}(n=1,2, \ldots, N)$ of their motion in own plane.

Due to the retardation kernels, the initial moment of inclusion interaction is implicitly exhibited in the derived BIEs, because the zero-values of corresponding integral terms before the arriving in actual inclusion of longitudinal wave scattered by the neighboring objects. Besides, the retardation in the arguments of the ISJs is limited by the time that the transverse wave travels between the most distant points belonging to the set of domains $S_{n}(n=1,2, \ldots, N)$. This result originates from the sharp Huygens principle for such geometrical system of 3-D obstacles. From Eqs. (10) and (13), or (14), (15) and (13) the statements by the BIEs of other 3-D problems on disk-shaped inclusions interaction can be obtained as the particular cases. So, the model of immovable interacting inclusions in the field of nonstationary elastic waves is approached in the limits $M_{n} \rightarrow \infty$. BIE formula-
tions of appropriate quasistatic problems [36] are achieved by the neglecting of inertial terms.

## 3 Regular and discrete analogues of BIEs

For the sake of brevity, let us demonstrate the time-domain solution algorithm for $N$ BIEs (14) corresponding to the antisymmetric problem for coplanar inclusions, when incident impulse wave has nonzero displacement component $u_{3}^{i n}$. Then from the antisymmetry conditions relative to the inclusions plane follows $\Delta \sigma_{j}^{(n)}=0, U_{j}^{(n)}=0, \Omega_{3}^{(n)}=0(j=$ $1,2, n=1,2, \ldots, N)$. Extension of proposed numerical approach to more complicated cases such as the symmetric problems for coplanar inclusions described by $2 N$ BIEs (15) and the general problem for arbitrarily oriented inclusions described by $3 N$ BIEs (10) can be fulfilled by the same scheme, and is connected with more cumbersome mathematical expressions only.

The BIEs (14) contain weakly-singular integrals in the sum terms with the number $k=n$. To isolate these singularities explicitly, the subtraction technique is applied to BIEs (14). This results in

$$
\begin{align*}
& \frac{\left(1+\gamma^{2}\right)}{2} \iint_{S_{n}} \frac{\Delta \sigma_{3}^{(n)}\left(\mathbf{y}^{(n)}, t\right)}{\left|\mathbf{x}^{(n)}-\mathbf{y}^{(n)}\right|} d S_{\mathbf{y}} \\
& \quad+\iint_{S_{n}} \frac{1}{\left|\mathbf{x}^{(n)}-\mathbf{y}^{(n)}\right|}\left\{\mathbf{B}_{1}^{\left|\mathbf{x}^{(n)}-\mathbf{y}^{(n)}\right|}\left[\Delta \sigma_{3}^{(n)}\left(\mathbf{y}^{(n)}, t\right)\right]\right. \\
& \left.\quad-\frac{\left(1+\gamma^{2}\right)}{2} \Delta \sigma_{3}^{(n)}\left(\mathbf{y}^{(n)}, t\right)\right\} d S_{\mathbf{y}} \\
& \quad+\sum_{k=1}^{N}\left(1-\delta_{k n}\right) \iint_{S_{k}} \frac{\mathbf{B}_{1}^{\left|\mathbf{x}^{(k n)}-\mathbf{y}^{(k)}\right|}\left[\Delta \sigma_{3}^{(k)}\left(\mathbf{y}^{(k)}, t\right)\right]}{\mid \mathbf{x}^{(k n)}-\mathbf{y}^{(k) \mid}} d S_{\mathbf{y}} \\
& =4 \pi G\left[-u_{3}^{i n}\left(\mathbf{x}^{(n)}, t\right)+U_{3}^{(n)}(t)-\Omega_{2}^{(n)}(t) x_{1}^{(n)},\right. \\
& \left.\quad+\Omega_{1}^{(n)}(t) x_{2}^{(n)}\right], \mathbf{x}^{(n)} \in S_{n}, n=1,2, \ldots, N \tag{16}
\end{align*}
$$

The second integral in the Eq. (16) is regular, as can be easily proved by analyzing the integrand in the limit $\mathbf{y}^{(n)} \rightarrow \mathbf{x}^{(n)}$. Therefore, from the point of view of numerical evaluation of this integral, it is sufficient to narrow the integration domain up to $S_{n}^{0}$ by elimination of a small region around the source point $\mathbf{x}^{(n)}$ from $S_{n}$. The integrals under the summation sign are ordinary due to the receiver and source points belonging to different inclusions domains. As to the characteristic part or the first integral of Eq. (16), it contains polar peculiarity and its density $\Delta \sigma_{3}^{(n)}$ characterizes the ISJ across the $n$-th inclusion surfaces at the actual time $t$. In accordance with the established structure of solutions of the integral equations with this type of kernels [37,38], the functions $\Delta \sigma_{3}^{(n)}$ for the circular disc-shaped inclusions with the radii $a_{n}(n=$
$1,2, \ldots, N)$ should be presented as

$$
\begin{align*}
\Delta \sigma_{3}^{(n)}\left(\mathbf{x}^{(n)}, t\right)= & \alpha_{3}^{(n)}\left(\mathbf{x}^{(n)}, t\right) / \sqrt{a_{n}^{2}-\left(x_{1}^{(n)}\right)^{2}-\left(x_{2}^{(n)}\right)^{2}}, \\
& \mathbf{x}^{(n)} \in S_{n}, n=1,2, \ldots, N, \tag{17}
\end{align*}
$$

where $\alpha_{3}^{(n)}(n=1,2, \ldots, N)$ are the new unknown functions, which are smooth and limited in the domains $S_{n}$.

