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Bounds on ratios of modified Bessel functions with complex arguments

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ABSTRACT

A simple uniform bound for the ratio of two modified spherical Bessel functions is derived. The arguments of the two functions are complex but their ratio is real.

1. Introduction

In a recent investigation in the context of acoustic scattering, we encountered a Volterra integral equation of the second kind,

$$u(x) - \int_x^\infty \mathcal{K}(x,y)\, u(y)\, \mathrm{d}y = f(x), \quad x > 1, \tag{1}$$

where $f(x)$ is given and $u(x)$ is to be found. The kernel is given by

$$\mathcal{K}(x,y) = \int_x^y \left(\frac{W(y)}{W(\eta)} \right)^2 \mathrm{d}\eta, \tag{2}$$

where

$$W(\xi) = \xi^{1/2} K_\nu(\mu\xi), \tag{3}$$

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K_ν is a modified Bessel function, and the parameters μ and ν will be specified shortly. Usually, Volterra integral equations of the second kind (such as (1)) can be solved by iteration. To justify this approach, a bound on $|\mathcal{K}(x,y)|$ is needed.

Fortunately, there is a recent paper by Baricz [2] containing a thorough review of the literature on known bounds on ratios of modified Bessel functions. From [2, eqn (3.6)],

$$\frac{K_\nu(x)}{K_\nu(y)} > \mathrm{e}^{y-x}\left(\frac{y}{x}\right)^{1/2}, \quad |\nu| > \frac{1}{2}, \quad 0 < x < y.$$

Hence

$$F(y,\eta) \equiv \frac{W(y)}{W(\eta)} = \frac{y^{1/2}K_\nu(\mu y)}{\eta^{1/2}K_\nu(\mu\eta)} < \mathrm{e}^{\mu(\eta-y)}, \quad 0 < \eta < y. \tag{4}$$

This holds for real μ with $\mu > 0$. Using the bound (4) in (2),

$$0 < \mathcal{K}(x,y) < \int_x^y \mathrm{e}^{2\mu(\eta-y)}\,\mathrm{d}\eta = \frac{1}{2\mu}\left(1 - \mathrm{e}^{-2\mu(y-x)}\right) \le \frac{1}{2\mu},$$

as $y \ge x$, and this uniform bound can be used to justify an iterative scheme.

Unfortunately, in the application we have in mind, the parameter μ is complex: it is essentially a Laplace transform variable, which explains why we are interested in the half-plane

$$\operatorname{Re}\mu > 0.$$

To see that some form of (4) may hold, use the standard asymptotic approximation [6, 10.25.3], $K_\nu(z) \sim \sqrt{\pi/(2z)}\,\mathrm{e}^{-z}$ as $z \to \infty$, $|\arg z| < \frac{3}{2}\pi$. Substitution in (3) gives

$$F(y,\eta) \sim \mathrm{e}^{\mu(\eta-y)} \quad \text{as } \mu \to \infty,$$

suggesting that (4) may be valid. On the other hand, $K_\nu(z)$ has zeros in the half-plane $\operatorname{Re} z < 0$ [7, §15.7], implying that we cannot expect a bound when $\operatorname{Re}\mu < 0$.

We are able to prove a natural generalization of (4) but only when the order $\nu = n + \frac{1}{2}$, where n is an integer. This special case, which corresponds to modified spherical Bessel functions, is exactly the case that arises when the three-dimensional wave equation is solved using Laplace transforms with respect to time and spherical polar coordinates. We leave the general case (with arbitrary positive ν) to future work.

The paper has two more sections. In Section 2, we reduce the problem to one involving Bessel polynomials and then to one in which the signs of the terms in a certain polynomial have to be determined. This problem is solved using properties of generalized hypergeometric series; these calculations are collected in Section 3. The bound itself follows readily, and is given by (13) at the end of Section 2.

