

On the far-field computation of acoustic radiation forces

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It is known that the steady acoustic radiation force on a scatterer due to incident time-harmonic waves can be calculated by evaluating certain integrals of velocity potentials over a sphere surrounding the scatterer. The goal is to evaluate these integrals using far-field approximations and appropriate limits. Previous derivations are corrected, clarified, and generalized. Similar corrections are made to textbook derivations of optical theorems. © 2017 Acoustical Society of America. https://doi.org/10.1121/1.5007721

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I. INTRODUCTION

Time-harmonic acoustic waves of small amplitude \mathcal{A} exert a steady second-order (proportional to \mathcal{A}^2) force on obstacles. This *acoustic radiation force* has been studied extensively; it may be used to levitate or manipulate small particles, for example. For reviews, see Refs. 1–3.

The computation of the radiation force can be reduced to the evaluation of an integral over a sphere of radius r, S_r . This sphere encloses the scatterer but is otherwise arbitrary: the value of the integral does not depend on r. The integrand involves quadratic combinations of first-order quantities. It is natural to try to simplify the calculation by letting $r \to \infty$, so that familiar far-field quantities appear: this was first done by Westervelt.^{4,5} However, it turns out that the limiting process is not straightforward.

To begin, recall the basic equations giving the radiation force in terms of the time-harmonic velocity potential Φ . For simplicity, consider axisymmetric motions only (implying that both the incident field and the geometry are axisymmetric about the z-axis) and compute the axial component of the radiation force, F_z . The relevant formulas involve contributions from the incident potential Φ_i and the scattered potential Φ_s , with $\Phi = \Phi_i + \Phi_s$.

At this point, all formulas are exact. Then far-field $(r \rightarrow \infty)$ approximations for Φ_s are introduced, followed by taking the limit as the sphere S_r expands.

First the two-dimensional (2-D) case is examined: scattering by a cylinder. The calculations are straightforward; leading-order far-field approximations for Φ_s suffice.

The three-dimensional (3-D) case is more complicated. An orders-of-magnitude argument shows that two contributions to F_z were omitted in earlier papers. One of these comes from a higher-order contribution to the far-field behavior of Φ_s : the leading-order term is proportional to 1/rbut the next term, proportional to $1/r^2$, should be retained. The second contribution comes from an angular derivative of Φ_s , tangential to S_r . Thus

$$F_z = \mathcal{O}(r) + \mathcal{N}(r) + o(1) \text{ as } r \to \infty,$$

where $\mathcal{O}(r)$ contains the old previously published contribution and $\mathcal{N}(r)$ contains the new contributions; all of these are defined by integrals over S_r . As F_z does not depend on r, it is natural to let $r \to \infty$. It is shown that $\mathcal{O}(r)$ has a well-defined limit, and that limit is computed for a wide class of incident fields Φ_i . This is done using a plane-wave representation (also known as a Herglotz wave function) for Φ_i containing a density function q; see Eq. (25) below. The resulting limit contains q and the far-field pattern f; see Eq. (29) below.

It is also shown that $\mathcal{N}(r) \to 0$ as $r \to \infty$. This is one of those situations where careful analysis does not lead to something new (apart from clarifying and justifying previous work).

Known results for incident plane waves and for incident Bessel beams are recovered.

In an Appendix, it is shown that similar criticisms can be made of some derivations of *optical* (or *forward scattering*) *theorems*. An optical theorem is proved for general incident fields defined using the density function q. It agrees with a formula found in Ref. 6.

II. BASIC EQUATIONS

The starting point is the following formula:

$$F_{z} = -\int_{S_{r}} \rho_{0} \langle u_{z} u_{r} \rangle \mathrm{d}S + \int_{S_{r}} \left\{ \frac{\rho_{0}}{2} \langle |\boldsymbol{u}|^{2} \rangle - \frac{1}{2\rho_{0}c_{0}^{2}} \langle p^{2} \rangle \right\} n_{z} \, \mathrm{d}S, \tag{1}$$

which agrees with Eqs. (3) and (4) in Ref. 7, Eq. (10) in Ref. 8, and Eq. (12) in Ref. 9, for example. This formula gives the component of the radiation force in the *z*-direction on the scatterer, which is assumed to be surrounded by inviscid compressible fluid of density ρ_0 .

In Eq. (1), u_z and u_r are components of the first-order velocity u, c_0 is the sound speed, p is the first-order pressure, S_r is a sphere (circle in two dimensions) of radius r enclosing the scatterer, and n_z is the z-component of the outward normal vector on S_r . The angled brackets denote time average over one period.

Introduce a first-order velocity potential ϕ , with $\phi = \operatorname{Re}\{\Phi e^{-i\omega t}\}$. Then

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$$u_z = \frac{\partial \phi}{\partial z}, \quad u_r = \frac{\partial \phi}{\partial r}, \quad |\boldsymbol{u}|^2 = |\nabla \phi|^2, \quad p = -\rho_0 \frac{\partial \phi}{\partial t}.$$

Time averaging gives

$$\langle u_z u_r \rangle = \frac{1}{2} \operatorname{Re} \left(\frac{\partial \Phi}{\partial r} \frac{\partial \Phi^*}{\partial z} \right), \quad \langle p^2 \rangle = \frac{1}{2} \rho_0^2 \omega^2 |\Phi|^2,$$

$$\langle |\boldsymbol{u}|^2 \rangle = \frac{1}{2} (\nabla \Phi) \cdot (\nabla \Phi^*),$$

where the asterisk denotes complex conjugation. Hence

$$F_{z} = -\rho_{0} \int_{S_{r}} \frac{1}{2} \operatorname{Re}\left(\frac{\partial \Phi}{\partial r} \frac{\partial \Phi^{*}}{\partial z}\right) \mathrm{d}S + \frac{\rho_{0}}{2} \int_{S_{r}} \frac{1}{2} \left\{ (\nabla \Phi) \cdot (\nabla \Phi^{*}) - k^{2} |\Phi|^{2} \right\} n_{z} \, \mathrm{d}S,$$

where $k = \omega/c_0$. This agrees with Eq. (11) in Ref. 8.