Substitution of representations (17) into Eqs. (16) leads to the two types of singularities in the resulting BIEs: the weak singularity at the receiver $\mathbf{x}^{(n)}$ and the "square-root" singularity on the contours of integration or inclusion domains $S_{n}(n=1,2, \ldots, N)$. Above circumstances cause two-stage regularization procedure. First the weakly-singular integrals, which are involved in Eqs. (16) and transformed by Eqs. (17), are interpreted in the sense

$$
\begin{align*}
& \iint_{S_{n}} \frac{\alpha_{3}^{(n)}\left(\mathbf{y}^{(n)}, t\right)}{\sqrt{a_{n}^{2}-\left(y_{1}^{(n)}\right)^{2}-\left(y_{2}^{(n)}\right)^{2}\left|\mathbf{x}^{(n)}-\mathbf{y}^{(n)}\right|} d S_{\mathbf{y}}} \\
& \quad=\pi^{2} \alpha_{3}^{(n)}\left(\mathbf{x}^{(n)}, t\right) \\
& \quad+\iint_{S_{n}^{0}} \frac{\alpha_{3}^{(n)}\left(\mathbf{y}^{(n)}, t\right)-\alpha_{3}^{(n)}\left(\mathbf{x}^{(n)}, t\right)}{\sqrt{a_{n}^{2}-\left(y_{1}^{(n)}\right)^{2}-\left(y_{2}^{(n)}\right)^{2}}\left|\mathbf{x}^{(n)}-\mathbf{y}^{(n)}\right|} d S_{\mathbf{y}}, \\
& \quad \mathbf{x}^{(n)} \in S_{n}, n=1,2, \ldots, N, \tag{18}
\end{align*}
$$

where the following exact values are used [36]:

$$
\begin{align*}
& \iint_{S_{n}} \frac{1}{\sqrt{a_{n}^{2}-\left(y_{1}^{(n)}\right)^{2}-\left(y_{2}^{(n)}\right)^{2}}\left|\mathbf{x}^{(n)}-\mathbf{y}^{(n)}\right|} d S_{\mathbf{y}}=\pi^{2} \\
& \mathbf{x}^{(n)} \in S, n=1,2, \ldots, N . \tag{19}
\end{align*}
$$

Next the variables $\xi^{(n)}\left(\xi_{1}^{(n)}, \xi_{2}^{(n)}\right)$ and $\eta^{(n)}\left(\eta_{1}^{(n)}, \eta_{2}^{(n)}\right)$ instead of variables $\mathbf{x}^{(n)}\left(x_{1}^{(n)}, x_{2}^{(n)}\right)$ and $\mathbf{y}^{(n)}\left(y_{1}^{(n)}, y_{2}^{(n)}\right)$ are introduced so that

$$
\begin{align*}
& \left\{\begin{array}{l}
x_{1}^{(n)}=a_{n} \sin \xi_{1}^{(n)} \cos \xi_{2}^{(n)}, \\
x_{2}^{(n)}=a_{n} \sin \xi_{1}^{(n)} \sin \xi_{2}^{(n)}, \\
\left\{\begin{array}{l}
y_{1}^{(n)}=a_{n} \sin \eta_{1}^{(n)} \cos \eta_{2}^{(n)} ; \\
y_{2}^{(n)}=a_{n} \sin \eta_{1}^{(n)} \sin \eta_{2}^{(n)},
\end{array}\right.
\end{array}\right. \text { n=1,2,..,N.}
\end{align*}
$$

The Jacobian of transformations (20) eliminates the "squareroot" singularity on the contours of inclusion domains, when $\eta_{1}^{(n)}=\pi / 2$. It should be mentioned, that by the change of variables (20) the circular inclusion domain $S_{n}$ is mapped onto mathematical rectangular domain $\tilde{S}_{n}$ : $\left\{0 \leq \xi_{1}^{(n)}, \quad \eta_{1}^{(n)} \leq \pi / 2 ; \quad 0 \leq \xi_{2}^{(n)}, \quad \eta_{2}^{(n)} \leq 2 \pi\right\}$.

Having applied the relations (17)-(20) to the BIEs (16), their regular analogues are obtained. Together with the dif-
ferential Eq. (13) of inclusions motion they form a complete system of $4 N$ equations, which can be written as:

$$
\begin{align*}
& f_{n}\left(\xi^{(n)}\right) \tilde{\alpha}_{3}^{(n)}\left(\xi^{(n)}, t\right) \\
& \quad+\iint_{\tilde{S}_{n}^{0}} \frac{a_{n} \sin \eta_{1}^{(n)}}{w_{n n}\left(\xi^{(n)}, \eta^{(n)}\right)} \mathbf{B}_{1}^{w_{n n}\left(\xi^{(n)}, \eta^{(n)}\right)}\left[\tilde{\alpha}_{3}^{(n)}\left(\eta^{(n)}, t\right)\right] d S_{\eta} \\
& \quad+\sum_{k=1}^{N}(1-\delta k n) \iint_{\tilde{S}_{k}} \frac{a_{k} \sin \eta_{1}^{(k)}}{w_{k n}\left(\xi^{(n)}, \eta^{(k)}\right)} \mathbf{B}_{1}^{w_{k n}\left(\xi^{(n)}, \eta^{(k)}\right)}\left[\tilde{\alpha}_{3}^{(k)}\left(\eta^{(k)}, t\right)\right] d S_{\eta} \\
& \quad-4 \pi G\left[U_{3}^{(n)}(t)+a_{n} \sin \xi_{1}^{(n)} \sin \xi_{2}^{(n)} \Omega_{2}^{(n)}(t)\right. \\
& \left.\quad-a_{n} \sin \xi_{1}^{(n)} \cos \xi_{2}^{(n)} \Omega_{1}^{(n)}(t)\right] \\
& \quad=-4 \pi G \tilde{u}_{3}^{i n}\left(\xi^{(n)}, t\right), \quad \xi^{(n)} \in \tilde{S}_{n}, n=1,2, \ldots, N, \\
& a_{n} \iint_{\tilde{S}_{n}} \tilde{\alpha}_{3}^{(n)}\left(\eta^{(n)}, t\right) \sin \eta_{1}^{(n)} d S_{\eta} \\
& \quad+M_{n} \frac{\mathrm{~d}^{2} U_{3}^{(n)}(t)}{\mathrm{d} t^{2}}=0, n=1,2, \ldots, N, \\
& \iint_{\tilde{S}_{n}} \tilde{\alpha}_{3}^{(n)}\left(\eta^{(n)}, t\right) \sin ^{2} \eta_{1}^{(n)}\left(\delta_{j 1} \sin \eta_{2}^{(n)}-\delta_{j 2} \cos \eta_{2}^{(n)}\right) d S_{\eta} \\
& \quad+\frac{1}{4} M_{n} \frac{\mathrm{~d}^{2} \Omega_{j}^{(n)}(t)}{\mathrm{d} t^{2}}=0, \quad j=1,2, n=1,2, \ldots, N . \tag{21}
\end{align*}
$$