2. A bound for modified spherical Bessel functions

In what follows, we suppose that the order

$$\nu = n + \tfrac{1}{2}, \quad n = 0, 1, 2, \ldots.$$

Let $F_n(y,\eta;\mu) = F(y,\eta)$ with $\nu = n+\frac{1}{2}$, $y > \eta > 1$ and μ complex, with $\operatorname{Re}\mu > 0$. By definition [6, 10.47.9]

$$k_n(z) = \sqrt{\frac{\pi}{2z}}\,K_{n+1/2}(z)$$

where k_n is a modified spherical Bessel function. Hence

$$F_n(y,\eta;\mu) = \frac{y\,k_n(\mu y)}{\eta\,k_n(\mu\eta)}. \tag{5}$$

As $k_0(z) = (\pi/2)\,\mathrm{e}^{-z}/z$, $F_0(y,\eta;\mu) = \mathrm{e}^{\mu(\eta-y)}$. More generally [6, 10.49.12]

$$k_n(z) = \frac{\pi}{2z}\,\mathrm{e}^{-z}L_n(z) \quad \text{with} \quad L_n(z) = \sum_{k=0}^{n} a_k^n (2z)^{-k},$$

where

$$a_k^n = \frac{(n+k)!}{k!\,(n-k)!}, \qquad k = 0,1,\ldots,n. \tag{6}$$

Thus $L_n(z)$ is a polynomial in $1/z$ of degree n (essentially, a *Bessel polynomial* [3]). Substitution in (5) gives

$$F_n(y,\eta;\mu) = \mathrm{e}^{\mu(\eta-y)}\,\frac{L_n(\mu y)}{L_n(\mu\eta)}. \tag{7}$$

We want to estimate this quantity in the complex μ-plane. We show that

$$|L_n(\mu y)| < |L_n(\mu\eta)| \quad \text{for } 0 < \eta < y \text{ and } \operatorname{Re}\mu > 0. \tag{8}$$

It is convenient to generalise the definition (6), replacing the integer n by a complex variable ω:

$$a_k^\omega = \frac{\Gamma(\omega+k+1)}{k!\,\Gamma(\omega-k+1)}.$$

In particular $a_0^\omega = 1$ and $a_k^n = 0$ when k is any integer greater than n. Then, with $\bar{\mu}$ denoting the complex conjugate of μ,

$$\begin{aligned}
|L_n(\mu y)|^2 &= L_n(\mu y)L_n(\bar{\mu}y) \\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_k^n a_j^n \frac{1}{\mu^k\bar{\mu}^j(2y)^{j+k}} = \sum_{k=0}^{\infty}\sum_{p=k}^{\infty} a_k^n a_{p-k}^n \frac{1}{\mu^k\bar{\mu}^{p-k}(2y)^p} \\
&= \sum_{p=0}^{\infty}\frac{1}{(2y)^p}\sum_{k=0}^{p} a_k^n a_{p-k}^n \frac{1}{\mu^k\bar{\mu}^{p-k}} = \sum_{p=0}^{\infty}\frac{1}{(2|\mu|y)^p} S_p^n,
\end{aligned} \tag{9}$$

say, where

$$S_p^\omega = \sum_{k=0}^{p} a_k^\omega a_{p-k}^\omega \frac{|\mu|^p}{\mu^k\bar{\mu}^{p-k}}. \tag{10}$$

Evidently, $|L_n(\mu y)|^2$ is a polynomial in $1/y$ of degree $2n$, so that (9) reduces to

$$|L_n(\mu y)|^2 = 1 + \sum_{p=1}^{n} \frac{1}{(2|\mu|y)^{2p}} S_{2p}^n + \sum_{p=0}^{n-1} \frac{1}{(2|\mu|y)^{2p+1}} S_{2p+1}^n.$$

Now, as μ is complex, write

$$\mu = |\mu|\,\mathrm{e}^{\mathrm{i}\delta}$$

for some real δ. Substituting in (10) gives

$$S_p^\omega = \sum_{k=0}^{p} a_k^\omega a_{p-k}^\omega\, \mathrm{e}^{\mathrm{i}(p-2k)\delta} = \sum_{j=0}^{p} a_{p-j}^\omega a_j^\omega\, \mathrm{e}^{-\mathrm{i}(p-2j)\delta}$$

whence

$$S_p^\omega(\delta) = \sum_{k=0}^{p} a_k^\omega a_{p-k}^\omega \cos\{(p-2k)\delta\}. \tag{11}$$