Suppose there is a given potential for the incident wave Φ_i and an unknown potential for the scattered waves Φ_s . Thus put $\Phi = \Phi_i + \Phi_s$ giving $F_z = F_z^i + F_z^s + F_z^c$, where F_z^i contains terms involving products of Φ_i with itself, F_z^s contains terms involving products of Φ_s with itself, and F_z^c contains cross terms. Noting that $\operatorname{Re} \mathcal{Z} = \frac{1}{2}(\mathcal{Z} + \mathcal{Z}^*)$ for any complex \mathcal{Z} ,

$$\begin{split} F_{z}^{i} &= -\frac{\rho_{0}}{4} \int_{S_{r}} \left\{ 2 \operatorname{Re} \left(\frac{\partial \Phi_{i}}{\partial r} \frac{\partial \Phi_{i}^{*}}{\partial z} \right) \right. \\ &- \left(|\nabla \Phi_{i}|^{2} - k^{2} |\Phi_{i}|^{2} \right) n_{z} \right\} \mathrm{d}S, \\ F_{z}^{s} &= -\frac{\rho_{0}}{4} \int_{S_{r}} \left\{ 2 \operatorname{Re} \left(\frac{\partial \Phi_{s}}{\partial r} \frac{\partial \Phi_{s}^{*}}{\partial z} \right) \right. \\ &- \left(|\nabla \Phi_{s}|^{2} - k^{2} |\Phi_{s}|^{2} \right) n_{z} \right\} \mathrm{d}S, \\ F_{z}^{c} &= -\frac{\rho_{0}}{4} \int_{S_{r}} \left\{ \frac{\partial \Phi_{i}}{\partial r} \frac{\partial \Phi_{s}^{*}}{\partial z} + \frac{\partial \Phi_{s}}{\partial r} \frac{\partial \Phi_{i}^{*}}{\partial z} \right. \\ &+ \frac{\partial \Phi_{i}^{*}}{\partial r} \frac{\partial \Phi_{s}}{\partial z} + \frac{\partial \Phi_{s}^{*}}{\partial r} \frac{\partial \Phi_{i}}{\partial z} \right\} \mathrm{d}S \\ &+ \frac{\rho_{0}}{2} \int_{S_{r}} \operatorname{Re} \left\{ (\nabla \Phi_{s}) \cdot (\nabla \Phi_{i}^{*}) - k^{2} \Phi_{s} \Phi_{i}^{*} \right\} n_{z} \mathrm{d}S \\ &= \frac{\rho_{0}}{2} \operatorname{Re} \int_{S_{r}} \left\{ \left[(\nabla \Phi_{s}) \cdot (\nabla \Phi_{i}^{*}) - k^{2} \Phi_{s} \Phi_{i}^{*} \right] n_{z} \\ &- \frac{\partial \Phi_{i}}{\partial r} \frac{\partial \Phi_{s}^{*}}{\partial z} - \frac{\partial \Phi_{s}}{\partial r} \frac{\partial \Phi_{i}^{*}}{\partial z} \right\} \mathrm{d}S. \end{split}$$

It is well known⁸ that $F_z^i \equiv 0$: there is no radiation force in the absence of scattering.

The formulas above for F_z^s and F_z^c are general: they do not depend on the shape or composition of the scatterer (which is inside S_r) and they do not depend on the form of the incident wave Φ_i . (Of course, the computed values of F_z^c and F_z^s do depend on all these factors.) Moreover, $F_z = F_z^s + F_z^c$ does not depend on the radius *r* (provided Φ_i is a regular solution of the governing Helmholtz equation). It is this *r*-independence that suggests simplifying the calculation by letting $r \to \infty$. The 2-D and 3-D cases are examined separately in Secs. III and IV.

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The 2-D case is straightforward. Simple far-field approximations for Φ_s lead to integrals that can be approximated using the method of stationary phase.

The 3-D case is more complicated. This paper is concerned with axisymmetric problems but it seems likely that more general problems can be treated using similar methods.

III. TWO DIMENSIONS

In two dimensions, S_r is a circle of radius r. In plane polar coordinates, $z = r \cos \theta$, $n_z = \cos \theta$, $dS = r d\theta$,

$$\frac{\partial \Phi}{\partial z} = \frac{\partial \Phi}{\partial r} \cos \theta - \frac{\sin \theta}{r} \frac{\partial \Phi}{\partial \theta},\tag{2}$$

$$(\nabla\Phi) \cdot (\nabla\Psi) = \frac{\partial\Phi}{\partial r} \frac{\partial\Psi}{\partial r} + \frac{1}{r^2} \frac{\partial\Phi}{\partial\theta} \frac{\partial\Psi}{\partial\theta}.$$
(3)