Here the values $r_{1}^{(n)}=r_{2}^{(n)}=a_{n} / 2(n=1,2, \ldots, N)$ of the radii of inertia for the circular disc-shaped massive inclusions having uniform distribution of densities are taken into account, the time-retardation operator $\mathbf{B}_{1}^{d}$ is defined by the relation (11), $\tilde{S}_{n}^{0}$ is the mapping of the domain $S_{n}^{0}$ due to the changing of variables (20) (in the domain $\tilde{S}_{n}^{0}$ the points $\xi^{(n)}$ and $\eta^{(n)}$ do not coincide). Also, $\tilde{\alpha}_{3}^{(n)}\left(\xi^{(n)}, t\right)=\alpha_{3}^{(n)}\left(\mathbf{x}^{(n)}, t\right)$ after substitution of Eqs. (20), similarly for the denotation $\tilde{u}_{3}^{i n}\left(\xi^{(n)}, t\right)$, the function $w_{k n}\left(\xi^{(n)}, \eta^{(k)}\right)$ is the distance between the points $\mathbf{x}^{(k n)}$ and $\mathbf{y}^{(k)}$ in terms of their images. Finally,

$$
\begin{equation*}
f_{n}\left(\xi^{(n)}\right)=\frac{\left(1+\gamma^{2}\right)}{2}\left[\pi^{2}-\iint_{\tilde{S}_{n}^{0}} \frac{a_{n} \sin \eta_{1}^{(n)}}{w_{n n}\left(\xi^{(n)}, \eta^{(n)}\right)} d S_{\eta}\right] . \tag{22}
\end{equation*}
$$

Discrete analogue of a system of the regularized Eq. (21) is constructed by the space collocation method in conjunction with the time-stepping scheme. Since the functions $\tilde{\alpha}_{3}^{(n)}(n=$ $1,2, \ldots, N)$ depend on both the spatial and temporal coordinates, the meshing of domains $\tilde{S}_{n}(n=1,2, \ldots, N)$ and the time interval $[0, T]$ is required. To this end each domain $\tilde{S}_{n}(n=1,2, \ldots, N)$ is divided into $Q_{n}$ rectangular elements $\tilde{S}_{n q}\left(q=1,2, \ldots, Q_{n}, \quad \tilde{S}_{n}=\tilde{S}_{n 1} \cup \tilde{S}_{n 2} \ldots \cup \tilde{S}_{n Q_{n}}\right)$ of the lengths $2 \pi / L_{n}$ and $\pi L_{n} /\left(2 Q_{n}\right)$, where $L_{n}$ is the given mesh parameter for the $n$-th domain, the time interval $[0, T]$ is divided into $K$ equidistant subintervals, i. e.
$t_{r}=r \Delta t(r=1,2, \ldots, K)$ denotes the time station at the $r$ - th time step, $\Delta t$ is the time increment.

Then the unknowns $\tilde{\alpha}_{3}^{(n)}(n=1,2, \ldots, N)$ are approximated by the interpolation functions as
$\tilde{\alpha}_{3}^{(n)}\left(\xi^{(n)}, t\right)=\sum_{q=1}^{P_{n}} \sum_{r=1}^{K} \tilde{\alpha}_{3 q r}^{(n)} \theta_{q}^{(n)}\left(\xi^{(n)}\right) \vartheta_{r}(t), n=1,2, \ldots, N$,
where $\tilde{\alpha}_{3 q r}^{(n)}$ is the value of the function $\tilde{\alpha}_{3}^{(n)}$ at the nodal point $\xi_{q}^{(n)}\left(\xi_{1 q}^{(n)}, \xi_{2 q}^{(n)}\right)$ on the $n$-th domain mesh at a time $t_{r}=r \Delta t, P_{n}$ is the total number of nodal points introduced in the domain $\tilde{S}_{n}, \theta_{q}^{(n)}$ and $\vartheta_{r}$ are the given spatial and temporal shape functions, respectively, with the properties $\theta_{q}^{(n)}\left(\xi_{i}^{(n)}\right)=\delta_{q i}, \vartheta_{r}\left(t_{j}\right)=\delta_{r j}$.

The remaining unknown functions, namely translation and rotation parameters for the inclusions $U_{3}^{(n)}, \quad \Omega_{j}^{(n)}(j=$ $1,2, n=1,2, \ldots, N)$, depend on the temporal coordinate only. Let $U_{3 r}^{(n)}, \quad \Omega_{j r}^{(n)}$ represent these functions, respectively, at the time $t_{r}$. The backward difference schemes are applied to approximate their accelerations, i.e.:

$$
\begin{align*}
\left.\frac{\mathrm{d}^{2} U_{3}^{(n)}}{\mathrm{d} t^{2}}\right|_{t=t_{r}}= & \frac{1}{(\Delta t)^{2}}\left[U_{3 r}^{(n)}-2 U_{3(r-1)}^{(n)}+U_{3(r-2)}^{(n)}\right], \\
& n=1,2, \ldots, N \\
\left.\frac{\mathrm{~d}^{2} \Omega_{j}^{(n)}}{\mathrm{d} t^{2}}\right|_{t=t_{r}}= & \frac{1}{(\Delta t)^{2}}\left[\Omega_{j r}^{(n)}-2 \Omega_{j(r-1)}^{(n)}+\Omega_{j(r-2)}^{(n)}\right], \\
& j=1,2, n=1,2, \ldots, N . \tag{24}
\end{align*}
$$

