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On acoustic and electric Faraday cages

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Two-dimensional problems involving many identical small circles are considered; the circles are the cross sections of parallel wires, modelling a cage or a grating. Both electrostatic and acoustic fields are considered. The main emphasis is on periodic configurations of N circles distributed evenly around a large circle (a ring). Foldy's theory is used for acoustic problems and then adapted for electrostatic problems. In both cases, circulant matrices are encountered: the fields can be calculated explicitly. Then, the limit $N \rightarrow \infty$ is studied. A connection between the *N*-circle problem and the limiting problem (fields exterior to the ring) is established, using known results on the convergence of a defective form of the trapezoidal rule, defective in that endpoint contributions are ignored, because the integrand has logarithmic singularities at those points. This shows that the solution of the limiting problem is approached very slowly, as $N^{-1} \log N$ as $N \to \infty$.

1. Introduction

Faraday's cage, first demonstrated in 1836, consisted of a room covered with metal foil: inhabitants are safe from external electrical discharges. The metal can have small holes or gaps, but then the protection is no longer perfect. We are interested in calculating the fields inside and outside the cage. We model the cage as a ring of thin parallel wires, and thus obtain a two-dimensional problem. We consider both electrostatic fields (governed by Laplace's equation) and acoustic waves (governed by the Helmholtz equation). The wires are assumed to be identical with circular cross sections, small compared with the ring diameter and the wavelength (for acoustic problems).

Problems involving many small identical circles have a long history. For an infinite periodic row of circles, we obtain a grating. The electrostatic problem, with an



infinite row of identical line charges, can be found in Maxwell's famous *Treatise* [1]. For discussions, see [2], [3, pp. 291–298], [4, p. 1236] and [5,6, $\S45\alpha$].

For the problem of scattering by an infinite periodic row of identical scatterers, we mention papers by Lamb [7], Ignatowsky [8] and Gans [9]. Larsen [10] gave a useful survey of the early work. Some of this work assumes that the scatterers are small, so that they are represented as line sources (wires), and we shall do the same: the main reason for doing this is that we are interested in other configurations of scatterers, and we are interested in configurations with many scatterers.

In fact, the problem of scattering by N circular objects can be solved exactly, using a method that combines separated solutions of the governing two-dimensional Helmholtz equation with appropriate addition theorems. The result is an infinite system of linear algebraic equations, a system that may be truncated and solved numerically. This method is described in detail in §4.5 of [11]; numerous references are also given there. Some authors have given results for scattering by N identical circles arranged with their centres equally spaced around a large circle [12–14].

We start in §2 with a short discussion of two static problems, with discrete charges along a straight line or around a circle. The charge strengths are specified, and the field can be calculated exactly, as is well known. For the infinite periodic row (§2*a*), the exponential decay of the field with distance from the row is notable. Moreover, the charge strengths can be adjusted, so that perfectly conducting wires are modelled. Analogous acoustic problems are discussed in §3.

In order to handle problems where there is a specified external field or a given incident wave, we need a method for calculating the charge or source strengths. In acoustics, such a method was given by Foldy [15]. It is a self-consistent method in which each scatterer scatters isotropically; this method, which is appropriate for sound-soft (Dirichlet) scatterers, is summarized in §4. The main application is to a ring (radius *b*) of *N* small scatterers (§5). Foldy's method leads to an $N \times N$ linear algebraic system. For equally spaced scatterers, the system matrix is a circulant matrix: an explicit solution follows. To understand the limiting behaviour as $N \to \infty$, a theorem of Sidi [16] is used; in effect, we encounter a defective trapezoidal rule with missing endpoint contributions. This reveals that the expected convergence to the solution of scattering by a large sound-soft circle of radius *b* is demonstrated, but the rate of convergence is very slow (as $N^{-1} \log N$).

Finally, we return to Faraday's cage in §6. We develop an electrostatic version of Foldy's method; as far as we know, this is both new and simple. We then apply this method to the cage of *N* parallel wires, and we show (slow) convergence as $N \rightarrow \infty$ to the solution of the appropriate problem for a large circle.

2. Two electrostatic problems

Electrostatic problems are governed by Laplace's equation. We consider two such problems, in two dimensions, one involving an infinite periodic row of line charges (§2*a*) and one involving a ring of line charges (§2*b*). Then, we shall consider analogous acoustic problems.

(a) Infinite row of electrostatic line charges

Consider the electrostatic problem of an infinite periodic row of identical line charges. Let us start with the potential

$$\Phi(X,Y) = Re\{\log\sin(X+iY)\} = \log|\sin(X+iY)| = \frac{1}{2}\log\frac{\cosh 2Y - \cos 2X}{2}.$$
 (2.1)

Evidently, $\Phi(X, Y)$ is an even function of Y with logarithmic singularities on the line Y = 0 at $X = 0, \pm \pi, \pm 2\pi, \ldots$; see equation (3.59) in [17] for Φ written as a sum of these logarithms. Using the expansion $\log (1 - z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}$ for |z| < 1 and

$$2(\cosh 2Y - \cos 2X) = e^{2|Y|}(1 - e^{-2|Y|}e^{2iX})(1 - e^{-2|Y|}e^{-2iX}),$$

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we obtain

$$\Phi(X,Y) = |Y| - \log 2 - \sum_{n=1}^{\infty} \frac{e^{-2n|Y|}}{n} \cos 2nX, \quad Y \neq 0.$$

In addition, using 4.19.7 from [18], we have

$$\log \sin z = \log z + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n}}{2n(2n)!} B_{2n} z^{2n}, \quad 0 < |z| < \pi,$$
(2.2)

(B_{2n} is a Bernoulli number) so that, near the origin, $\Phi(X, Y) \sim \frac{1}{2} \log (X^2 + Y^2)$.