Hence

$$F_{z}^{s} = -\frac{\rho_{0}r}{4} \int_{0}^{2\pi} 2\operatorname{Re}\left[\frac{\partial\Phi_{s}}{\partial r}\left(\frac{\partial\Phi_{s}}{\partial r}\cos\theta - \frac{\sin\theta}{r}\frac{\partial\Phi_{s}^{*}}{\partial\theta}\right)\right] \mathrm{d}\theta$$
$$+\frac{\rho_{0}r}{4} \int_{0}^{2\pi} \left(\left|\frac{\partial\Phi_{s}}{\partial r}\right|^{2} + \frac{1}{r^{2}}\left|\frac{\partial\Phi_{s}}{\partial\theta}\right|^{2} - k^{2}|\Phi_{s}|^{2}\right)\cos\theta\,\mathrm{d}\theta$$
$$= -\frac{\rho_{0}r}{4} \int_{0}^{2\pi} \left(\left|\frac{\partial\Phi_{s}}{\partial r}\right|^{2} - \frac{1}{r^{2}}\left|\frac{\partial\Phi_{s}}{\partial\theta}\right|^{2} + k^{2}|\Phi_{s}|^{2}\right)\cos\theta\,\mathrm{d}\theta$$
$$+ \frac{\rho_{0}}{2} \int_{0}^{2\pi} \operatorname{Re}\left(\frac{\partial\Phi_{s}}{\partial r}\frac{\partial\Phi_{s}^{*}}{\partial\theta}\right)\sin\theta\,\mathrm{d}\theta. \tag{4}$$

This formula is exact. Now, allow r to become large. The radiation condition gives

$$\Phi_s(r,\theta) = \frac{f(\theta)}{\sqrt{kr}} e^{ikr} + O(r^{-3/2}) \quad \text{as } r \to \infty,$$
(5)

where *f* is the far-field pattern. It follows that the second integral in Eq. (4) is $O(r^{-1})$ as $r \to \infty$. Similarly, the middle term in the first integrand is negligible. Hence

$$F_z^s = -\frac{\rho_0 k}{2} \int_0^{2\pi} |f(\theta)|^2 \cos \theta \,\mathrm{d}\theta,\tag{6}$$

in agreement with the first term in Eq. (13) of Ref. 8.

Next, the cross terms give

$$F_{z}^{c} = -\frac{\rho_{0}r}{2}\operatorname{Re}\int_{0}^{2\pi} \left(\frac{\partial\Phi_{s}}{\partial r}\frac{\partial\Phi_{i}^{*}}{\partial r} - \frac{1}{r^{2}}\frac{\partial\Phi_{s}}{\partial\theta}\frac{\partial\Phi_{i}^{*}}{\partial\theta} + k^{2}\Phi_{s}\Phi_{i}^{*}\right)$$
$$\times \cos\theta \,\mathrm{d}\theta + \frac{\rho_{0}}{2}\operatorname{Re}\int_{0}^{2\pi} \left(\frac{\partial\Phi_{i}}{\partial r}\frac{\partial\Phi_{s}^{*}}{\partial\theta} + \frac{\partial\Phi_{s}}{\partial r}\frac{\partial\Phi_{i}^{*}}{\partial\theta}\right)\sin\theta \,\mathrm{d}\theta.$$

Using Eq. (5), the second integral is $O(r^{-1/2})$ as $r \to \infty$ (assuming that Φ_i and its derivatives do not grow with r). Similarly, the middle term in the first integrand is negligible, leaving

$$F_{z}^{c} = -\frac{\rho_{0}}{2} \lim_{r \to \infty} \int_{0}^{2\pi} r \operatorname{Re}\left(\left[ik\frac{\partial\Phi_{i}^{*}}{\partial r} + k^{2}\Phi_{i}^{*}\right] \times \frac{f(\theta) \operatorname{e}^{ikr}}{\sqrt{kr}}\right) \cos\theta \,\mathrm{d}\theta.$$

$$(7)$$

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This expression agrees with the second and third terms in Eq. (13) of Ref. 8 once the approximation $\partial \Phi_i^* / \partial z = (\partial \Phi_i^* / \partial r) \cos \theta + O(r^{-1})$ [see Eq. (2)] is used.

Now, the limit in Eq. (7) must exist because F_z^c does not depend on *r*. However, the limiting process seems delicate at first sight.

To examine further, choose a simple incident wave, a plane wave propagating at angle α to the *z*-axis with potential $\Phi_i = e^{ikr\cos(\theta - \alpha)}$. Substituting in Eq. (7) gives

$$F_{z}^{c} = -\frac{\rho_{0}k}{2} \lim_{r \to \infty} \left(\sqrt{kr} \operatorname{Re} \int_{0}^{2\pi} f(\theta) \times (1 + \cos[\theta - \alpha]) e^{ikr(1 - \cos[\theta - \alpha])} \cos \theta \, \mathrm{d}\theta \right).$$

To estimate the integral, use the method of stationary phase.¹⁰ Assuming for simplicity that $0 < \alpha < \pi$, the points of stationary phase are at $\theta = \alpha$ and $\theta = \alpha + \pi$. At the second of these, $1 + \cos(\theta - \alpha) = 0$ so that the dominant contribution comes from $\theta = \alpha$. That contribution is [using Eq. (6.1.5) in Ref. 10]

$$\sqrt{2\pi/(kr)} 2f(\alpha) \cos \alpha e^{i\pi/4} + O(r^{-1}),$$

whence

$$F_z^c = -\rho_0 k \sqrt{2\pi} \cos \alpha \operatorname{Re}\{f(\alpha) e^{i\pi/4}\}.$$
(8)

Then $F_z = F_z^c + F_z^s$ with F_z^s given by Eq. (6).