By substituting Eqs. (23) and (24) into Eqs. (21), considered at each collocation point $\xi_{q}^{(n)}\left(q=1,2, \ldots, P_{n}, n=\right.$ $1,2, \ldots, N)$ and at each time-step, we arrive at a system of $3 N+P_{1}+P_{2}+\ldots+P_{N}$ linear algebraic equations, which is recurrent relative to the time index $r$ and has the form

$$
\begin{align*}
& \sum_{i=1}^{P_{n}}\left[h_{q i r r}^{(n)} \tilde{\alpha}_{3 i r}^{(n)}+\sum_{k=1}^{N}\left(1-\delta_{k n}\right) b_{q i r r}^{(k n)} \tilde{\alpha}_{3 i r}^{(k)}\right] \\
& \quad-4 \pi G\left[U_{3 r}^{(n)}+a_{n} \sin \xi_{1 q} \sin \xi_{2 q} \Omega_{1 r}^{(n)}-a_{n} \sin \xi_{1 q} \cos \xi_{2 q} \Omega_{2 r}^{(n)}\right] \\
& = \\
& \quad-4 \pi G \tilde{u}_{3}^{i n}\left(\xi_{q}^{(n)}, t_{r}\right) \\
& \quad-\sum_{i=1}^{P_{n}} \sum_{l=1}^{r-1}\left[h_{q i r l}^{(n)} \tilde{\alpha}_{3 i l}^{(n)}+\sum_{k=1}^{N}\left(1-\delta \delta_{k n}\right) b_{q i r l}^{(k n)} \tilde{\alpha}_{3 i l}^{(k)}\right], \\
& \quad q=1,2, \ldots, P_{n}, n=1,2, \ldots, N, r=1,2, \ldots, K, \\
& \sum_{i=1}^{P_{n}} c_{i}^{(n)} \tilde{\alpha}_{3 i r}^{(n)}+\frac{M_{n}}{a_{n}(\Delta t)^{2}} U_{3 r}^{(n)}=\frac{M_{n}}{a_{n}(\Delta t)^{2}}\left(2 U_{3(r-1)}^{(n)}-U_{3(r-2)}^{(n)}\right), \\
& n=1,2, \ldots, N, r=1,2, \ldots, K, \\
& \sum_{i=1}^{P_{n}} c_{j i}^{(n)} \tilde{\alpha}_{3 i r}^{(n)}+\frac{M_{n}}{4(\Delta t)^{2}} \Omega_{j r}^{(n)}=\frac{M_{n}}{4(\Delta t)^{2}}\left(2 \Omega_{j(r-1)}^{(n)}-\Omega_{j(r-2)}^{(n)}\right),  \tag{25}\\
& j=1,2, n=1,2, \ldots, N, r=1,2, \ldots, K .
\end{align*}
$$

Here the coefficients $h_{q i r l}^{(n)}, b_{q i r l}^{(k n)}, c_{i}^{(n)}, c_{j i}^{(n)}$ are given by the formulas

$$
\begin{align*}
h_{q i r l}^{(n)}= & f_{3}\left(\xi_{q}^{(n)}\right) \delta_{q i} \delta \delta_{r l}+a_{n} \iint_{\tilde{S}_{n q}^{0}} \frac{\sin \eta_{1}^{(n)}}{w_{n n}\left(\xi_{q}^{(n)}, \eta^{(n)}\right)} \\
& \left.\theta_{i}^{(n)}\left(\eta^{(n)}\right)\left\{\mathbf{B}_{1}^{w_{n n}\left(\xi_{q}^{(n)}, \eta^{(n)}\right)}\left[\vartheta_{l}(t)\right]\right\}\right|_{t=t_{r}} d S_{\eta}, \\
b_{q i r l}^{(k n)}= & a_{k} \iint \frac{\sin \eta_{1}^{(k)}}{w_{k n}\left(\xi_{q}^{(n)}, \eta^{(k)}\right)} \\
c_{i}^{(n)}= & \iint_{\tilde{S}_{k}} \theta_{i}^{(n)}\left(\eta^{(n)}\right) \sin \eta_{1}^{(n)} d S_{\eta}, \\
c_{j i}^{(n)}= & \iint_{\tilde{S}_{n}} \theta_{i}^{(n)}\left(\eta^{(n)}\right) \sin ^{2} \eta_{1}^{(n)}\left(\delta_{1 j} \sin \eta_{2}^{(n)}-\delta_{2 j} \cos \eta_{2}^{(n)}\right) d S_{\eta}^{(n)} .
\end{align*}
$$

In relations (26) the doubly-connected domain of integration $\tilde{S}_{n q}^{0}$ is defined by the elimination from domain $\tilde{S}_{n}$ of a small surrounding the collocation point $\xi_{q}^{(n)}$.

Thus, the problem on the time variations of functions $\tilde{\alpha}_{3}^{(n)}$ and $U_{3}^{(n)}, \quad \Omega_{1}^{(n)}, \quad \Omega_{2}^{(n)}(n=1,2, \ldots, N)$ characterizing the "matrix-inclusions" loading transfer and motion of the inclusions, respectively, is reduced to a step by step solution of the recurrent system of Eq. (25). Then the solutions at time step $r=1$ are used further for the computation of above functions at time step $r=2$, etc. The values within the first step involve also the initial conditions. It should be mentioned the special computational properties of influence matrices of a system (25) caused by the presence of retarded-type operators in the coefficients (26) or physically by the traveling nature of generated wave fields. For instance, the matrix on the left side of the first $P_{1}+P_{2}+\ldots+P_{N}$ Eq. (25) is well-conditioned with the domination of diagonal elements due to the regularization (18), besides it has sparse form due to the zero-elements with the indexes $q$ and $i$ for which $w_{k n}\left(\xi_{q}^{(n)}, \eta_{i}^{(k)}\right)>c_{1} \Delta t$. Concerning the matrices on the right side of these equations, their elements $h_{\text {qirl }}^{(n)}=0, b_{\text {qirl }}^{(k n)}=0$ with the indexes $r$ and $l$ for which $t_{r}-D / c_{2} \geq t_{l+1}$, where $D$ is the distance $w_{k n}\left(\xi_{q}^{(n)}, \eta_{i}^{(k)}\right)$ between the most distant nodal points $\xi_{q}^{(n)}$ and $\eta_{i}^{(k)}$ on the set of domains $\tilde{S}_{n}(n=1,2, \ldots, N)$. It means the stabilization of computing time and memory in the marching algorithm after the $p$-th time step, which is defined by $p \Delta t c_{2}>D>(p-1) \Delta t c_{2}$. As the consequence, the errors are not accumulated when long duration wave excitations of the system are considered.