This formula confirms that $S_p^n(\delta)$ is real but it does not reveal its sign. To determine the sign of S_p^n, we express $\cos\{(p-2k)\delta\}$ as a polynomial in $\cos\delta$, and then show that S_p^n has a similar representation in which all the coefficients in the polynomial are positive. We do this in Section 3, making use of generalized hypergeometric series ${}_pF_q$ and their properties. The final formula (obtained by combining (22) and (25) below) is

$$S_p^n(\delta) = \sum_{j=0}^{[p/2]} \frac{p!\,(2\cos\delta)^{p-2j}}{j!\,(p-2j)!}\, a_{p-j}^n, \tag{12}$$

where $[p/2] = m$ when $p = 2m$ or $p = 2m+1$. Thus $S_{2p}^n(\delta)$ is positive for all δ, but $S_{2p+1}^n(\delta)$ is positive if and only if $\cos\delta > 0$; here is where the restriction to $\operatorname{Re}\mu > 0$ enters.

Returning to (8), we have

$$|L_n(\mu\eta)|^2 - |L_n(\mu y)|^2 = \sum_{p=1}^{2n} \frac{1}{(2|\mu|)^p} \left(\frac{1}{\eta^p} - \frac{1}{y^p}\right) S_p^n,$$

which is seen to be positive when $y > \eta > 0$. Hence (5) and (7) give the main result,

$$\left|\frac{y\,k_n(\mu y)}{\eta\,k_n(\mu\eta)}\right| < \mathrm{e}^{\mu_\mathrm{r}(\eta-y)} \tag{13}$$

where $\mu_\mathrm{r} = \operatorname{Re}\mu > 0$ and $0 < \eta < y$. Interestingly, Lin and Santosa [4, eqn (5.10)] have conjectured a similar bound for K_n,

$$\left|\frac{K_n(\mu y)}{K_n(\mu\eta)}\right| < \mathrm{e}^{\mu_\mathrm{r}(\eta-y)} \tag{14}$$

where $\mu_\mathrm{r} > 0$ and $0 < \eta < y$: “its proof remains completely open”.

3. Evaluation of S_p^n

It is convenient to consider two cases, depending on whether the integer p is even or odd.

3.1. Evaluation of S_{2p}^n

Let us start with S_{2p}^ω,

$$S_{2p}^\omega = \sum_{k=0}^{2p} a_k^\omega a_{2p-k}^\omega \cos\{(2p-2k)\delta\}.$$

We write $\cos 2m\delta$ as a polynomial in $\cos^2\delta$, noting that $\cos n\theta = T_n(\cos\theta)$ defines the Chebyshev polynomial T_n. In detail [5, p. 24],

$$T_{2m}(x) = (-1)^m + \sum_{j=0}^{m-1} c_j^{(2m)} x^{2m-2j}, \quad m = 0, 1, 2, \ldots$$

where the sum is absent when $m = 0$ and

$$c_j^{(2m)} = m\,(-1)^j \frac{2^{2m-2j}(2m-j-1)!}{j!\,(2m-2j)!}, \quad m = 1, 2, 3, \ldots.$$

Hence

$$S_{2p}^\omega = \sum_{k=0}^{2p} a_k^\omega a_{2p-k}^\omega (-1)^{p+k} + \sum_{\substack{k=0\\k\neq p}}^{2p} a_k^\omega a_{2p-k}^\omega \left(\cos\{(2p-2k)\delta\} - (-1)^{p+k}\right).$$

Denote the first sum by A_{2p}^ω; it does not depend on δ. Then, splitting the second sum,

$$S_{2p}^\omega = A_{2p}^\omega + \sum_{k=0}^{p-1} a_k^\omega a_{2p-k}^\omega \sum_{j=0}^{p-k-1} c_j^{(2p-2k)} x^{2p-2k-2j} + \sum_{k=p+1}^{2p} a_k^\omega a_{2p-k}^\omega \sum_{j=0}^{k-p-1} c_j^{(2k-2p)} x^{2k-2p-2j},$$

with $x = \cos\delta$. Put $k = p - q$ in the first sum and $k = p + q$ in the second sum, giving

$$\begin{aligned} S_{2p}^\omega &= A_{2p}^\omega + 2\sum_{q=1}^{p} a_{p-q}^\omega a_{p+q}^\omega \sum_{j=0}^{q-1} c_j^{(2q)} x^{2q-2j} = A_{2p}^\omega + 2\sum_{q=1}^{p} a_{p-q}^\omega a_{p+q}^\omega \sum_{m=1}^{q} c_{q-m}^{(2q)} x^{2m} \\ &= A_{2p}^\omega + 2\sum_{m=1}^{p} x^{2m} \sum_{q=m}^{p} a_{p-q}^\omega a_{p+q}^\omega c_{q-m}^{(2q)}. \end{aligned}$$