IV. THREE DIMENSIONS

In three dimensions, S_r is a sphere of radius r. Use spherical polar coordinates, r, θ , ϕ , with $z = r \cos \theta$. For simplicity, suppose that the problem is axisymmetric, with no dependence on ϕ . Then $n_z = \cos \theta$, $dS = 2\pi r^2 \sin \theta \, d\theta$, and Eqs. (2) and (3) remain valid. Hence

$$F_{z}^{s} = -\frac{\pi\rho_{0}r^{2}}{2}\int_{0}^{\pi} \left(\left| \frac{\partial\Phi_{s}}{\partial r} \right|^{2} - \frac{1}{r^{2}} \left| \frac{\partial\Phi_{s}}{\partial\theta} \right|^{2} + k^{2} |\Phi_{s}|^{2} \right) \\ \times \cos\theta \sin\theta \,\mathrm{d}\theta \\ + \pi\rho_{0}r \int_{0}^{\pi} \mathrm{Re} \left(\frac{\partial\Phi_{s}}{\partial r} \frac{\partial\Phi_{s}^{*}}{\partial\theta} \right) \sin^{2}\theta \,\mathrm{d}\theta.$$
(9)

This formula is exact. Now, allow r to become large. The radiation condition gives

$$\Phi_s(r,\theta) = \frac{f(\theta)}{ikr} e^{ikr} + O(r^{-2}) \quad \text{as } r \to \infty,$$
(10)

where *f* is the (axisymmetric) far-field pattern. It follows that the integrand in the second integral in Eq. (9) is $O(r^{-2})$ as $r \to \infty$. Similarly, the middle term in the first integrand is negligible. Hence, letting $r \to \infty$,

$$F_z^s = -\pi\rho_0 \int_0^\pi |f(\theta)|^2 \cos\theta \sin\theta \,\mathrm{d}\theta. \tag{11}$$

This agrees with the I_1 term in Eq. (3) of Ref. 11. It appears in many other papers going back at least as far as Westervelt [see Eq. (2) in Ref. 5].

Next, consider the contribution from the cross terms:

$$\begin{split} F_z^c &= \pi \rho_0 r^2 \operatorname{Re} \int_0^{\pi} \left(\frac{\partial \Phi_s}{\partial r} \frac{\partial \Phi_i^*}{\partial r} + \frac{1}{r^2} \frac{\partial \Phi_s}{\partial \theta} \frac{\partial \Phi_i^*}{\partial \theta} - k^2 \Phi_s \Phi_i^* \right) \\ &\times \cos \theta \sin \theta \, \mathrm{d}\theta - \pi \rho_0 r^2 \operatorname{Re} \\ &\times \int_0^{\pi} \left\{ \frac{\partial \Phi_i}{\partial r} \left(\frac{\partial \Phi_s^*}{\partial r} \cos \theta - \frac{\sin \theta}{r} \frac{\partial \Phi_s^*}{\partial \theta} \right) \right. \\ &+ \frac{\partial \Phi_s}{\partial r} \left(\frac{\partial \Phi_i^*}{\partial r} \cos \theta - \frac{\sin \theta}{r} \frac{\partial \Phi_i^*}{\partial \theta} \right) \right\} \sin \theta \, \mathrm{d}\theta \\ &= -\pi \rho_0 r^2 \operatorname{Re} \int_0^{\pi} \left(\frac{\partial \Phi_s}{\partial r} \frac{\partial \Phi_i^*}{\partial r} - \frac{1}{r^2} \frac{\partial \Phi_s}{\partial \theta} \frac{\partial \Phi_i^*}{\partial \theta} + k^2 \Phi_s \Phi_i^* \right) \\ &\times \cos \theta \sin \theta \, \mathrm{d}\theta + \pi \rho_0 r \operatorname{Re} \\ &\times \int_0^{\pi} \left(\frac{\partial \Phi_i}{\partial r} \frac{\partial \Phi_s^*}{\partial \theta} + \frac{\partial \Phi_s}{\partial r} \frac{\partial \Phi_i^*}{\partial \theta} \right) \sin^2 \theta \, \mathrm{d}\theta. \end{split}$$

Using the far-field approximation, Eq. (10), one difference from the 2-D case is seen immediately: the second integral is not negligible because the integrand is $O(r^{-1})$ as $r \to \infty$. However, in the first integral, the middle term is negligible, as before. Also two terms can be combined exactly using

$$\frac{\partial \Phi_i^*}{\partial z} = \frac{\partial \Phi_i^*}{\partial r} \cos \theta - \frac{\sin \theta}{r} \frac{\partial \Phi_i^*}{\partial \theta},$$

whence

$$F_{z}^{c} = -\pi\rho_{0}r^{2}\operatorname{Re}\int_{0}^{\pi} \left(\frac{\partial\Phi_{s}}{\partial r}\frac{\partial\Phi_{i}^{*}}{\partial z} + k^{2}\Phi_{s}\Phi_{i}^{*}\cos\theta\right)\sin\theta\,\mathrm{d}\theta$$
$$+\pi\rho_{0}r\operatorname{Re}\int_{0}^{\pi}\frac{\partial\Phi_{i}^{*}}{\partial r}\frac{\partial\Phi_{s}}{\partial\theta}\sin^{2}\theta\,\mathrm{d}\theta + O(r^{-1}), \quad r \to \infty.$$
(12)

If the approximation Eq. (10) is inserted in the first integral in Eq. (12), it is found to be approximately

$$-\pi\rho_0 r \operatorname{Re} \int_0^{\pi} e^{ikr} f(\theta) \left(\frac{\partial \Phi_i^*}{\partial z} - ik \Phi_i^* \cos \theta \right) \sin \theta \, \mathrm{d}\theta = F_1^c,$$
(13)

say; this agrees with Eq. (6) in Ref. 7, Eqs. (3), (5), and (6) in Ref. 11, and Eq. (14) in Ref. 9, taking the definition in Eq. (10) into account. However, this calculation ignores the second integral in Eq. (12). Moreover, the first integral in Eq. (12) is multiplied by r^2 , suggesting that the far-field approximation of Eq. (10) should be refined.