In the further numerical analysis piecewise-constant spatial and linear temporal approximations of unknowns are
used, what is provided by the spline-type shape functions

$$
\begin{align*}
& \theta_{q}^{(n)}\left(\xi^{(n)}\right)=\left\{\begin{array}{ll}
1, & \xi^{(n)} \in \tilde{S}_{n q} ; \\
0, & \xi^{(n)} \notin \tilde{S}_{n q},
\end{array} \quad n=1,2, \ldots, N,\right. \\
& \vartheta_{r}(t)= \begin{cases}1-|t-r \Delta t| / \Delta t, & |t-r \Delta t| \leq \Delta t ; \\
0, & |t-r \Delta t|>\Delta t .\end{cases} \tag{27}
\end{align*}
$$

Then the value $\tilde{\alpha}_{3 q r}$ represents the unknown function at the nodal point $\xi_{q}^{(n)}$ in the geometrical center of $q$-th element, the total number of nodal or collocation points in Eqs. (25) is equal to $Q_{1}+Q_{2}+\ldots+Q_{N}$, and the domain $\tilde{S}_{n q}^{0}$ with the eliminated surrounding of $\tilde{S}_{q}^{(n)}$-point is chosen as $\tilde{S}_{n q}^{0}=\tilde{S}_{n} \backslash \tilde{S}_{n q}$. The assumptions (27) lead to analytical evaluation of time convolution integrals (see Appendix 1) existing in the coefficients $h_{\text {qirl }}^{(n)}$ and $b_{q i r l}^{(k n)}$ due to the operator $\mathbf{B}_{1}^{R}$ (11) and standard Gaussian numerical evaluation of space integrals in these coefficients, analytical determination of the coefficients $c_{j i}^{(n)}$ and $c_{j i}^{(n)}$ (see Appendix 2), and enable the calculation of the matrix with the singular elements one time only during the first time step, because the coefficients $h_{\text {qirr }}^{(n)}$ and $b_{q i r r}^{(k n)}$ became not dependent on the time index $r$. Together with the subelement technique for the more accurate satisfying the causality or zero-values conditions ahead of the wave fronts by the subdivision of source boundary element onto active and inactive parts with respect to their influence on the receiver nodal point during actual time step, these improvements provide the stable and effective calculations of time-dependencies for the relatively small time increment $\Delta t$ [15].

In addition to the above parameters, important characteristics of stresses in the inclusion vicinities are stress intensity factors (SIFs) [39]. Once the time-dependent functions $\tilde{\alpha}_{3}^{(n)}(n=1,2, \ldots, N)$ are determined by solving Eqs. (25), the dynamic SIFs can be easily obtained without using special boundary elements at the inclusion fronts. It follows from the proper representation of the ISJs in the form (17) and (22). In the considered antisymmetric problem only the mode-II dynamic SIFs are non-zero. These characterize the normal (tension or compression) stresses in the inclusion domains or the shear stresses in the complementary domain in the plane $x_{3}^{(n)}=0$. The mode-II dynamic SIF $K_{2}^{(n)}$ for the $n$-th inclusion can be computed by the relation:

$$
K_{2}^{(n)}\left(\varphi^{(n)}, t\right)=-\left.\frac{1}{4(1-v) \sqrt{a_{n}}} \tilde{\alpha}_{3}^{(n)}\left(\xi^{(n)}, t\right)\right|_{\xi_{1}^{(n)}}=\pi / 2 ;,
$$

where $\varphi^{(n)}$ is the angular coordinate of the point at the front of $n$-th inclusion accounting from the axis $O^{(n)} x_{1}^{(n)}$.


Fig. 2 Schematic configuration and motion of two interacting discshaped inclusions as the rigid units under nonstationary elastic wave incidence

## 4 Numerical results

As demonstrative example of the application of described time-domain BEM, the transient antisymmetric problem for the pair of coplanar rigid disc-shaped inclusions of the same radius $a_{1}=a_{2}=a$, which centers are located on the distance $2 a+f$ with $f$ as the gap between the inclusions (see Fig. 2) in 3-D elastic matrix is considered. The inclusions of both equal and different masses $M_{1}$ and $M_{2}$ are involved into analysis, the latter case can be caused by their different material densities and thicknesses or aspect ratios. It is assumed perpendicular incidence of plane longitudinal elastic wave on the inclusions, and two types of incident wave scenario are foreseen to reach in the limits of physically expected and early investigated situations for the results verification, and to show more clearly the inertial effects of obstacles interaction. First type of wave has step-like time profile with the following displacement distribution:

$$
\begin{align*}
& u_{3}^{i n}\left(\mathbf{x}^{(n)}, t\right) \\
& = \begin{cases}U_{0} \frac{\left(c_{1} t-x_{3}^{(n)}\right)^{2}}{c_{1}^{2} t_{*}^{2}} \exp \left(-\frac{2\left(c_{1} t-x_{3}^{(n)}\right)}{c_{1} t_{*}}\right), & c_{1} t-x_{3}^{(n)} \leq c_{1} t_{*} ; \\
U_{0} \exp (-2), & c_{1} t-x_{3}^{(n)}>c_{1} t_{*},\end{cases} \tag{29}
\end{align*}
$$

where $U_{0}$ is constant magnitude coefficient, $t_{*}=2 a / c_{2}$ is the time when the pulse enters the stationary regime. Second one is characterized by the hill-like profile with the same magnitude and is given by the displacement
$u_{3}^{i n}\left(\mathbf{x}^{(n)}, t\right)=U_{0} \frac{\left(c_{1} t-x_{3}^{(n)}\right)^{2}}{c_{1}^{2} t_{*}^{2}} \exp \left(-\frac{2\left(c_{1} t-x_{3}^{(n)}\right)}{c_{1} t_{*}}\right)$.

The normalized displacements $\bar{u}^{i n}=u_{3}^{i n} / U_{0}$ in the inclusions plane $x_{3}^{(1)}=x_{3}^{(2)}=0$ for the above incident waves are plotted in Fig. 3. These waves arrive to both inclusions at the same time $t=0$ which is fixed as the initial moment.

