ON MIXED BOUNDARY-VALUE PROBLEMS IN A WEDGE

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[Received 8 December 2016. Revise 16 March 2017. Accepted 30 May 2017]

Summary

Mixed boundary-value problems for Laplace's equation inside a wedge-shaped region are formulated and solved. There is a homogeneous Neumann condition on both straight sides of the wedge except for one finite piece of one side where a Dirichlet condition is imposed. Solutions are sought with specified logarithmic behaviour at both the tip of the wedge and at infinity. Exact solutions are constructed by solving an integral equation.

1. Introduction

In the first volume of the *Quarterly Journal of Mechanics and Applied Mathematics*, there is a paper by C. J. Tranter on 'The use of the Mellin transform in finding the stress distribution in an infinite wedge' (**1**). Indeed, this has become the traditional method for solving boundary-value problems in wedge-shaped regions. For the method to work, the Mellin transform of the unknown function,

$$U(z, \theta) = \int_0^\infty r^{z-1} u(r, \theta)\, dr$$

must exist for z in a strip in the complex z-plane, $\sigma_0 < \operatorname{Re} z < \sigma_\infty$. Unfortunately, there may not always be such a strip.z For example, in the context of Laplace's equation, simple solutions such as $u = 1$ and $u = \log r$ do not have Mellin transforms. On the other hand, if $u(r, \theta) = O(\log r)$ as $r \to 0$ and $u(r, \theta) = O(r^{-\gamma})$ as $r \to \infty$, with $\gamma > 0$, then we can take $\sigma_0 = 0$ and $\sigma_1 = \gamma$.

The difficulty described above is easily remedied for pure boundary-value problems, such as the Neumann problem for Laplace's equation (section 3). But our interest is in mixed boundary-value problems; in a wedge $0 < \theta < \alpha$, we place homogeneous Neumann conditions on both $\theta = 0$ and $\theta = \alpha$ except for one piece of $\theta = 0$, $a < r < b$, where a Dirichlet condition is imposed. This can be seen as an anti-plane elasticity problem with a 'punch' on one face of the wedge; indeed, our main interest is with plane-strain elasticity problems and composite wedges, but the present harmonic problem may have some independent interest.

As one might expect, the behaviour of u near $r = 0$ and as $r \to \infty$ has to be restricted. We allow logarithmic growth at both locations, with $u \sim B_0 \log r$ as $r \to 0$ and $u \sim B_\infty \log r$ as $r \to \infty$. The constants B_0 and B_∞ can be specified arbitrarily. Once that has been done, the mixed boundary-value problem is uniquely solvable. We construct the solution, explicitly, by solving an integral equation. The article ends with some remarks on analogous problems arising in plane elastostatics.

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Q. Jl Mech. Appl. Math, Vol. 70. No. 4
Advance Access publication 29 July 2017. doi:10.1093/qjmam/hbx013

The problem of solving Laplace's equation in a wedge, with a variety of boundary conditions, can be solved by a variety of methods. One possibility is to open out the wedge into a half-plane using a conformal mapping. (Later, essentially this mapping is used to simplify an integral equation; see (4.24) below.) We do not use this approach because we are mainly interested in plane-strain elasticity problems (although such problems may be solved using Kolosov–Muskhelishvili potentials (**2**)) and because we want to keep control of the singular behaviour near $r = 0$ and as $r \to \infty$. Another possibility would be to introduce an appropriate Green function, as done by Williams (**3**). A direct treatment using Mellin transforms seems preferable; this is reminiscent of how D. S. Jones simplified the application of the Wiener–Hopf technique (**4**, p. vii).