It is known that (see Corollary 3.8 in Ref. 12, for example)

$$\Phi_s(r,\theta) = \frac{e^{ikr}}{ikr} \left(f(\theta) + \frac{f_1(\theta)}{ikr} \right) + O(r^{-3})$$
(14)

as $r \to \infty$, where

$$f_1(\theta) = \frac{\mathrm{i}}{2\sin\theta} \frac{\mathrm{d}}{\mathrm{d}\theta} \left(\sin\theta \frac{\mathrm{d}f}{\mathrm{d}\theta}\right).$$

(In fact, $\Phi_s(r, \theta)$ can be reconstructed exactly for all $r > r_0$ from $f(\theta)$, where $r = r_0$ is the smallest sphere enclosing the scatterer; this is known as the Atkinson–Wilcox theorem.) Hence

$$\frac{\partial \Phi_s}{\partial r} = \frac{\mathrm{e}^{\mathrm{i}kr}}{r} \left(f + \frac{1}{\mathrm{i}kr} (f_1 - f) \right) + O(r^{-3}) \tag{15}$$

as $r \to \infty$. Substituting in Eq. (12) gives

$$F_z^c = F_1^c + F_2^c + O(r^{-1}),$$

where F_1^c is defined by Eq. (13) and

$$F_{2}^{c} = \frac{\pi\rho_{0}}{k} \operatorname{Im} \int_{0}^{\pi} e^{ikr} \left((f - f_{1}) \frac{\partial \Phi_{i}^{*}}{\partial z} + ikf_{1} \Phi_{i}^{*} \cos \theta + f' \frac{\partial \Phi_{i}^{*}}{\partial r} \sin \theta \right) \sin \theta \, \mathrm{d}\theta.$$
(16)

It will be shown that, in general, $F_2^c \to 0$ as $r \to \infty$.

A. Two examples

As before, the value of F_z^c does not depend on r, so that the limit $r \to \infty$ exists but, apparently, the limiting operation is delicate. To investigate, start with the simplest example, an axisymmetric incident plane wave, $\Phi_i = e^{ikz}$. From Eq. (13),

$$F_1^c = -\pi\rho_0 kr \operatorname{Im} \int_0^{\pi} \Psi(\theta) f(\theta) \left(1 + \cos\theta\right) \sin\theta \, \mathrm{d}\theta,$$

where $\Psi(\theta) = e^{ikr(1-\cos\theta)}$. As $\Psi'(\theta) = ikr\Psi(\theta)\sin\theta$,

$$\begin{split} F_1^c &= \pi \rho_0 \operatorname{Re} \int_0^{\pi} \Psi'(\theta) \left(1 + \cos \theta\right) f(\theta) \, \mathrm{d}\theta \\ &= -2\pi \rho_0 \operatorname{Re} \{f(0)\} \\ &- \pi \rho_0 \operatorname{Re} \int_0^{\pi} \Psi(\theta) \, \frac{\mathrm{d}}{\mathrm{d}\theta} \left\{ \left(1 + \cos \theta\right) f(\theta) \right\} \mathrm{d}\theta, \end{split}$$

after an integration by parts. The remaining integral $\rightarrow 0$ as $r \rightarrow \infty$; this can be shown using the method of stationary phase.¹⁰

A similar calculation shows that $F_2^c \to 0$ as $r \to \infty$, whence

$$F_z = -2\pi\rho_0 \operatorname{Re}\{f(0)\} - \pi\rho_0 \int_0^\pi |f(\theta)|^2 \cos\theta \sin\theta \,\mathrm{d}\theta.$$
(17)

The optical theorem, Eq. (A6), can be used to express $\operatorname{Re}{f(0)}$ in another way. In particular, if the scatterer is lossless, this gives

$$F_z = \pi \rho_0 \int_0^\pi |f(\theta)|^2 (1 - \cos \theta) \sin \theta \, \mathrm{d}\theta.$$
(18)

This agrees with Eq. (13) in Ref. 11.

For a second example, consider an incident axisymmetric Bessel beam, defined by

$$\Phi_i = \mathrm{e}^{\mathrm{i}kz\cos\beta} J_0(k\varrho\sin\beta),\tag{19}$$

where β is a parameter, $\rho = r \sin \theta$, and $z = r \cos \theta$ are cylindrical polar coordinates, and J_0 is a Bessel function. Substitution in Eq. (13) gives

$$F_1^c = -\pi \rho_0 kr \operatorname{Im} \int_0^{\pi} e^{ikr} g(\theta) \Phi_i^* d\theta$$
$$= -\frac{1}{2} \rho_0 kr \operatorname{Im} \int_0^{\pi} \int_{-\delta}^{2\pi-\delta} g(\theta) e^{ikr\phi(\theta,\psi)} d\psi d\theta, \qquad (20)$$

where $g(\theta) = f(\theta) (\cos \beta + \cos \theta) \sin \theta$,

$$\varphi(\theta,\psi) = 1 - \cos\theta\cos\beta + \sin\theta\sin\beta\cos\psi$$

and a convenient integral representation for J_0 ,

$$J_0(w) = \frac{1}{2\pi} \int_{-\delta}^{2\pi-\delta} \mathrm{e}^{\mathrm{i}w\cos\psi} \, d\psi, \quad 0 < \delta < \pi$$