If the axes $O^{(1)} x_{2}^{(1)}$ and $O^{(2)} x_{2}^{(2)}$ are chosen so that they coincide with the line connecting the centers of inclusions, natural symmetry of the problem concerning the plane


Fig. 3 Time profiles of normalized displacements $\bar{u}^{i n}$ of the incident wave
$x_{1}^{(1)}=x_{1}^{(2)}=0$ takes place for the considered geometrical configuration and disturbing field. It leads to zero rotation angles $\Omega_{2}^{(1)}=\Omega_{2}^{(2)}=0$ and description of inclusions kinetics by the translations $U_{3}^{(n)}$ and rotations $\Omega_{1}^{(n)}(n=1,2)$ only, as displayed in Fig. 2. Concerning the SIFs $K_{2}^{(n)}(n=1,2)$, due to the inclusions interaction they change from point to point around the inclusions fronts, but as the consequence of problem symmetry is $K_{2}^{(1)}\left(\varphi^{(1)}, t\right)=K_{2}^{(1)}\left(2 \pi-\varphi^{(1)}, t\right)$ and $K_{2}^{(2)}\left(\varphi^{(2)}, t\right)=K_{2}^{(2)}\left(2 \pi-\varphi^{(2)}, t\right)$. Besides, in the particular case $M_{1}=M_{2}$ both inclusions are under the same conditions, and the subsequent parametrical simplification occurs as the equalities $U_{3}^{(1)}=U_{3}^{(2)}, \Omega_{2}^{(1)}=\Omega_{2}^{(2)}, K_{2}^{(1)}\left(\varphi^{(1)}, t\right)=$ $K_{2}^{(2)}\left(\pi-\varphi^{(2)}, t\right)$.

For the calculation 176 rectangular boundary elements with the lengths $\pi / 22$ and $\pi / 12$ are used on each domain $\tilde{S}_{1}$ and $\tilde{S}_{2}$, the subelement mesh is formed by $5 \times 5$ subdivision, the time increment is chosen as $\Delta t=0.16 a / c_{2}$. Poisson's ratio of matrix material is taken as $v=0.3$. The time-dependences of normalized translation $\bar{U}=$ $U_{3}^{(1)} / U_{0}$, rotation $\bar{\Omega}=\Omega_{2}^{(1)} a / U_{0}$ and dynamic SIF $\bar{K}=$ $(1-\nu) \sqrt{a} K_{2}^{(1)} /\left(\pi G U_{0}\right)$ of one of the inclusions are depicted on the Figs. 4, 5, 6, 7, 8, 9, 10 and 11. In addition the displayed dynamic SIF $\bar{K}$ corresponds to the inclusion front point $A$ nearest to the other inclusion (see Figure 2) as the most sensitive point relative to the interaction effects. Then the generality of the analysis is provided by the inverse character of results as to the neighboring object, and considering the presence near the actual inclusion with the dimensionless mass $\bar{M}_{1}=M_{1} /\left(\rho a^{3}\right)$ of the other inclusion with smaller, equal or larger dimensionless mass $\bar{M}_{2}=M_{2} /\left(\rho a^{3}\right)$. For comparison purposes, the behavior of pertinent parameters for a single inclusion subjected to the same transient excitation, when the interaction between


Fig. 4 Histories of normalized dynamic SIF $\bar{K}$ of immovable inclusions for various gaps $f$ between them under incident wave with steplike profile. Points denoted by stars correspond to the known analytical static solutions


Fig. 5 Histories of normalized dynamic SIF $\bar{K}$ of inclusion with fixed mass $\bar{M}_{1}$ for various masses $\bar{M}_{2}$ of neighboring inclusion under incident wave with step-like profile
the objects is neglected, is also showed in the Figs. 4, 5, 6, $7,8,9,10$ and 11 .

Two approaches are applied for the validation of proposed numerical method. Since no results for transient disturbing functions for the multiple disc-shaped inclusions embedded in an elastic matrix are reported in the literature, first the static equivalents of the problems are considered as the particular cases, where the analytical solutions exist. These static limits are achieved under the incidence on the immovable inclusions of step-like wave for large time values (see Fig. 4), and correspond to the problems on the response of immovable inclusions on the primary field of constant displacements in the matrix (or the same on the constant vertical


Fig. 6 Histories of normalized translation $\bar{U}$ of the inclusion with fixed mass $\bar{M}_{1}$ for various masses $\bar{M}_{2}$ of neighboring inclusion under incident wave with hill-like profile


Fig. 7 Histories of normalized rotation angle $\bar{\Omega}$ of the inclusion with fixed mass $\bar{M}_{1}$ for various masses $\bar{M}_{2}$ of neighboring inclusion under incidence wave with hill-like profile
translations of inclusions in the matrix). Then exact solution for a single inclusion [39] and approximate solution in a form of series by the small geometrical parameter for two remote inclusions [36] are available. Excellent agreement of numerical dynamic solutions and analytical static solutions is found both for single and multiple inclusions situations. On the other hand, it is anticipated that convergence of the proposed solution depends on both space and time discretization meshes. In the previous direct time-domain BEM analyses [10,20], the marching algorithm instability at small time increment relative to the normalized size or area of the selected boundary elements have been mentioned and explained. With this reason, the scheme with the gradual decreasing the time step at the fixed and sufficiently fine boundary element mesh, and the achievement of deviations


Fig. 8 Histories of normalized dynamic SIF $\bar{K}$ of inclusion with fixed mass $\bar{M}_{1}$ for various masses $\bar{M}_{2}$ of neighboring inclusion under incident wave with hill-like profile


Fig. 9 Histories of normalized translation $\bar{U}$ of the interacting inclusions with equal masses for various gaps $f$ between them under incident wave with hill-like profile
between the results for two consecutive attempts less than $1 \%$ for the considered time interval and involved input data as the criterion of the process completion is used. A number of parametric studies within this convergence estimation tactics show that the desired accuracy of results and the computational efficiency is provided with the introduced above time step.

The kinematic interaction of "matrix-multiple inclusions" system is exhibited by the oscillations in time of inclusions motion or matrix compliance parameters (Figs. 6, 7, 9, 10) and matrix stress intensification parameters (Figs. 4, 5, 8, 11) with the peaks attenuation and reaching the static values at large time. In accordance with the causality principle, before arriving the longitudinal wave scattered by the neighboring


Fig. 10 Histories of normalized rotation angle $\bar{\Omega}$ of the interacting inclusions with equal masses for various gaps $f$ between them under incident wave with hill-like profile


Fig. 11 Histories of normalized dynamic SIF $\bar{K}$ of interacting inclusions with equal masses for various gaps $f$ between them under incident wave with hill-like profile
inclusion, namely in the interval $0 \leq t \leq f / c_{1}$, the histories for the single inclusion and pair of inclusions coincide. Obviously, this interval is longer for the most distant objects. The change of sign in the involved parameters is observed for the movable inclusions. It means the opposite transitions and rotations of inclusions as to their initial positions in the plane $x_{3}^{(1)}=x_{3}^{(2)}=0$, as well as arising both tensile and compressional stresses in the inclusions domains, depending on the time.