Next, we construct the potential V(x, y) by

$$V(x,y) = \frac{Ey}{2} + \frac{Ed}{2\pi} \left\{ \Phi\left(\frac{\pi x}{d}, \frac{\pi y}{d}\right) + \log 2 \right\} + C$$
(2.3)

where *E*, *d* and *C* are constants. Thus

$$V(x,y) = \begin{cases} Ey + C - \frac{Ed}{2\pi} \sum_{n=1}^{\infty} \frac{e^{-2n\pi y/d}}{n} \cos \frac{2n\pi x}{d}, & y > 0, \\ C - \frac{Ed}{2\pi} \sum_{n=1}^{\infty} \frac{e^{2n\pi y/d}}{n} \cos \frac{2n\pi x}{d}, & y < 0. \end{cases}$$

This potential has logarithmic singularities on y = 0 at $x = 0, \pm d, \pm 2d, \ldots$. Moreover, it generates a uniform field as $y \to \infty$ (grad $V \to (0, E)$, a constant vector) but an exponentially small field in y < 0.

Now, consider an infinite periodic row of identical wires, each with a circular cross section of radius *a*. Assume that $a \ll d$, the spacing between the wires. The potential *V* contains an arbitrary constant *C*, which we can adjust in order to satisfy, approximately, the boundary condition on each wire. For the small circle of radius *a* centred at the origin, use of (2.1), (2.2) and (2.3) gives a Fourier series in θ for $V(a \cos \theta, a \sin \theta)$. Hence

$$\int_{-\pi}^{\pi} V(a\cos\theta, a\sin\theta)a\,\mathrm{d}\theta = aEd\log\frac{2\pi a}{d} + 2\pi aC,\tag{2.4}$$

and this vanishes with the choice

$$C = -\frac{Ed}{2\pi} \log \frac{2\pi a}{d}.$$
(2.5)

We can interpret this choice as ensuring that the boundary condition V = 0 on the small circle r = a is satisfied in an average sense. By periodicity, the same condition is satisfied on all the other small circles in the grating.

(b) Ring of electrostatic line charges

Suppose we have *N* identical line charges, equally spaced around a circle. This electrostatic problem is discussed on page 290 of [3], page 1235 of [4] and in §44 of [5], for example.

Suppose the circle is centred at the origin and has radius *b*. Let $h = 2\pi/N$ be the angular spacing between adjacent charges. Then, using plane polar coordinates, *r* and θ , the *j*th charge is located at r = b, $\theta = \theta_j = jh$.

As $z^N - 1 = \prod_{j=1}^N (z - e^{ijh})$, the potential generated by the *N* charges is proportional to *V*, where

$$V(r,\theta) - C = \frac{1}{N} \sum_{j=1}^{N} \log \left| \left(\frac{r}{b} \right) e^{i\theta} - e^{ijh} \right| = \frac{1}{N} \log \left| \left(\frac{r}{b} \right)^{N} e^{iN\theta} - 1 \right|$$
$$= \frac{1}{2N} \log \left\{ 1 - 2 \left(\frac{r}{b} \right)^{N} \cos N\theta + \left(\frac{r}{b} \right)^{2N} \right\}$$

and *C* is a constant. Thus, V = C at r = 0, and $V \sim \log(r/b) + C$ as $r \to \infty$ for fixed *N*.

Let us examine the potential near one of the charges. Without loss of generality (by symmetry), we choose the charge at r = b, $\theta = 0$, and consider a small circle of radius *a* centred at that charge. Near z = 1, we put z = 1 + w with $|w| \ll 1$, whence

$$z^{N} - 1 = (1 + w)^{N} - 1 = Nw \left\{ 1 + (N - 1)\left(\frac{w}{2}\right) + \dots + \frac{w^{N-1}}{N} \right\}$$

and

$$\log (z^N - 1) = \log Nw + \sum_{n=1}^{\infty} c_n w^n,$$

for certain (real) coefficients c_n and sufficiently small |w|. Thus, with $w = (r_0/b)e^{i\varphi}$,

$$\int_{-\pi}^{\pi} V|_{r_0=a} a \,\mathrm{d}\varphi = 2\pi a \left\{ \frac{1}{N} \log \frac{Na}{b} + C \right\},\,$$

and this vanishes with the choice

$$C = -\frac{1}{N}\log\frac{Na}{b} = -\frac{d}{2\pi b}\log\frac{2\pi a}{d}, \text{ where } d = bh = \frac{2\pi b}{N}$$

is the arclength between adjacent charges; cf. (2.5).