has been used. The integral in Eq. (20) can be estimated for large *kr* using the 2-D method of stationary phase.¹⁰ Assume for simplicity that $0 < \beta < \pi$. Then there is just one relevant point of stationary phase, and it is at $(\theta, \psi) = (\beta, \pi) = x_0$, say. (Here, $g(0) = g(\pi) = g(\pi - \beta) = 0$ has been used to eliminate other potential stationary-phase points.) Then, using Eq. (8.4.44) in Ref. 10, the double integral asymptotes to

$$\frac{2\pi}{kr}\frac{g(\mathbf{x}_0)}{\sqrt{|\det A|}}\exp\left\{ikr\varphi(\mathbf{x}_0)+i(\pi/4)\operatorname{sig} A\right\}$$

as $r \to \infty$, where the 2×2 matrix *A* has entries $A_{ij} = \partial^2 \varphi / \partial \xi_i \partial \xi_j$ evaluated at \mathbf{x}_0 , $\xi_1 = \theta$, $\xi_2 = \psi$, and sig *A* is the signature of *A*, equal to the number of positive eigenvalues minus the number of negative eigenvalues. Some calculation gives

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \beta \end{pmatrix}, \quad \det A = \sin^2 \beta, \quad \operatorname{sig} A = 2,$$

 $g(\mathbf{x}_0) = g(\beta) = 2f(\beta) \cos \beta \sin \beta$ and $\varphi(\mathbf{x}_0) = \varphi(\beta, \pi) = 0$. Then, letting $r \to \infty$, Eq. (20) gives

$$F_1^c = -2\pi\rho_0 \operatorname{Re}\{f(\beta)\}\cos\beta.$$
(21)

In the limit $\beta \to 0$, the plane-wave result, Eq. (17), is recovered as expected.

A similar calculation shows that $F_2^c \to 0$ as $r \to \infty$, whence

$$F_z = -2\pi\rho_0 \operatorname{Re}\{f(\beta)\}\cos\beta - \pi\rho_0 \int_0^{\pi} |f(\theta)|^2 \cos\theta\sin\theta \,\mathrm{d}\theta.$$
(22)

As for plane-wave incidence, an appropriate form of the optical theorem, Eq. (A7), can be used to express $\operatorname{Re}\{f(\beta)\}\$ in another way. In particular, if the scatterer is lossless,

$$F_z = \pi \rho_0 \int_0^\pi |f(\theta)|^2 (\cos\beta - \cos\theta) \sin\theta \,\mathrm{d}\theta, \tag{23}$$

which agrees with Eq. (8) in Ref. 6.

B. Justification and generalization

Return to F_1^c , defined by Eq. (13), which is rewritten as an integral over the unit sphere Ω ,

$$F_1^c = -\frac{\rho_0 r}{2} \operatorname{Re} \int_{\Omega} e^{ikr} f(\hat{\boldsymbol{r}}) \left(\frac{\partial \Phi_i^*}{\partial z} - ik \Phi_i^* \, \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{z}} \right) \mathrm{d}\Omega, \quad (24)$$

where $\hat{r} = r/r$ and \hat{z} is a unit vector in the *z*-direction. The goal is to calculate $\lim_{r\to\infty} F_1^c$.

For a general incident wave, express Φ_i using a planewave representation,⁶

$$\Phi_i(\mathbf{r}) = \frac{1}{4\pi} \int_{\Omega} q(\hat{\mathbf{s}}) \,\mathrm{e}^{\mathrm{i}k\mathbf{r}\cdot\hat{\mathbf{s}}} \,\mathrm{d}\Omega(\hat{\mathbf{s}}),\tag{25}$$

where the density function q is specified on Ω . This kind of integral representation goes back to Whittaker¹³ but is often known as a *Herglotz wave function*. Substituting in Eq. (24) gives

$$F_1^c = -\frac{\rho_0 k r}{8\pi} \operatorname{Im} \int_{\Omega} q^*(\hat{s}) \int_{\Omega} e^{i k r (1 - \hat{r} \cdot \hat{s}) \times (\hat{s} + \hat{r}) \cdot \hat{z} f(\hat{r}) \, \mathrm{d}\Omega(\hat{r}) \, \mathrm{d}\Omega(\hat{s}).}$$
(26)

The inner integral can be estimated for large kr using the 2-D method of stationary phase.¹⁰ Doing this more generally leads to what is known as the *Jones lemma* (see Appendix XII in Ref. 14): for smooth functions *G*,

$$\int_{\Omega} G(\hat{\boldsymbol{r}}) e^{-ikr\cdot\hat{\boldsymbol{s}}} d\Omega(\hat{\boldsymbol{r}}) = \frac{2\pi i}{kr} \{ G(\hat{\boldsymbol{s}}) e^{-ikr} - G(-\hat{\boldsymbol{s}}) e^{ikr} \} + o(r^{-1}) \quad \text{as } r \to \infty.$$
(27)

When this result is applied to F_1^c , Eq. (26), it is found that

$$F_{1}^{c} = -\frac{\rho_{0}}{2} \operatorname{Re} \int_{\Omega} q^{*}(\hat{s}) f(\hat{s}) \, \hat{s} \cdot \hat{z} \, \mathrm{d}\Omega(\hat{s}) + o(1)$$
(28)

as $r \to \infty$. This result was also derived in Ref. 6.

A similar analysis for F_2^c , defined by Eq. (16), shows that $F_2^c \to 0$ as $r \to \infty$.