Finally, put $q = m + k$ and $m = p - j$ giving

$$S_{2p}^\omega = A_{2p}^\omega + 2\sum_{j=0}^{p-1} x^{2p-2j} \sum_{k=0}^{j} a_{j-k}^\omega a_{2p+k-j}^\omega c_k^{(2p+2k-2j)}. \tag{15}$$

Denote the double sum by B_{2p}^ω; we shall evaluate A_{2p}^ω and B_{2p}^ω below.

3.1.1. Evaluation of A^ω_{2p}

The first term in (15) is A^ω_{2p}, defined by

$$A^\omega_{2p} = \sum_{k=0}^{2p} a^\omega_k a^\omega_{2p-k} (-1)^{p+k} = (-1)^p \sum_{k=0}^{\infty} C_k,$$

say, with

$$C_k = (-1)^k a^\omega_k a^\omega_{2p-k} = \frac{(-1)^k \Gamma(\omega+k+1)\,\Gamma(\omega+2p-k+1)}{k!\,\Gamma(\omega-k+1)\,(2p-k)!\,\Gamma(\omega-2p+k+1)}.$$

In particular

$$C_0 = \frac{\Gamma(\omega+2p+1)}{(2p)!\,\Gamma(\omega-2p+1)}. \tag{16}$$

Some calculation gives

$$\frac{C_{k+1}}{C_k} = \frac{(-1)(\omega+k+1)(\omega-k)(2p-k)}{(k+1)(\omega+2p-k)(\omega-2p+k+1)} = \frac{(k+\omega+1)(k-\omega)(k-2p)}{(k+1)(k-\omega-2p)(k+\omega-2p+1)}.$$

Equivalently,

$$(k+1)(k-\omega-2p)(k+\omega-2p+1)C_{k+1} = (k+\omega+1)(k-\omega)(k-2p)C_k,$$

which implies that $C_k = 0$ for $k \geq 2p+1$. Hence A^ω_{2p} can be expressed as a generalized hypergeometric series (see [1, p. 61]),

$$A^\omega_{2p} = C_0(-1)^p\, {}_3F_2(a_1, a_2, a_3; b_1, b_2; 1)$$

where

$$a_1 = \omega+1, \quad a_2 = -\omega, \quad a_3 = -2p \quad b_1 = -\omega-2p, \quad b_2 = \omega-2p+1.$$

We notice that $a_1 + b_1 = a_2 + b_2 = a_3 + 1$: the series is *well poised* [6, 16.4.1] and so it can be summed using Dixon's formula. Thus, using [6, 16.4.5] (with $k = p$, $b = a_1$ and $c = a_2$ therein) and (16),

$$\begin{aligned} A^\omega_{2p} &= \frac{(-1)^p\,\Gamma(\omega+2p+1)}{(2p)!\,\Gamma(\omega-2p+1)} \frac{[(2p)!]^2\Gamma(p+\omega+1)\,\Gamma(p-\omega)}{(p!)^2\Gamma(2p-\omega)\,\Gamma(\omega+2p+1)} \\ &= \frac{(-1)^p(2p)!\,\Gamma(p+\omega+1)\,\Gamma(p-\omega)}{(p!)^2\Gamma(1-[2p-\omega])\,\Gamma(2p-\omega)}. \end{aligned}$$

If we use the reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$, once with $z = 2p-\omega$ and once with $z = p-\omega$, we obtain

$$A^\omega_{2p} = \frac{(2p)!\,\Gamma(\omega+p+1)}{(p!)^2\Gamma(\omega-p+1)}.$$

In particular, when $\omega = n$, we obtain

$$A^n_{2p} = \sum_{k=0}^{2p} a^n_k a^n_{2p-k} (-1)^{p+k} = \frac{(2p)!}{p!}\, a^n_p, \tag{17}$$

using (6). Thus $A^n_{2p} > 0$.