Inside the ring, where r < b, put $r = b - \rho$. Then, for large *N*,

$$V - C \simeq -\frac{1}{N} \left(\frac{r}{b}\right)^N \cos N\theta = -\frac{1}{N} e^{N \log(1 - \left[\rho/b\right])} \cos N\theta \simeq -\frac{1}{N} e^{-N\rho/b} \cos N\theta,$$

assuming we are near the ring ($\rho/b \ll 1$). This shows exponential decay as N increases.

If we want to place the ring of wires in an ambient field, we can still approximate the resulting potential using a ring of line charges, but we have to calculate the strength of each charge. This is carried out in §6.

3. Two acoustic problems

Acoustic problems are governed by the Helmholtz equation. We consider two such problems, in two dimensions, one involving an infinite periodic row of line sources (§3*a*) and one involving a ring of line sources (§3*b*). The main difference compared with the electrostatic problems of §2 is that if we want to represent small scatterers then we do not obtain exponential decay.

(a) Infinite row of acoustic line sources

The acoustic version of the problem considered in §2*a* starts with an infinite periodic row of identical sources. Let $r_i = (jd, 0)$ and

$$U(x,y) = \sum_{j=-\infty}^{\infty} H_0(k|\boldsymbol{r}-\boldsymbol{r}_j|),$$

where $H_0 \equiv H_0^{(1)}$ is a Hankel function. (The suppressed time dependence is $e^{-i\omega t}$.) Much is known about the evaluation of such formulae; see Linton's survey [19]. In particular, from equation (3.1) in [19],

$$U(x,y) = \frac{2}{kd} e^{ik|y|} - 4i \sum_{n=1}^{\infty} \frac{e^{-\gamma_n|y|}}{\gamma_n d} \cos \frac{2n\pi x}{d}, \quad y \neq 0,$$

where $\gamma_n = \sqrt{(2\pi n/d)^2 - k^2}$, and we have assumed that $kd < 2\pi$. Thus, if we define

$$u_1(x,y) = e^{-iky} - \frac{1}{2}kdU(x,y),$$
(3.1)

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we find that

$$u_1(x,y) = \begin{cases} e^{-iky} - e^{iky} + 2ikd\sum_{n=1}^{\infty} \frac{e^{-\gamma_n y}}{\gamma_n d} \cos \frac{2n\pi x}{d}, & y > 0, \\ 2ikd\sum_{n=1}^{\infty} \frac{e^{\gamma_n y}}{\gamma_n d} \cos \frac{2n\pi x}{d}, & y < 0. \end{cases}$$

This represents perfect reflection of a wave at normal incidence to the infinite row of sources; the field is exponentially small for y < 0. However, we have not investigated the behaviour near each source in the row. To do this, we use the expansion

$$U(x, y) = H_0(kr) + \sum_{n = -\infty}^{\infty} \tau_n(kd) J_n(kr) e^{in\theta}, \quad 0 < r < d,$$

where τ_n is a *lattice sum* (see equation (3.1) in [20]); τ_n depends on *kd* but not on *r* or θ . Thus

$$\int_{-\pi}^{\pi} U|_{r=a} a \, \mathrm{d}\theta = 2\pi a \{ H_0(ka) + \tau_0(kd) J_0(ka) \}$$

and, using (3.1) and $\int_{-\pi}^{\pi} e^{ika \sin \theta} d\theta = 2\pi J_0(ka)$,

$$\int_{-\pi}^{\pi} u_1|_{r=a} a \, \mathrm{d}\theta = \pi a \{ 2J_0(ka) - kd [H_0(ka) + \tau_0(kd)J_0(ka)] \}.$$

We cannot make this vanish, unlike with the electrostatic problem; see (2.4) and (2.5). Thus, although u_1 is the acoustic analogue of the Maxwell potential (2.3), it does not contain an adjustable parameter that can be used in order to represent an infinite row of small sound-soft circular scatterers.

Instead, we introduce an additional parameter, \mathcal{R} , and define

$$u_2(x,y) = e^{-iky} + \frac{1}{2}kd\mathcal{R}U(x,y).$$
(3.2)

We have, omitting exponentially small terms,

$$u_2(x,y) \simeq \begin{cases} e^{-iky} + \mathcal{R} e^{iky}, & y > 0, \\ (1+\mathcal{R})e^{-iky}, & y < 0. \end{cases}$$

In addition,

$$\int_{-\pi}^{\pi} u_2|_{r=a} a \, \mathrm{d}\theta = \pi a \{ 2J_0(ka) + \mathcal{R}kd[H_0(ka) + \tau_0(kd)J_0(ka)] \}.$$

Setting this quantity to zero gives

$$\mathcal{R} = \frac{-2J_0(ka)}{kd[H_0(ka) + \tau_0(kd)J_0(ka)]},$$
(3.3)

a known result for the reflection coefficient for a plane wave at normal incidence to a periodic row of small circular scatterers. This result goes back to Ignatowsky [8]; see [21,22] or §3 of [23]. Note that there is a transmitted wave, $(1 + R)e^{-iky}$, in the region y < 0.