From the comparison of curves behavior on the Figs. 4 and 5 for the sophisticated unmovable inclusions and realistic movable ones in the field of step-like incident wave follows that the SIF amplitudes are much less in the second case. But for this transient excitation the difference between the SIFs for single and interacting inclusions is not considerable (the
same is for the characteristics of inclusions motion), therefore subsequent illustrations concern in details the response of inclusions on the more abrupt incident wave with the hilllike profile.

First the time-dependencies of vertical translations (Fig. 6), rotations (Fig. 7) and SIF (Fig. 8) of the inclusions are revealed upon their mass ratio. The inhomogeneities are located at close distance with the gap $f=0.2 a$. The dimensionless mass of actual inclusion is selected as $\bar{M}_{1}=12$, and three masses of associated inclusion are considered: $\bar{M}_{2}=6 ; 12 ; 18$. Mutual influence of rigid disc-shaped inclusions causes decreasing the peaks of their translation $\bar{U}$ in comparison with the single inclusion case, in addition these peaks are smaller and reached later in time under neighborhood of actual inclusion with the more massive inclusion. Opposite tendency takes place for the peaks of rotation angle $\bar{\Omega}$ of ones. However, this rotational degree of freedom leads to the more smooth behavior in time and less peaks of dynamic SIF, hence movable inclusions cumulatively demonstrate reinforcing properties.

Next above histories are examined upon the distance between the inclusions, which is chosen as $f=0.2 a ; 1.2 a$; 2.5a. The dimensionless masses of inclusions are assumed the same and equal to $\bar{M}_{1}=\bar{M}_{2}=20$. Unlike the inclusion rotations on Fig. 10, where their maximums are smaller for the bigger gap between the interacting object, non-unique inertia effects are observed for the translations (Fig. 9) and dynamic SIF (Fig. 11) of the inclusions. So, the peak values of these quantities can both increase and decrease with increasing the gap $f$, furthermore the SIF peaks overshooting as to the single inclusion configuration is fixed at some gaps. As the inclusion-inclusion distance further increases, the $\bar{U}$ - and $\bar{K}$-curves approach to the corresponding curves for a single inclusion and the $\bar{\Omega}$-values approach to zero, which indicates that the inclusions interaction can be neglected.

The examples with a few nonstationary disturbed discshaped inclusions in an elastic matrix give the answers concerning the histories of local or close field parameters in the corresponding particulate composites. Then described above numerical scheme can be effectively realized by the desktop computational possibilities within the reasonable CPU time (in the considered cases of two inclusions CPU time was about 70 seconds for the involved meshes and time interval). Obviously, analysis of wave transmission in such composites demands taking into account the interaction of many inclusions. Then fast multipole algorithms can be incorporated into the step by step scheme to avoid the difficulties connected with the inevitable large matrix problems. Also some modifications in marching procedure are needed when simulating rapidly changing transient processes, when small time increment should be used to provide the accurate results.

## 5 Conclusions

Time-domain boundary integral formulation of three-dimensional elastodynamic problems of elastic waves propagation in a matrix containing system of rigid disc-shaped movable inclusions is developed. Within this main task the several subtasks are fulfilled.

1. Basing on superposition principle and integral representations of displacements due to the contribution of each inclusion, the BIEs with weakly-singular retarded-type kernels are derived for the general inclusion shapes and locations. In the obtained BIEs, the ISJs across the inclusions as function of time are unknown integral densities, while translations and rotations of inclusions are presented as the free terms and satisfy associated equations of motion for a rigid unit. Time retardation in an argument of unknown functions is limited by the period of traveling of transverse elastic wave between the most distant points belonged to the considered conglomerate of inclusions.
2. Proposed time-stepping/collocation approach to the numerical solution of BIEs differs from the classical schemes by multiplicative extraction from the solution of smooth function by accounting implicitly of solution behavior near the inclusion edges. Besides the smooth properties, these new functions yield direct determination of dynamic stress intensity factors in the inclusion vicinities.
3. Also improved regularization procedures are applied $a$ priori the discretization, which foresee analytical evaluation of regularizing integrals and not need binding the singular point with the local polar coordinate system.
4. Basing on linear temporal and constant space approximations of the solution, the analytical values of convolution integrals as well as subelement technique for the calculation of integrals over "active" part of the source boundary elements are incorporated into the numerical scheme.

All these factors increase the solution accuracy and calculation efficiency, in particular suggest the savings in computer memory and time under extension of actual time diapason, and allow the computing long duration transient processes without numerical error accumulation.

Observed numerically inertial effects in the system "mat-rix-disc-shaped rigid inclusions" can be generalized as the following.

1. A pair of coplanar inclusions subjected to transient elastic wave incidence is distinguished by the damping effects of inclusion interaction, especially in the cases of closely located objects. They lie in the reduction of translation magnitudes in the system of massive inclusions as the
rigid units in comparison with the single inclusion configuration.
2. Movability of inclusions, in particular their rotations due to the mutual influence, yields also the peaks reducing and more monotonic character in the time dependencies of dynamic SIF in the inclusions vicinities.

Noted phenomena confirm that the improved both stiffness and strength dynamic properties of composite materials can be achieved by using rigid thin-walled inclusions as the structural elements.

Since the proposed algorithm is completely formulated in the time-domain, it forms a basis for generalization of time-domain BEM analysis of disk-shaped rigid inclusions interaction from three-dimensional isotropic to anisotropic and piezoelectric matrixes with involving appropriate fundamental solutions [40,41], in two-dimensional case of cracked solids this possibility have been demonstrated earlier [42,43]. Extension of the time-domain BEM analysis on multiple thinwalled elastic inclusions can be also provided by using spring contact models [35] for such inclusions.

Acknowledgments The research was partially supported by the Scientific and Technology Center in Ukraine (STCU) and National Academy of Sciences of Ukraine (Project No. 5726), the work of V. V. Mykhas'kiv was supported by the Fulbright Foundation within Scholar Visiting Program 2012-2013.