3.1.2. Evaluation of B_{2p}^{ω}

The second term in (15) is B_{2p}^{ω}, defined by

$$B_{2p}^{\omega}(\delta) = 2\sum_{j=0}^{p-1} x^{2p-2j} \sum_{k=0}^{j} a_{j-k}^{\omega} a_{2p+k-j}^{\omega} c_k^{(2p+2k-2j)},$$

where $x = \cos\delta$. We have

$$a_{j-k}^{\omega} a_{2p+k-j}^{\omega} c_k^{(2p+2k-2j)} = \frac{\Gamma(\omega+j-k+1)\Gamma(\omega+2p+k-j+1)(p+k-j)(-1)^k 2^{2p-2j}(2p+k-2j-1)!}{(j-k)!\,\Gamma(\omega-j+k+1)(2p+k-j)!\,\Gamma(\omega-2p-k+j+1)k!\,(2p-2j)!}.$$

Hence

$$B_{2p}^{\omega} = \sum_{j=0}^{p-1} \frac{(2x)^{2p-2j}}{(2p-2j)!} \sum_{k=0}^{j} C_k, \tag{18}$$

say, with

$$C_k = 2\frac{\Gamma(\omega+j-k+1)\Gamma(\omega+2p+k-j+1)(p+k-j)(-1)^k(2p+k-2j-1)!}{(j-k)!\,\Gamma(\omega-j+k+1)(2p+k-j)!\,\Gamma(\omega-2p-k+j+1)k!}.$$

In particular,

$$C_0 = \frac{(2p-2j)!\,\Gamma(\omega+j+1)\Gamma(\omega+2p-j+1)}{(2p-j)!\,j!\,\Gamma(\omega-j+1)\,\Gamma(\omega-2p+j+1)}. \tag{19}$$

Proceeding as in Section 3.1.1, we calculate

$$\frac{C_{k+1}}{C_k} = \frac{(k-j)(k+p-j+1)(k-\omega+2p-j)(k+\omega+2p-j+1)(k+2p-2j)}{(k+2p-j+1)(k+p-j)(k+\omega-j+1)(k-\omega-j)(k+1)}.$$

It follows that $C_{j+1} = 0$ and then $C_k = 0$ for $k \geq j+1$, implying that the k-sum in (18) can be written in terms of a generalized hypergeometric series:

$$\sum_{k=0}^{j} C_k = C_0\, {}_5F_4(a_1, a_2, a_3, a_4, a_5;\, b_1, b_2, b_3, b_4; 1), \tag{20}$$

where

$$a_1 = p-j+1,\ a_2 = -j,\ a_3 = -\omega+2p-j,\ a_4 = \omega+2p-j+1,\ a_5 = 2p-2j,$$
$$b_1 = p-j,\ b_2 = 2p-j+1,\ b_3 = \omega-j+1,\ b_4 = -\omega-j.$$

We notice that $a_1 + b_1 = a_2 + b_2 = a_3 + b_3 = a_4 + b_4 = a_5 + 1$: the series is well poised. In addition, $a_1 = b_1 + 1$: the series is *very well poised* [6, §16.4(i)]. Hence we can use the Rogers–Dougall formula [6, 16.4.9] (with $a = a_5$, $b = a_3$, $c = a_4$ and $d = a_2$ therein). Thus, denoting the right-hand side of (20) by $C_0\, {}_5F_4(1)$, we find that

$${}_5F_4(1) = \frac{(2p-j)!\,\Gamma(\omega-j+1)\,\Gamma(-\omega-j)\,\Gamma(-2p+j)}{(2p-2j)!\,\Gamma(-2p)\,\Gamma(\omega+1)\,\Gamma(-\omega)}.$$

Multiplying by C_0, (19),

$$\sum_{k=0}^{j} C_k = \frac{\Gamma(\omega+j+1)\Gamma(\omega+2p-j+1)}{j!\,\Gamma(\omega-2p+j+1)} \frac{\Gamma(-\omega-j)\,\Gamma(-2p+j)}{\Gamma(-2p)\,\Gamma(\omega+1)\,\Gamma(-\omega)}.$$