The lattice sum τ_0 is defined by

$$\tau_0(kd) = \sum_{\substack{m = -\infty \\ m \neq 0}}^{\infty} H_0(|m|kd) = 2 \sum_{m=1}^{\infty} H_0(mkd).$$
(3.4)

For small *kd* (which implies that *ka* is also small), we have

$$\tau_0(kd) \sim \frac{2}{kd} - 1 - \frac{2i\gamma}{\pi} - \frac{2i}{\pi} \log \frac{kd}{4\pi} = \frac{2}{kd} + \frac{2i\beta}{\pi} - \frac{2i}{\pi} \log \frac{kd}{2\pi},$$
(3.5)

where $\gamma \simeq 0.5772$ is Euler's constant and β is the complex constant occurring in the asymptotic approximation

$$H_0^{(1)}(w) = \left(\frac{2i}{\pi}\right) (\log w - \beta) + O(w^2 \log w) \quad \text{as } w \to 0;$$
(3.6)

thus, $\beta = \log 2 - \gamma + i\pi/2$. (The estimate (3.5) can be obtained from equation (2.47) in [19] or equation (3.37) in [20].) Using (3.6) for $H_0(ka)$ and $J_0(ka) \sim 1$ as $ka \to 0$, we obtain

$$\mathcal{R} \simeq \frac{1}{i\Lambda - 1}$$
 with $\Lambda = \frac{kd}{\pi} \log \frac{d}{2\pi a}$

which agrees with equation (94) of Lamb [7] and equation (24) of Gans [9], for example. In particular, as $kd \rightarrow 0$, $\mathcal{R} \sim -1$, so that we recover perfect reflection from a sound-soft flat surface in this limit.

In order to have a more systematic method, applicable to other geometries, we use Foldy's method (§4). Before doing that, we consider a ring of identical acoustic sources.

(b) Ring of acoustic line sources

In addition to the notation of §2*b*, let r_j be the position vector of the *j*th source, so that $r_j = b(\cos jh, \sin jh)$. Then, the acoustic version of the problem considered in §2*b* leads to the outgoing (radiating) wave function

$$U(\mathbf{r}) = \frac{1}{N} \sum_{j=1}^{N} H_0(k|\mathbf{r} - \mathbf{r}_j|).$$

(For three studies of this problem, see [24–26].) At the centre of the ring, $U(\mathbf{0}) = H_0(kb)$. In the far field,

$$U(\mathbf{r}) = U(r,\theta) \sim \sqrt{\frac{2}{\pi}} e^{-i\pi/4} \frac{e^{ikr}}{\sqrt{kr}} F_N(\theta) \quad \text{as } r \to \infty,$$
(3.7)

where

$$F_N(\theta) = \frac{1}{N} \sum_{j=1}^{N} e^{-ikb\cos(\theta - jh)}$$
(3.8)

is the far-field pattern and $h = 2\pi/N$. Now, the function $\mathcal{E}(\tau) = e^{-ikb\cos(\theta - \tau)}$ is an entire 2π -periodic function of τ . It follows that [27]

$$F_N = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{E}(\tau) \,\mathrm{d}\tau + E_N,$$

where $E_N \rightarrow 0$ exponentially fast as $N \rightarrow \infty$. (One can think of this result as being a consequence of the Euler–Maclaurin formula or the exponential convergence of the trapezoidal rule for smooth periodic functions.) Thus,

$$F_N \sim \frac{1}{2\pi} \int_0^{2\pi} e^{-ikb\cos\tau} d\tau = J_0(kb) \text{ as } N \to \infty,$$
 (3.9)

with no dependence on θ . For an alternative proof, use the Jacobi–Anger expansion (see §10.12 in [18] or equation (2.18) in [11])

$$e^{i\rho\cos\theta} = \sum_{n=-\infty}^{\infty} i^n J_n(\rho) e^{in\theta}$$
(3.10)

in (3.8), whence

$$F_N = \frac{1}{N} \sum_{n=-\infty}^{\infty} (-\mathbf{i})^n J_n(kb) \, \mathrm{e}^{-\mathbf{i}n\theta} \sum_{j=1}^N \mathrm{e}^{\mathbf{i}njh}$$
$$= J_0(kb) + 2 \sum_{p=1}^{\infty} (-\mathbf{i})^{pN} J_{pN}(kb) \cos pN\theta$$

after using discrete orthogonality,

$$\sum_{j=1}^{N} \varpi^{nj} = \begin{cases} N, & n = 0 \mod N, \\ 0, & \text{otherwise,} \end{cases}$$
(3.11)

where $\varpi = e^{2\pi i/N} = e^{ih}$. But, $J_{\nu}(x)$ decays exponentially as $\nu \to \infty$ for fixed *x* (see 10.9.1 in [18]), and so we obtain the result (3.9).

These results are interesting but we are more interested in scattering problems. For such problems, the strengths of the sources are not all the same and, indeed, they have to be calculated. We do this using Foldy's method.