2. Mellin transforms

Define the Mellin transform of f by

$$F(z) = \mathcal{M}\{f\} = \int_0^\infty r^{z-1} f(r)\, dr, \tag{2.1}$$

where $z = \sigma + i\tau$ is a complex variable. Typically, $F(z)$ is an analytic function of z in a strip, $\sigma_0 < \sigma < \sigma_\infty$, where the numbers σ_0 and σ_∞ are determined by the behaviour of $f(r)$ as $r \to 0$ and $r \to \infty$, respectively. For example, if

$$f(r) \sim f_0 r^\beta \quad \text{as } r \to 0 \quad \text{and} \quad f(r) \sim f_\infty r^{-\gamma} \quad \text{as } r \to \infty, \tag{2.2}$$

then $\sigma_0 = -\beta$ and $\sigma_\infty = \gamma$.

The inverse Mellin transform is

$$f(r) = \frac{1}{2\pi i} \int_{\mathrm{Br}} r^{-z} F(z)\, dz, \tag{2.3}$$

where Br is a Bromwich contour in the strip of analyticity; in other words, when f satisfies (2.2), Br is a contour $\sigma = c$, $-\infty < \tau < \infty$ with $\sigma_0 < c < \sigma_\infty$.

3. Neumann problem

3.1 *Formulation*

Suppose that $u(r, \theta)$ satisfies Laplace's equation, $\nabla^2 u = 0$, in a two-dimensional wedge-shaped region, $r > 0$, $0 < \theta < \alpha$, where r and θ are plane polar coordinates. *We henceforth assume that all lengths have been scaled so that r is dimensionless: this is important because we will encounter* $\log r$.

There are Neumann boundary conditions on the two sides of the wedge,

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = 0, \quad \theta = \alpha, \quad r > 0, \tag{3.1}$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = f(r), \quad \theta = 0, \quad r > 0, \tag{3.2}$$

where f is a specified function. (Sufficient conditions on f in terms of $F = \mathcal{M}\{f\}$ are readily derived so as to justify the subsequent analysis.) Evidently, the solution of the problem, u, is not unique because we can always add $\mathcal{A} + \mathcal{B} \log r$, where $\mathcal{A}$ and $\mathcal{B}$ are arbitrary constants.

Suppose that $u \sim B_0 \log r$ as $r \to 0$ and $u \sim B_\infty \log r$ as $r \to \infty$. Green's theorem then gives

$$\alpha(B_\infty - B_0) = \int_0^\infty f(r)\,dr = F(1), \tag{3.3}$$

using (2.1). Equation (3.3) is a relation between B_0, B_∞ and f. If we interpret u as a velocity potential for an incompressible flow, (3.3) expresses mass conservation: fluid entering through the straight wall at $\theta = 0$ must leave through the tip at $r = 0$ or flow out to infinity.

Now, to have a uniquely solvable boundary-value problem for u, we specify that

$$u(r,\theta) \to 0 \quad \text{as } r \to \infty,\ 0 < \theta < \alpha. \tag{3.4}$$

Near the tip, we have

$$u(r,\theta) = A_0 + B_0 \log r + o(1) \quad \text{as } r \to 0, \tag{3.5}$$

where, from (3.3),

$$B_0 = -F(1)/\alpha. \tag{3.6}$$

The constant A_0 will be calculated (see below) and is not arbitrary. (It is analogous to the so-called *blockage coefficient* in other potential flows.) Once u has been determined, we can add any arbitrary constant and any multiple of $\log r$; for example, subtracting $B_0 \log r$ would give a solution that is bounded at $r = 0$ (but logarithmically large at infinity).

3.2 *Solution*

The function $r^{-z} \cos(z(\theta - \alpha))$ satisfies Laplace's equation and the boundary condition at $\theta = \alpha$, (3.1), for any choice of z. Therefore, we consider the (Mellin) superposition,

$$u(r,\theta) = \frac{1}{2\pi i} \int_{\mathrm{Br}} g(z)\, r^{-z} \cos(z(\theta - \alpha))\,dz, \tag{3.7}$$

where g is unspecified and Br is an appropriate Bromwich contour; both will be chosen later. Then the boundary condition at $\theta = 0$, (3.2), gives

$$f(r) = \frac{1}{2\pi i} \int_{\mathrm{Br}} z g(z)\, r^{-z-1} \sin z\alpha\,dz, \qquad r > 0.$$