Now

$$\frac{\Gamma(\omega+j+1)\,\Gamma(-\omega-j)}{\Gamma(\omega+1)\,\Gamma(-\omega)} = \frac{\sin(\pi[-\omega])}{\sin(\pi[-\omega-j])} = (-1)^j$$

whence

$$\sum_{k=0}^{j} C_k = \frac{(-1)^j\Gamma(\omega+2p-j+1)\,\Gamma(-2p+j)}{j!\,\Gamma(\omega-2p+j+1)\,\Gamma(-2p)}.$$

Furthermore, we have

$$\frac{\Gamma(\omega+2p-j+1)}{\Gamma(\omega-2p+j+1)} = (2p-j)!\,a_{2p-j}^{\omega}$$

and

$$\begin{aligned}\frac{\Gamma(-2p+j)}{\Gamma(-2p)} &= (-2p)(-2p+1)\cdots(-2p+j-1)\\ &= (-1)^j(2p)(2p-1)\cdots(2p-j+1) = \frac{(-1)^j(2p)!}{(2p-j)!}.\end{aligned}$$

Hence, we obtain a very simple formula for the k-sum,

$$\sum_{k=0}^{j} C_k = \frac{(2p)!}{j!}\,a_{2p-j}^{\omega}. \tag{21}$$

3.1.3. Synthesis

Combining (15), (17), (18) and (21),

$$\begin{aligned}S_{2p}^{n}(\delta) = A_{2p}^{n} + B_{2p}^{n} &= \frac{(2p)!}{p!}\,a_p^n + \sum_{j=0}^{p-1}\frac{(2x)^{2p-2j}}{(2p-2j)!}\frac{(2p)!}{j!}\,a_{2p-j}^{n}\\ &= \sum_{j=0}^{p}\frac{(2p)!\,(2\cos\delta)^{2p-2j}}{j!\,(2p-2j)!}\,a_{2p-j}^{n},\end{aligned} \tag{22}$$

which is clearly positive for all δ.

3.2. Evaluation of S_{2p+1}^{n}

From (11), we have

$$\begin{aligned}S_{2p+1}^{\omega} &= \sum_{k=0}^{2p+1} a_k^{\omega}a_{2p+1-k}^{\omega}\cos\{(2p-2k+1)\delta\}\\ &= \sum_{k=0}^{p} a_k^{\omega}a_{2p+1-k}^{\omega}\cos\{(2p-2k+1)\delta\} + \sum_{k=p+1}^{2p+1} a_k^{\omega}a_{2p+1-k}^{\omega}\cos\{(2k-2p-1)\delta\}.\end{aligned}$$

Put $k = p - q$ in the first sum and $k = p + q + 1$ in the second sum, giving

$$S^{\omega}_{2p+1} = 2\sum_{q=0}^{p} a^{\omega}_{p-q} a^{\omega}_{p+q+1} T_{2q+1}(x),$$

with $x = \cos\delta$. Expanding the Chebyshev polynomial [5, p. 24],

$$T_{2m+1}(x) = \sum_{j=0}^{m} c_j^{(2m+1)} x^{2m+1-2j}, \quad m = 0, 1, 2, \ldots,$$

with

$$c_j^{(2m+1)} = (2m+1)(-1)^j \frac{2^{2m-2j}(2m-j)!}{j!\,(2m+1-2j)!}.$$

Hence

$$\begin{aligned} S^{\omega}_{2p+1} &= 2x\sum_{q=0}^{p} a^{\omega}_{p-q} a^{\omega}_{p+q+1} \sum_{j=0}^{q} c_j^{(2q+1)} x^{2q-2j} = 2x\sum_{q=0}^{p} a^{\omega}_{p-q} a^{\omega}_{p+q+1} \sum_{m=0}^{q} c_{q-m}^{(2q+1)} x^{2m} \\ &= 2x\sum_{m=0}^{p} x^{2m} \sum_{q=m}^{p} a^{\omega}_{p-q} a^{\omega}_{p+q+1} c_{q-m}^{(2q+1)} = 2x\sum_{j=0}^{p} x^{2p-2j} \sum_{k=0}^{j} a^{\omega}_{j-k} a^{\omega}_{2p+k-j+1} c_k^{(2p+2k-2j+1)}. \end{aligned}$$