4. Foldy's method

Foldy's 1945 paper [15] gives us a general theory for multiple scattering by randomly distributed scatterers (see §8.3 of [11] for details). Within this theory, there is a deterministic method for scattering by N obstacles, assuming that the scattering is *isotropic*. This means that, near the *j*th scatterer, B_j , the scattered field is approximated by

$$A_i G(\mathbf{r} - \mathbf{r}_i)$$

where B_j is centred at r_j , A_j is an unknown amplitude and G is the free-space Green's function. Foldy worked in three dimensions, but, as we are interested in two-dimensional problems, we take

$$G(\mathbf{r}) = H_0(k|\mathbf{r}|).$$

Then, following Foldy, we represent the total field as

$$u(\mathbf{r}) = u_{\rm inc}(\mathbf{r}) + \sum_{j=1}^{N} A_j G(\mathbf{r} - \mathbf{r}_j).$$
(4.1)

The so-called external field is

$$u_n(\mathbf{r}) \equiv u(\mathbf{r}) - A_n G(\mathbf{r} - \mathbf{r}_n) = u_{\rm inc}(\mathbf{r}) + \sum_{\substack{j=1\\ i \neq n}}^N A_j G(\mathbf{r} - \mathbf{r}_j).$$
(4.2)

It can be regarded as the field incident on B_n in the presence of all the other scatterers.

Now, we characterize the scattering properties of the scatterers by

$$A_n = g_n u_n(\mathbf{r}_n). \tag{4.3}$$

This makes the strength of the scattered wave from B_n , A_n , proportional to the external field acting on it, $u_n(\mathbf{r}_n)$. Foldy calls g_n the *scattering coefficient* for B_n . Thus, the scattered field is determined by the value of the 'external field' at the centre of the scatterer, \mathbf{r}_n , together with the quantity g_n (see §4*a*).

So, evaluating (4.2) at r_n gives, after using (4.3),

$$g_n^{-1}A_n = u_{\rm inc}(\mathbf{r}_n) + \sum_{\substack{j=1\\j\neq n}}^N A_j G(\mathbf{r}_n - \mathbf{r}_j), \quad n = 1, 2, \dots, N,$$
(4.4)

which is a linear system of algebraic equations for A_j . Then, the total field is given by (4.1). These are Foldy's 'fundamental equations of multiple scattering' [15].

(a) The scattering coefficient

The coefficient g_n characterizes how the *n*th scatterer scatters waves in isolation. For identical scatterers, we write $g_n \equiv g$. Then, for a scatterer at the origin, given that we have already assumed

that the scattering is isotropic, the total field near the scatterer is given by

$$u(\mathbf{r}) \simeq u_{\rm inc}(\mathbf{r}) + g u_{\rm inc}(\mathbf{0}) G(\mathbf{r}), \tag{4.5}$$

where $u_{inc}(r)$ is the incident field. Conservation of energy implies that *g* must satisfy (see §8.3.1 of [11])

$$Re(g) + |g|^2 = 0. (4.6)$$

If the scatterer is a sound-soft circle of radius *a*, C_0 , we can impose the boundary condition u = 0 on average, $\int_{C_0} u \, d\theta = 0$, giving

$$g = -\frac{J_0(ka)}{H_0(ka)};$$
 (4.7)

this choice satisfies (4.6) exactly. The choice

$$g = -\frac{1}{H_0(ka)} \tag{4.8}$$

seems natural for small *ka* (because $J_0(ka) \sim 1$ as $ka \rightarrow 0$), but this *g* does not satisfy (4.6) exactly; this error may be important when *N* is large.

Another choice comes from the low-frequency asymptotics for scattering by one object. For a circle, with $ka \ll 1$, we have (see §8.3.1 of [11])

$$g = \frac{1}{2}i\pi(\log ka - \beta)^{-1},$$

where β is defined by (3.6). This choice also satisfies (4.6) exactly.

To use Foldy's method for any configuration of identical scatterers, a choice for g must be made. We comment on this choice later.

(b) Application to the infinite grating

For a plane wave at normal incidence to a grating of small circles along the *x*-axis, as studied in $\S3a$, we have (cf. (4.1))

$$u(\mathbf{r}) = u(x, y) = \mathrm{e}^{-\mathrm{i}ky} + \sum_{j=-\infty}^{\infty} A_j H_0(k|\mathbf{r} - \mathbf{r}_j|),$$

where the coefficients are given by (4.4), which becomes

$$g^{-1}A_n = 1 + \sum_{\substack{j=-\infty \ j\neq n}}^{\infty} A_j H_0(|n-j|kd), \quad n = 0, \pm 1, \pm 2, \dots$$

The symmetry of the problem gives $A_j = A_0$ for all j, whence $g^{-1}A_0 = 1 + A_0\tau_0$, where τ_0 is a lattice sum, (3.4). Thus, $A_0 = [g^{-1} - \tau_0]^{-1}$, which gives the reflection coefficient in agreement with (3.3), provided we define g by (4.7).

For oblique incidence and for semi-infinite gratings, see [23].

5. Scattering by a ring of identical small scatterers

Recalling the notation of §§2*b* and 3*b*, we suppose we have *N* identical small scatterers, equally spaced around a circle of radius *b*. The incident field is $u_{inc}(r)$. The distance between the *j*th and