As $\mathcal{M}\{r f(r)\} = F(z+1)$, we obtain $F(z+1) = z g(z) \sin z\alpha$, whence (3.7) gives

$$u(r,\theta) = \frac{1}{2\pi i} \int_{\mathrm{Br}} F(z+1)\, r^{-z} \frac{\cos(z(\theta - \alpha))}{z \sin z\alpha}\,dz. \tag{3.8}$$

The integrand has a double pole at $z = 0$ and simple poles at $z = \pm\sigma_n$, where $\sigma_n = n\pi/\alpha$, $n = 1, 2, \ldots$. There may be additional singularities due to the presence of $F(z+1)$.

The main idea now is to move the inversion contour Br in (3.8), picking up residue contributions as we encounter poles. Moving to the left will generate an expansion that is useful near $r = 0$, whereas moving to the right will produce an expansion that is effective for large r.

To start, we have to specify where to locate Br in (3.8). Because of (3.4) and (3.5), we choose Br just to the right of $z = 0$. Near $z = 0$, $F(z+1) \sim F(1) + zF'(1)$, $r^{-z} = \exp(-z \log r) \sim 1 - z \log r$ and the integrand in (3.8) is approximately

$$\frac{F(1)}{\alpha z^2} + \frac{F'(1) - F(1)\log r}{\alpha z}.$$

Moving the contour to the left, the residue at $z = 0$ gives (3.5) with $B_0 = -F(1)/\alpha$ (in agreement with (3.6)) and $A_0 = F'(1)/\alpha$.

Residues from the simple poles at $z = -\sigma_n$ generate terms

$$\frac{(-1)^{n+1}}{n\pi} r^{\sigma_n} F(1-\sigma_n) \cos(\sigma_n(\theta - \alpha)).$$

There may also be residue contributions from F. For example, suppose that $f(r) = f_0$ for $0 < r < r_0$ and is zero otherwise, where f_0 and r_0 are constants. Then

$$F(z+1) = \int_0^{r_0} f_0 r^z \, dr = \frac{f_0 r_0^{z+1}}{z+1}$$

has a simple pole at $z = -1$. This contributes a term

$$\frac{f_0}{\sin\alpha} r \cos(\theta - \alpha)$$

to the expansion of u.

If we move Br in (3.8) to the right, we encounter simple poles at $z = +\sigma_n$ with residue contributions

$$\frac{(-1)^{n+1}}{n\pi} r^{-\sigma_n} F(1+\sigma_n) \cos(\sigma_n(\theta - \alpha))$$

(noting that the contour around the pole is traversed clockwise). The resulting expansion contains only negative powers of r, so that (3.4) is satisfied.

Moving the contour requires some justification but this is easily done using simple estimates for large $|\tau|$ of the integrand in (3.8) for $0 < \theta < \alpha$. For example, a sufficient condition would be $|F(z+1)| = o(z)$ as $\tau = \operatorname{Im} z \to \pm\infty$ in $-1 \le \sigma \le \sigma_0$.

4. Mixed problem

4.1 *Formulation and uniqueness*

Let us change the boundary condition on one side of the wedge, $\theta = 0$. Thus $w(r, \theta)$ satisfies Laplace's equation, $\nabla^2 w = 0$, for $r > 0$, $0 < \theta < \alpha$, with a Neumann condition at $\theta = \alpha$,

$$\frac{1}{r}\frac{\partial w}{\partial \theta} = 0, \quad \theta = \alpha, \quad r > 0. \tag{4.1}$$

There are mixed boundary conditions at $\theta = 0$,

$$\frac{1}{r}\frac{\partial w}{\partial \theta} = 0, \quad \theta = 0, \quad 0 < r < a \quad \text{and} \quad r > b, \tag{4.2}$$

and

$$w(r, 0) = w_0(r), \quad a < r < b, \tag{4.3}$$

where w_0 is a specified function. Recall that a, b and r are dimensionless.