## Appendix 1: Analytical values of retarded-type operator acting on linear shape function

Evaluation of operator $\mathbf{B}_{1}^{R}$ involved in Eq. (26) with the shape function (27) gives

$$
\begin{equation*}
\left.\mathbf{B}_{1}^{R}\left[\vartheta_{l}(t)\right]\right|_{t=t_{r}=r \Delta t}=B_{1}^{r-l+1}+B_{2}^{r-l}, \tag{31}
\end{equation*}
$$

where after the temporal integration different expressions take place for the coefficients $B_{1}^{H}$ and $B_{2}^{H}(H=r-l+$ $1, r-l$ ) depending on the distance $R$. With the denotation $E=H-1$ they are:

Case I: $R<E c_{2} \Delta t$ or $R>H c_{1} \Delta t$, then

$$
\begin{equation*}
B_{1}^{H}=B_{2}^{H}=0 ; \tag{32}
\end{equation*}
$$

Case II: $E c_{2} \Delta t<R<H c_{2} \Delta t$ and $R<E c_{1} \Delta t$, then

$$
\begin{align*}
B_{1}^{H} & =\frac{1}{2} H-\frac{2}{3} \frac{R}{\Delta t c_{2}}+\frac{1}{2} \frac{(\Delta t)^{2} c_{2}^{2}}{R^{2}} E^{2}\left(1+\frac{1}{3} E\right), \\
B_{2}^{H} & =-\frac{1}{2} E+\frac{2}{3} \frac{R}{\Delta t c_{2}}-\frac{1}{6} \frac{(\Delta t)^{2} c_{2}^{2}}{R^{2}} E^{3} \tag{33}
\end{align*}
$$

Case III: $E c_{1} \Delta t<R<H c_{2} \Delta t$, then

$$
\begin{align*}
& B_{1}^{H}=\frac{1}{2} H-\frac{2}{3} \frac{R}{\Delta t c_{2}}+\frac{1}{2} \gamma^{2}\left(H-\frac{2}{3} \frac{R}{\Delta t c_{1}}\right) ; \\
& B_{2}^{H}=-\frac{1}{2} E+\frac{2}{3} \frac{R}{\Delta t c_{2}}-\frac{1}{2} \gamma^{2}\left(E-\frac{2}{3} \frac{R}{\Delta t c_{1}}\right) ; \tag{34}
\end{align*}
$$

Case IV: $E c_{1} \Delta t<R<H c_{1} \Delta t$ and $R>H c_{2} \Delta t$, then

$$
\begin{align*}
B_{1}^{H}= & -\frac{1}{2} \gamma^{2}\left(H-\frac{2}{3} \frac{R}{\Delta t c_{1}}\right)-\frac{1}{6} H^{3} \frac{(\Delta t)^{2} c_{2}^{2}}{R^{2}} \\
B_{2}^{H}= & -\frac{1}{2} \gamma^{2}\left(E-\frac{2}{3} \frac{R}{\Delta t c_{1}}\right) \\
& -\frac{1}{2} \frac{(\Delta t)^{2} c_{2}^{2}}{R^{2}} H^{2}\left(1-\frac{1}{3} H\right) \tag{35}
\end{align*}
$$

Case V: $H c_{2} \Delta t<R<E c_{1} \Delta t$, then

$$
\begin{align*}
B_{1}^{H} & =-\frac{1}{6} \frac{(\Delta t)^{2} c_{2}^{2}}{R^{2}}(3 E+1) \\
B_{2}^{H} & =-\frac{1}{6} \frac{(\Delta t)^{2} c_{2}^{2}}{R^{2}}(E+2 H) \tag{36}
\end{align*}
$$

## Appendix 2: Coefficients of resulting system of linear algebraic equations for piecewise-constant spatial approximation

From Eqs. (26) and (27) it follows

$$
\begin{align*}
& c_{i}^{(n)}=\iint_{\tilde{S}_{n i}} \sin \eta_{1}^{(n)} d S_{\eta}, \\
& c_{j i}^{(n)}=\iint_{\tilde{S}_{n i}} \sin ^{2} \eta_{1}^{(n)}\left(\delta_{1 j} \sin \eta_{2}^{(n)}-\delta_{2 j} \cos \eta_{2}^{(n)}\right) d S_{\eta}, \\
& \tilde{S}_{n i}:\left\{\pi(m-1) L_{n} /\left(2 Q_{n}\right) \leq \eta_{1}^{(n)} \leq \pi m L_{n} /\left(2 Q_{n}\right) ;\right. \\
& \left.\quad 2 \pi(l-1) / L_{n} \leq \eta_{2}^{(n)} \leq 2 \pi l / L_{n}\right\}, \\
& m=1,2, \ldots, Q_{n} / L_{n} ; \quad n=1,2, \ldots, L_{n} ; \\
& \quad i=(m-1) L_{n}+l . \tag{37}
\end{align*}
$$

After integration we obtain

$$
\begin{align*}
c_{i}^{(n)}= & \frac{2 \pi}{L_{n}}\left\{\cos \left[\pi(m-1) L_{n} /\left(2 Q_{n}\right)\right]-\cos \left[\pi m L_{n} /\left(2 Q_{n}\right)\right]\right\}, \\
c_{j i}^{(n)}= & \frac{1}{4} \delta_{1 j}\left\{\cos \left[2 \pi(l-1) / L_{n}\right]\right. \\
& \left.-\cos \left[2 \pi l / L_{n}\right]\right\}\left\{\sin \left[\pi(m-1) L_{n} / Q_{n}\right]\right. \\
& \left.-\sin \left[\pi m L_{n} / Q_{n}\right]+\pi L_{n} / Q_{n}\right\} \\
& +\frac{1}{4} \delta_{2 j}\left\{\sin \left[2 \pi(l-1) / L_{n}\right]\right. \\
& \left.-\sin \left[2 \pi l / L_{n}\right]\right\}\left\{\sin \left[\pi(m-1) L_{n} / Q_{n}\right]\right. \\
& \left.-\sin \left[\pi m L_{n} / Q_{n}\right]+\pi L_{n} / Q_{n}\right\} . \tag{38}
\end{align*}
$$

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