*n*th scatterers is

$$|\mathbf{r}_n - \mathbf{r}_j| = b\sqrt{2 - 2\cos(\theta_n - \theta_j)} = 2b \left| \sin\left(\frac{[n-j]\pi}{N}\right) \right|,\tag{5.1}$$

where $\theta_i = jh = 2\pi j/N$. Then, with $g_n \equiv g$, the $N \times N$ Foldy system (4.4) simplifies to

$$\sum_{j=1}^{N} C_{n-j} A_j = f_n, \quad n = 1, 2, \dots, N,$$
(5.2)

where $f_n = -u_{inc}(\mathbf{r}_n)$,

$$C_0 = -g^{-1}, \quad C_j = H_0\left(2kb\left|\sin\left(\frac{j\pi}{N}\right)\right|\right), \quad j \neq 0 \text{ mod } N$$
(5.3)

and C_j is *N*-periodic: $C_{j+mN} = C_j, m = \pm 1, \pm 2, ...$

Richmond [28] gave numerical solutions of (5.2) with N = 30, ka = 0.05, $kb = 2\pi$ and g defined by (4.8). (He also gave results for other configurations of the small circles.) Wilson [29] also solved (5.2), both numerically and analytically; A_j was determined exactly as an infinite series. Later, Vescovo [30] showed that A_j could be found much more simply by noting that the system matrix in (5.2) is a circulant matrix. This means that the linear system can be solved explicitly, essentially by using the discrete Fourier transform. Thus, let $\varpi = e^{2\pi i/N}$. Multiply (5.2) by ϖ^{mn} , sum over nand use the N-periodicity of C_n . This gives

$$\tilde{A}_m = \frac{f_m}{\tilde{C}_m},\tag{5.4}$$

where

$$\tilde{A}_m = \sum_{j=1}^N A_j \varpi^{mj}, \quad \tilde{f}_m = \sum_{j=1}^N f_j \varpi^{mj} \quad \text{and} \quad \tilde{C}_m = \sum_{j=1}^N C_j \varpi^{mj}.$$
(5.5)

Finally, invert the discrete Fourier transform of $\{A_i\}$, $\{\tilde{A}_m\}$, using (3.11), to give

$$A_n = \frac{1}{N} \sum_{j=1}^N \tilde{A}_j \varpi^{-nj}.$$

For more details, see appendix B of [31].

Having determined A_n , we can calculate the field everywhere, using (4.1). For example, the total field at the origin is given by

$$u(\mathbf{0}) = u_{\rm inc}(\mathbf{0}) + H_0(kb) \sum_{j=1}^N A_j = u_{\rm inc}(\mathbf{0}) + \tilde{A}_0 H_0(kb),$$
(5.6)

where we have used $(5.5)_1$. The far field of the *N*-scatterer cluster is given by (3.7), with *U* replaced by the scattered field, u_{sc} , and far-field pattern

$$F_N(\theta) = \sum_{j=1}^N A_j \,\mathrm{e}^{-\mathrm{i}kb\cos{(\theta-jh)}} = \sum_{n=-\infty}^\infty \tilde{A}_n(-\mathrm{i})^n J_n(kb) \,\mathrm{e}^{-\mathrm{i}n\theta}\,,\tag{5.7}$$

where we have used (3.10).

Let us calculate \tilde{A}_m from (5.4) when there is an incident plane wave,

$$u_{\rm inc}(\mathbf{r}) = e^{ikx} = e^{ikr\cos\theta}$$

Then, as $f_j = -u_{inc}(r_j)$, (5.5)₂ gives

$$\tilde{f}_m = -\sum_{j=1}^N \mathrm{e}^{\mathrm{i}kb\cos jh} \,\mathrm{e}^{\mathrm{i}mjh}$$

whence, as in §3b,

$$\frac{1}{N}\tilde{f}_m \sim -\frac{1}{2\pi} \int_0^{2\pi} e^{ikb\cos\theta} e^{im\theta} d\theta = -i^m J_m(kb)$$
(5.8)

as $N \rightarrow \infty$, with exponential convergence.

The behaviour of \tilde{C}_m is quite different. From (5.3) and (5.5)₃,

$$\tilde{C}_m = -\frac{1}{g} + \sum_{j=1}^{N-1} \varpi^{mj} H_0\left(2kb \left|\sin\left(\frac{j\pi}{N}\right)\right|\right) = -\frac{1}{g} + \sum_{j=1}^{N-1} v(jh),\tag{5.9}$$

where

$$v(\theta) \equiv e^{im\theta} H_0\left(2kb \left| \sin\left(\frac{\theta}{2}\right) \right| \right)$$

is 2π -periodic in θ but has logarithmic singularities at $\theta = 0$ and $\theta = 2\pi$. The sum on the righthand side of (5.9) looks like the trapezoidal rule in which the endpoint contributions have been 'ignored'; fortunately, the properties of such sums have been analysed by Sidi [16]. Using his theorem 2.3(b), we find that

$$\frac{1}{N}\tilde{C}_m = -\frac{1}{Ng} + \frac{1}{2\pi} \int_0^{2\pi} v(\theta) \,\mathrm{d}\theta + O\left(\frac{\log N}{N}\right) \quad \text{as } N \to \infty.$$

(In other words, the quadrature error is $O(h \log h)$ as $h \to 0$, which is poor!) In fact, Sidi's analysis gives an asymptotic expansion of the error in terms of the asymptotic behaviour of $v(\theta)$ near the endpoints; it yields

$$\frac{1}{N}\tilde{C}_m \sim -\frac{1}{Ng} + \frac{1}{2\pi} \int_0^{2\pi} v(\theta) \,\mathrm{d}\theta + \frac{2\mathrm{i}}{\pi N} \log N \quad \text{as } N \to \infty,$$

showing that the error estimate is sharp for our problem.