Define $v(r)$ by

$$v(r) = \frac{1}{r}\frac{\partial w}{\partial \theta} \quad \text{evaluated at } \theta = 0, \quad a < r < b.$$

Of course, this quantity is unknown at present; it will be found by solving an integral equation. As in section 3.1, if $w \sim B_0 \log r$ as $r \to 0$ and $w \sim B_\infty \log r$ as $r \to \infty$, Green's theorem gives the constraint

$$\alpha(B_\infty - B_0) = \int_a^b v(r)\, dr. \tag{4.4}$$

Suppose that, in more detail, w has the following behaviour:

$$w(r, \theta) = B_0 \log r + A_0 + O(r^{\pi/\alpha}), \quad r \to 0 \tag{4.5}$$

$$w(r, \theta) = B_\infty \log r + A_\infty + O(r^{-\pi/\alpha}), \quad r \to \infty. \tag{4.6}$$

Then, if we specify *both* B_0 and B_∞, w is unique. This can be proved using a standard argument as follows. Suppose there are two solutions, w_1 and w_2. Their difference, $w_1 - w_2 = \phi$, say, satisfies homogeneous boundary conditions. Moreover, $\phi(r, \theta)$ has the form (4.5) near $r = 0$ but with $B_0 = 0$, and the form (4.6) for large r but with $B_\infty = 0$. Apply Green's theorem to ϕ,

$$\int_D |\text{grad}\, \phi|^2\, dA = \int_S \phi \frac{\partial \phi}{\partial n}\, ds,$$

where D is a piece of the wedge bounded by circular arcs $r = r_0$ and $r = r_\infty$, $0 < \theta < \alpha$, $\partial\phi/\partial n$ is the outward normal derivative of ϕ, and S is the boundary of D. Using the known behaviour of ϕ on S, we let $r_0 \to 0$ and $r_\infty \to \infty$. We infer that $|\text{grad}\, \phi| = 0$ everywhere in the wedge. Then, as $\phi = 0$ on $\theta = 0$ for $a < r < b$, we must have $\phi \equiv 0$ so that $w_1 = w_2$.

Return to w, satisfying $\nabla^2 w = 0$ in the wedge, together with (4.1), (4.3), (4.5) and (4.6), where both B_0 and B_∞ are specified. Then the constants A_0 and A_∞ are uniquely defined but unknown.

4.2 *Reduced problem*

The function w does not have a Mellin transform. Therefore, motivated by section 3, define

$$u(r, \theta) = w(r, \theta) - B_\infty \log r - A_\infty, \tag{4.7}$$

so that $\nabla^2 u = 0$ in the wedge, together with (3.1), (3.2), (3.4),

$$u(r, \theta) = \mathcal{A} + \mathcal{B} \log r + o(1) \quad \text{as } r \to 0, \tag{4.8}$$

and

$$u(r, 0) = u_0(r), \quad a < r < b, \tag{4.9}$$

where

$$\mathcal{A} = A_0 - A_\infty, \quad \mathcal{B} = B_0 - B_\infty, \tag{4.10}$$

$$u_0(r) = w_0(r) - B_\infty \log r - A_\infty, \tag{4.11}$$

$$f(r) = \begin{cases} 0, & 0 < r < a \text{ and } r > b, \\ v(r), & a < r < b, \end{cases} \tag{4.12}$$

and $\mathcal{B}$ is related to v by (4.4).

To find u, we proceed as in section 3.2. We write $u(r, \theta)$ as (3.7),

$$u(r, \theta) = \frac{1}{2\pi i} \int_{\mathrm{Br}_+} g(z)\, r^{-z} \cos\left(z(\theta - \alpha)\right) dz, \tag{4.13}$$

where the Bromwich contour is just to the right of $z = 0$ and

$$g(z) = \frac{F(z+1)}{z \sin z\alpha} = \frac{1}{z \sin z\alpha} \int_a^b t^z\, v(t)\, dt. \tag{4.14}$$

We see that $g(z)$ has a double pole at $z = 0$ and simple poles at $z = \pm n\pi/\alpha$, $n = 1, 2, \ldots$.