Examining the terms in the k-sum, we find

$$\begin{aligned} & a^{\omega}_{j-k} a^{\omega}_{2p+k-j+1} c_k^{(2p+2k-2j+1)} \\ &\quad = \frac{\Gamma(\omega+j-k+1)\,\Gamma(\omega+2p+k-j+2)(2p+2k-2j+1)(-1)^k 2^{2p-2j}(2p+k-2j)!}{(j-k)!\,\Gamma(\omega-j+k+1)\,(2p+k-j+1)!\,\Gamma(\omega-2p-k+j)k!\,(2p-2j+1)!}. \end{aligned}$$

Hence

$$S^{\omega}_{2p+1} = \sum_{j=0}^{p} \frac{(2x)^{2p+1-2j}}{(2p-2j+1)!} \sum_{k=0}^{j} C_k \tag{23}$$

with

$$C_k = \frac{\Gamma(\omega+j-k+1)\,\Gamma(\omega+2p+k-j+2)(2p+2k-2j+1)(-1)^k(2p+k-2j)!}{(j-k)!\,\Gamma(\omega-j+k+1)\,(2p+k-j+1)!\,\Gamma(\omega-2p-k+j)k!}.$$

In particular

$$C_0 = \frac{(2p-2j+1)!\,\Gamma(\omega+j+1)\,\Gamma(\omega+2p-j+2)}{(2p-j+1)!\,j!\,\Gamma(\omega-j+1)\,\Gamma(\omega-2p+j)}. \tag{24}$$

Some calculation gives

$$\frac{C_{k+1}}{C_k} = \frac{(k+\omega+2p-j+2)(k+p-j+\frac{3}{2})(k+2p-2j+1)(k-j)(k-\omega+2p-j+1)}{(k-\omega-j)(k+p-j+\frac{1}{2})(k+\omega-j+1)(k+2p-j+2)(k+1)}.$$

It follows that the k-sum can be written as (20) with

$$a_1 = p - j + \tfrac{3}{2},\ a_2 = -j,\ a_3 = -\omega + 2p - j + 1,\ a_4 = \omega + 2p - j + 2,\ a_5 = 2p - 2j + 1,$$
$$b_1 = p - j + \tfrac{1}{2},\ b_2 = 2p - j + 2,\ b_3 = \omega - j + 1,\ b_4 = -\omega - j.$$

Again, the series is very well poised, and so it can be summed with the Rogers–Dougall formula:

$${}_5F_4(1) = \frac{(2p - j + 1)!\,\Gamma(\omega - j + 1)\Gamma(-\omega - j)\Gamma(-2p + j - 1)}{(2p - 2j + 1)!\,\Gamma(-2p - 1)\Gamma(1 + \omega)\Gamma(-\omega)}.$$

Multiplying by C_0, (24),

$$\begin{aligned}\sum_{k=0}^{j} C_k &= \frac{\Gamma(\omega + 2p - j + 2)\,\Gamma(\omega + j + 1)\Gamma(-\omega - j)\Gamma(-2p + j - 1)}{j!\,\Gamma(\omega - 2p + j)\,\Gamma(1 + \omega)\Gamma(-\omega)\,\Gamma(-2p - 1)} \\ &= a^{\omega}_{2p+1-j}\frac{(2p + 1 - j)!\,(-1)^j\,\Gamma(-2p + j - 1)}{j!\,\Gamma(-2p - 1)} \\ &= \frac{(2p + 1)!}{j!}\,a^{\omega}_{2p+1-j}.\end{aligned}$$

Hence, from (23),

$$S^n_{2p+1}(\delta) = \sum_{j=0}^{p} \frac{(2p + 1)!(2x)^{2p+1-2j}}{j!\,(2p + 1 - 2j)!}\,a^{\omega}_{2p+1-j}, \tag{25}$$

which is positive when $\cos\delta > 0$, that is, when $\operatorname{Re}\mu > 0$.

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