Reverting to the earlier notation, combine Eqs. (11) and (28) and obtain

$$F_z = -\pi\rho_0 \operatorname{Re} \int_0^{\pi} \{q^*(\theta) + f^*(\theta)\} f(\theta) \cos\theta \sin\theta \,\mathrm{d}\theta.$$
(29)

C. Axisymmetric Bessel beam: Reprise

Here the formula Eq. (29) is verified for the Bessel beam, Eq. (19). It has the expansion

$$\Phi_{i}(r,\theta) = e^{ikr\cos\beta\cos\theta}J_{0}(kr\sin\beta\sin\theta)$$

= $\sum_{n=0}^{\infty} (2n+1)i^{n}j_{n}(kr)P_{n}(\cos\theta)P_{n}(\cos\beta),$
(30)

where j_n is a spherical Bessel function and P_n is a Legendre polynomial. This expansion is a special case of a formula given by Watson [put $\nu = \frac{1}{2}$ in Eq. (9) on p. 370 of Ref. 16]; see also Eq. (B2) in Ref. 17. For a quick proof, note that $\Phi_i(r, \theta)$ is an axisymmetric solution of the Helmholtz equation, so it is sufficient to check Eq. (30) on the axis where $\theta = 0$, using $J_0(0) = 1$, $P_n(1) = 1$ and the known expansion for $e^{ikr\cos\beta}$ (Theorem 3.19 in Ref. 15).

Axisymmetry also implies that $q(\hat{s}) = q(\psi)$ with $\hat{s} \cdot \hat{z} = \cos \psi$. Then, on the axis, Eq. (25) gives

$$\Phi_i(r,0) = \frac{1}{2} \int_0^{\pi} q(\psi) \,\mathrm{e}^{\mathrm{i}kr\cos\psi} \sin\psi \,\mathrm{d}\psi.$$

Expanding the exponential and then comparing with Eq. (30) evaluated at $\theta = 0$ gives

$$q_n \equiv \int_0^{\pi} q(\theta) P_n(\cos \theta) \sin \theta \, \mathrm{d}\theta = 2 P_n(\cos \beta). \tag{31}$$

Now, although $q(\theta)$ does not have a convergent Legendre expansion, the far-field pattern does have such an expansion:

$$f(\theta) = \sum_{n=0}^{\infty} f_n P_n(\cos \theta).$$
(32)

Substitution in Eq. (29) leads to

$$\int_{0}^{\pi} q^{*}(\theta) f(\theta) \cos \theta \sin \theta \, \mathrm{d}\theta = \sum_{n=0}^{\infty} f_{n} I_{n}, \tag{33}$$

where (as q is real)

$$I_n = \int_0^{\pi} q(\theta) P_n(\cos \theta) \cos \theta \sin \theta \, \mathrm{d}\theta.$$

Evaluation gives $I_0 = q_1$ and

$$(2n+1)I_n = (n+1)q_{n+1} + nq_{n-1}, \quad n = 1, 2, 3, ...,$$

using $(2n+1)tP_n(t) = (n+1)P_{n+1}(t) + nP_{n-1}(t)$. Substituting in Eq. (33), using Eq. (31), the sum is recognized as $2f(\beta)$. Hence Eq. (29) reduces to Eq. (22).

V. CONCLUSIONS

A number of formulas for the acoustic radiation force on a scatterer have been derived. The derivations take account of contributions that had been ignored previously but, nevertheless, existing formulas are recovered in the limit as the radius of the surrounding sphere tends to infinity. Formulas for this limit were found for general incident waves, Eq. (29), and these were verified for plane waves and for Bessel beams. Generalizations beyond the axial component of the force and to arbitrary scatterers should be feasible.

Similar derivations have been given of optical theorems. Again, these derivations correct and clarify previous work, although they do not change the final formulas in the theorems themselves.

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APPENDIX: OPTICAL THEOREMS

Applying Green's theorem to Φ and Φ^* in the region between the sphere S_r and the scatterer gives

$$\operatorname{Im} \int_{S_r} \Phi^* \frac{\partial \Phi}{\partial r} \, \mathrm{d}S = \operatorname{Im} \int_S \Phi^* \frac{\partial \Phi}{\partial n} \, \mathrm{d}S,\tag{A1}$$

where the normal vector on the scatterer's surface, *S*, points outwards. Denote the right-hand side of Eq. (A1) by I_{ab} ; it represents energy absorbed by the scatterer. For lossless scatterers (for example, if $\Phi = 0$ on *S* or $\partial \Phi / \partial n = 0$ on *S*), $I_{ab} = 0$. Put $\Phi = \Phi_i + \Phi_s$ and expand the left-hand side of Eq. (A1). Assuming Φ_i satisfies the Helmholtz equation everywhere inside S_r , Im $\int_{S_r} \Phi_i^* (\partial \Phi_i / \partial r) \, dS = 0$. Hence

$$\operatorname{Im} \int_{S_r} \Phi_s^* \frac{\partial \Phi_s}{\partial r} \, \mathrm{d}S + \operatorname{Im} \int_{S_r} \left(\Phi_i^* \frac{\partial \Phi_s}{\partial r} - \Phi_s \frac{\partial \Phi_i^*}{\partial r} \right) \mathrm{d}S = I_{\mathrm{ab}}.$$
(A2)