As the term -1/(Ng) is comparable to the error estimate, we absorb it and obtain

$$\frac{1}{N}\tilde{C}_m \sim \frac{1}{2\pi} \int_0^{2\pi} e^{im\theta} H_0\left(2kb \left|\sin\left(\frac{\theta}{2}\right)\right|\right) d\theta + O\left(\frac{\log N}{N}\right)$$
$$= \frac{2}{\pi} \int_0^{\pi/2} H_0(2kb \sin \vartheta) \cos 2m\vartheta \, d\vartheta = J_m(kb)H_m(kb) + O(N^{-1}\log N)$$
(5.10)

as $N \to \infty$, using equations (6) and (7) from §6.681 in [32]. It is worth noting that (5.10) gives an estimate of \tilde{C}_m for *fixed* m as $N \to \infty$. It is also worth noting that the choice of the scattering coefficient g does not affect the leading-order estimate of \tilde{C}_m .

Combining the estimate (5.10) with (5.8) gives

$$\tilde{A}_n = -\frac{\mathrm{i}^n}{H_n(kb)} + O(N^{-1}\log N) \quad \text{as } N \to \infty.$$
(5.11)

If the leading approximation is substituted in (5.7), we obtain exactly the far-field pattern for scattering by a sound-soft cylinder of radius *b* (see equation (4.10) in [11], for example), but this limit is approached very slowly. Similarly, when the total field at the origin is calculated using (5.6), we find $u(\mathbf{0}) = O(N^{-1} \log N)$ as $N \to \infty$, having used $u_{inc}(\mathbf{0}) = 1$ and (5.11).

6. Another electrostatic problem: the Faraday cage

We return to electrostatics, governed by Laplace's equation. We start with general considerations for the basic boundary-value problem (§6*a*). Then, we give an electrostatic version of Foldy's method (§6*b*). Finally, we apply this method to the cage problem in §6*c*.

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(a) Exterior Dirichlet problem for a multi-connected domain

In general, we have to solve the exterior Dirichlet problem when there are *N* conductors (closed curves) C_j , j = 1, 2, ..., N. We write the total potential as

$$V = V_{\rm amb} + V'$$
,

where V_{amb} is the given ambient potential when there are no conductors, $\nabla^2 V' = 0$ in the region exterior to the conductors, \mathcal{D} , and V = 0 ($V' = -V_{amb}$) on \mathcal{C}_j , j = 1, 2, ..., N. In addition, there is a far-field condition, which we take as requiring that V' be bounded at infinity. It is known that this boundary-value problem has exactly one solution [33]. Moreover, it is known that V' can be written as the real part of an analytic function, $V' = Re \Phi(z)$, where

$$\Phi(z) = \Phi_0(z) + \sum_{j=1}^N A_j \log (z - z_j),$$
(6.1)

 $\Phi_0(z)$ is analytic and single valued in \mathcal{D} , z_i is an arbitrary point inside \mathcal{C}_i and the condition

$$\sum_{j=1}^{N} A_j = 0$$
 (6.2)

must be satisfied if V' is to be bounded at infinity. For more details and computational results based on this formulation, see [34].

(b) A Foldy-type method for Laplace's equation

We shall develop a static version of Foldy's acoustic method (§4). For small conductors (wires), we write the potential V as

$$V(\mathbf{r}) = V_{\rm amb}(\mathbf{r}) + C + \sum_{j=1}^{N} A_j G_0(\mathbf{r} - \mathbf{r}_j),$$
(6.3)

where $G_0(\mathbf{r}) = \log(|\mathbf{r}|/L)$, *L* is a length scale, *C* is a constant to be found, and the coefficients A_j are to be determined subjected to (6.2). Thus, compared with (6.1), we are taking Φ_0 to be an unknown constant.

Next, we require an analogue of (4.3). The external field on the *n*th wire in the cage is

$$V_n(\mathbf{r}) = V_{\text{amb}}(\mathbf{r}) + C + \sum_{\substack{j=1\\j\neq n}}^N A_j G_0(\mathbf{r} - \mathbf{r}_j) = V - A_n G_0(\mathbf{r} - \mathbf{r}_n).$$
(6.4)

Suppose all the wires are identical with circular cross section of radius *a*. Take L = a so that $G_0(r - r_n) = 0$ when *r* locates a point on the *n*th wire, C_n . As V = 0 on C_n too, (6.4) gives $V_n = 0$ on C_n , which we approximate by $V_n(r_n) = 0$, n = 1, 2, ..., N. Thus,

$$\sum_{\substack{j=1\\j\neq n}}^{N} A_j G_0(\mathbf{r}_n - \mathbf{r}_j) = f_n, \quad n = 1, 2, \dots, N,$$
(6.5)

where $f_n = -V_{amb}(\mathbf{r}_n) - C$. We regard (6.5) as a linear system for the coefficients A_j , even though f_n contains the unknown constant C; however, we also have the constraint (6.2).