Abandoning the restriction to axisymmetry, amend Eq. (14) to

$$\Phi_s(\mathbf{r}) = \frac{e^{ikr}}{ikr} \left(f(\hat{\mathbf{r}}) + \frac{f_1(\hat{\mathbf{r}})}{ikr} \right) + O(r^{-3})$$
(A3)

as $r = |\mathbf{r}| \to \infty$, where $\hat{\mathbf{r}} = \mathbf{r}/r$. Using the leading-order approximation for Φ_s in Eq. (A2) and letting $r \to \infty$ gives

$$\frac{1}{k} \int_{\Omega} |f(\hat{\boldsymbol{r}})|^2 \mathrm{d}\Omega(\hat{\boldsymbol{r}}) + \lim_{r \to \infty} \mathrm{Im} \int_{S_r} \left(\Phi_i^* \frac{\partial \Phi_s}{\partial r} - \Phi_s \frac{\partial \Phi_i^*}{\partial r} \right) \mathrm{d}S = I_{\mathrm{ab}},$$
(A4)

where Ω is the unit sphere. Denote the second integral by I_2 ; as its integrand is $O(r^{-1})$ but $dS = r^2 d\Omega$, retain both terms in Eq. (A3). Hence

$$I_{2} = r \int_{\Omega} e^{ikr} \left\{ \left(f + \frac{f_{1} - f}{ikr} \right) \Phi_{i}^{*} - \frac{1}{ik} \left(f + \frac{f_{1}}{ikr} \right) \frac{\partial \Phi_{i}^{*}}{\partial r} \right\} d\Omega + O(r^{-1})$$
(A5)

as $r \to \infty$.

For a plane wave, $\Phi_i = e^{ikz}$. If the scatterer is axisymmetric, $f(\hat{r}) = f(\theta)$. Then I_2 simplifies to

$$\frac{2\pi}{\mathrm{i}k}\int_0^{\pi} \left\{ f + \frac{f_1 - f}{\mathrm{i}kr} + \left(f + \frac{f_1}{\mathrm{i}kr} \right) \cos \theta \right\} \Psi'(\theta) \mathrm{d}\theta + O(r^{-1})$$

as $r \to \infty$, with $\Psi(\theta) = \exp \{ikr(1 - \cos \theta)\}$. After an integration by parts, let $r \to \infty$ whence Eq. (A4) gives

$$\int_0^{\pi} |f(\theta)|^2 \sin \theta \, \mathrm{d}\theta + 2 \operatorname{Re}\left\{f(0)\right\} = \frac{k}{2\pi} I_{\mathrm{ab}}.$$
 (A6)

This formula is usually known as the *optical theorem* or the *forward scattering theorem*.

For the Bessel beam, Eq. (19), and an axisymmetric scatterer, we can use the same stationary-phase method as used to derive Eq. (21); this shows that $I_2 = 4\pi i k^{-1} f(\beta) + O(r^{-1})$ as $r \to \infty$. Hence Eq. (A4) gives

$$\int_0^{\pi} |f(\theta)|^2 \sin \theta \, \mathrm{d}\theta + 2 \operatorname{Re}\left\{f(\beta)\right\} = \frac{k}{2\pi} I_{\mathrm{ab}}.$$
 (A7)

This result is equivalent to Eq. (2) in Ref. 18. It reduces to Eq. (A6) when $\beta \rightarrow 0$, as expected.

For a more general incident wave, write Φ_i using the plane-wave representation, Eq. (25). Substituting in Eq. (A5) gives

$$I_{2} = \frac{r}{4\pi} \int_{\Omega} q^{*}(\hat{s}) \int_{\Omega} e^{ikr(1-\hat{r}\cdot\hat{s})} (1+\hat{r}\cdot\hat{s})f(\hat{r}) \,\mathrm{d}\Omega(\hat{r}) \,\mathrm{d}\Omega(\hat{s}) + \frac{1}{4\pi \mathrm{i}k} \int_{\Omega} q^{*}(\hat{s}) \int_{\Omega} e^{ikr(1-\hat{r}\cdot\hat{s})} \times \left\{ (1+\hat{r}\cdot\hat{s})f_{1}(\hat{r}) - f(\hat{r}) \right\} \mathrm{d}\Omega(\hat{r}) \,\mathrm{d}\Omega(\hat{s}).$$

The inner integrals can be estimated for large kr using the Jones lemma, Eq. (27), giving

$$I_2 = \frac{\mathrm{i}}{k} \int_{\Omega} q^*(\hat{s}) f(\hat{s}) \,\mathrm{d}\Omega(\hat{s}) + o(1) \quad \text{as } r \to \infty$$

(with no dependence on f_1) and then Eq. (A4) gives

$$\int_{\Omega} |f(\hat{\boldsymbol{r}})|^2 \mathrm{d}\Omega(\hat{\boldsymbol{r}}) + \mathrm{Re} \int_{\Omega} q^*(\hat{\boldsymbol{r}}) f(\hat{\boldsymbol{r}}) \,\mathrm{d}\Omega(\hat{\boldsymbol{r}}) = k I_{\mathrm{ab}}.$$
 (A8)

This is a generalization of the optical theorem. It holds for arbitrary scatterers. It agrees with Eq. (9) in Ref. 6.

For a specific example, consider the scattering of an axisymmetric Bessel beam, Eq. (19), by an axisymmetric scatterer. Then if the far-field pattern is expanded as Eq. (32), substitution in the second integral in Eq. (A8) gives

$$\int_{\Omega} q^*(\hat{\boldsymbol{r}}) f(\hat{\boldsymbol{r}}) \, \mathrm{d}\Omega(\hat{\boldsymbol{r}}) = 4\pi \sum_{n=0}^{\infty} f_n P_n(\cos\beta) = 4\pi f(\beta),$$

using the Legendre coefficients of q, given by Eq. (31). Hence Eq. (A8) gives Eq. (A7), as before.

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