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(c) Application to a ring of wires

We consider a ring of wires, equally spaced around a circle of radius b, using the notation used earlier. In particular, we have (5.1) and then the system (6.5) simplifies to (5.2) in which

$$C_0 = 0, \quad C_j = \log\left(2\left|\sin\left(\frac{j\pi}{N}\right)\right|\frac{b}{a}\right), \quad j \neq 0 \mod N$$
 (6.6)

and C_j is *N*-periodic. As in §5, the system matrix is a circulant matrix, so that the explicit solution is given by (5.4) and (5.5). These formulae determine A_j in terms of *C*; we determine *C* using (6.2). Note that (6.2) reduces to $\tilde{A}_0 = 0$.

For r > b, expanding (6.3) gives

$$V(r,\theta) = V_{\rm amb}(r,\theta) + C - \sum_{n=1}^{\infty} \frac{b^n}{nr^n} \{ (Re\,\tilde{A}_n)\cos n\theta + (Im\,\tilde{A}_n)\sin n\theta \};$$
(6.7)

for r < b, interchange r and b. In particular, at the centre of the ring,

$$V(\mathbf{0}) = V_{\rm amb}(\mathbf{0}) + C$$

This simple result does not require the wires to be equally spaced around the ring.

Let us take a particular ambient field, $V_{amb} = Ex$, where *E* is a constant. From (5.5)₂,

$$\tilde{f}_m = -\sum_{j=1}^N (Eb\cos\theta_j + C)\varpi^{mj} = -\sum_{j=1}^N \left(\frac{Eb}{2}\varpi^{(m-1)j} + \frac{Eb}{2}\varpi^{(m+1)j} + C\varpi^{mj}\right).$$

Thus, (3.11) gives $\tilde{f}_0 = -NC$, $\tilde{f}_1 = \tilde{f}_{N-1} = -\frac{1}{2}NEb$ and $\tilde{f}_m = 0$ for m = 2, 3, ..., N-2. From (5.5)₃ and (6.6), we have

$$\tilde{C}_m = \sum_{j=1}^{N-1} \varpi^{mj} \log\left(2\left|\sin\left(\frac{j\pi}{N}\right)\right| \frac{b}{a}\right) = \sum_{j=1}^{N-1} v_0(jh),$$

where

$$v_0(\theta) = e^{im\theta} \log\left(2\left|\sin\left(\frac{\theta}{2}\right)\right| \frac{b}{a}\right)$$

We require \tilde{C}_0 , \tilde{C}_1 and $\tilde{C}_{N-1} = \tilde{C}_1^*$, the complex conjugate of \tilde{C}_1 . Using Sidi's theorem [16], as in §5, we obtain

$$\frac{1}{N}\tilde{C}_m \sim \frac{1}{2\pi} \int_0^{2\pi} v_0(\theta) \,\mathrm{d}\theta + \frac{1}{N}\log N$$

with an error that is $O(N^{-1})$ as $N \to \infty$. Evaluating the integral, using equations (1) and (3) from §4.384 in [32], we find that $\tilde{C}_0 \sim N \log (b/a) + \log N$ and $\tilde{C}_1 \sim -\frac{1}{2}N + \log N$. Hence, from (5.4),

$$\tilde{A}_0 \sim \frac{C}{\log(a/b)}, \quad \tilde{A}_1 \sim Eb\left(1 + \frac{2}{N}\log N\right) \quad \text{and} \quad \tilde{A}_{N-1} \sim Eb\left(1 + \frac{2}{N}\log N\right).$$

The constraint (6.2), $\tilde{A}_0 = 0$, gives C = 0. Then, (6.7) gives

$$V(r,\theta) = Er\cos\theta - \frac{Eb^2}{r}\left(1 + \frac{2}{N}\log N + O(N^{-1})\right)\cos\theta + O\left(\frac{\cos\left(N-1\right)\theta}{Nr^{N-1}}\right),$$

for large *N* and large *r*, and

$$V(r,\theta) \sim -2E \frac{\log N}{N} r \cos \theta$$

for large *N* and r < b. In both cases, the leading estimates (namely $Ex(1 - b^2/r^2)$ and zero) agree exactly with the solution of the corresponding electrostatic problem for a circle of radius *b*.

7. Conclusion

We have discussed several problems where fields (electrostatic or acoustic) interact with many identical small objects (wires). For an infinite periodic row of wires (a grating), known results were recalled and rederived. Such problems can be solved by writing down formulae for the fields generated by an infinite periodic row of line singularities or sources and then adjusting the value of one free parameter. This is a kind of inverse method, well known to Maxwell, Lamb and other Victorian scientists.

For more complicated geometrical arrangements of wires, a more complicated method is required. For the acoustic scattering problems, we used Foldy's method, where each of the N wires is represented by a source of unknown strength and then these strengths are determined by solving a linear algebraic $N \times N$ system. For rings of wires, this method was used by Richmond [28]. For other geometrical arrangements, there are recent applications of Foldy's method to 'quantum corrals' [35,36] and to imaging problems [37].

For scattering by a ring of N equally spaced wires, the source strengths were obtained explicitly, using properties of circulant matrices. This permitted an asymptotic estimate of the fields as N became large. Intuitively, we might expect that, in this limit, the ring behaves like a single circular scatterer. This expectation was confirmed but, perhaps surprisingly, the limit was found to be approached slowly (as $N^{-1} \log N$). Similar results were obtained for the analogous electrostatic problem (Faraday's cage), using an electrostatic version of Foldy's method.

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