Near $z = 0$, the integrand in (4.13) is approximately

$$\frac{1}{z^2\alpha} \int_a^b v(t)\, dt + \frac{1}{z\alpha} \int_a^b v(t) \log t\, dt - \frac{\log r}{z\alpha} \int_a^b v(t)\, dt,$$

so that moving the contour to the left confirms (4.8) with

$$\mathcal{A} = \frac{1}{\alpha} \int_a^b v(t) \log t\, dt \quad \text{and} \quad \mathcal{B} = -\frac{1}{\alpha} \int_a^b v(t)\, dt. \tag{4.15}$$

Moving the contour further to the left, the next pole encountered is at $z = -\pi/\alpha$. It contributes the next term to (4.8),

$$\mathcal{C}\, r^{\pi/\alpha} \cos(\pi\theta/\alpha) \quad \text{with} \quad \mathcal{C} = -\frac{1}{\pi} \int_a^b t^{-\pi/\alpha} v(t)\, dt. \tag{4.16}$$

4.3 *Deriving an integral equation*

From (4.9) and (4.13), we have

$$u_0(r) = \frac{1}{2\pi i} \int_{\mathrm{Br}_+} g(z)\, r^{-z} \cos(z\alpha)\, dz, \quad a < r < b.$$

Substituting for g from (4.14), we obtain an integral equation for v,

$$u_0(r) = \int_a^b K(r/t)\, v(t)\, dt, \quad a < r < b, \tag{4.17}$$

where the kernel is defined by

$$K(\tau) = \frac{1}{2\pi i} \int_{\mathrm{Br}_+} \frac{\cos z\alpha}{z \sin z\alpha}\, \tau^{-z}\, dz. \tag{4.18}$$

Moving the contour to the left gives

$$K(\tau) = -\frac{\log \tau}{\alpha} - \sum_{n=1}^{\infty} \frac{1}{n\pi} \left(\tau^{\pi/\alpha}\right)^n = \frac{1}{\pi} \log\left(\tau^{-\pi/\alpha} - 1\right), \quad |\tau| < 1. \tag{4.19}$$

Moving the contour to the right instead gives

$$K(\tau) = -\sum_{n=1}^{\infty} \frac{1}{n\pi} \left(\tau^{-\pi/\alpha}\right)^n = \frac{1}{\pi} \log\left(1 - \tau^{-\pi/\alpha}\right), \quad |\tau| > 1. \tag{4.20}$$

Combining (4.19) and (4.20) gives

$$K(r/t) = -\frac{\log r}{\alpha} + \frac{1}{\pi} \log\left|t^{\pi/\alpha} - r^{\pi/\alpha}\right|, \quad r > 0, \quad t > 0. \tag{4.21}$$

Substituting in (4.17), together with (4.11), gives

$$w_0(r) - B_\infty \log r - A_\infty = -\frac{\log r}{\alpha} \int_a^b v(t)\, dt + \frac{1}{\pi} \int_a^b v(t) \log\left|t^{\pi/\alpha} - r^{\pi/\alpha}\right| dt.$$

But, using (4.4), the first integral is equal to $\alpha(B_\infty - B_0)$, whence

$$\frac{1}{\pi} \int_a^b v(t) \log\left|t^{\pi/\alpha} - r^{\pi/\alpha}\right| dt = q(r), \quad a < r < b, \tag{4.22}$$

where

$$q(r) = w_0(r) - B_0 \log r - A_\infty. \tag{4.23}$$

The integral equation for v, (4.22), is to be solved subject to (4.4). Note that the constant A_∞ occurring in (4.23) is also to be determined.

We obtained the integral equation (4.22) by considering a reduced problem for $u = w - A_\infty - B_\infty \log r$, with $u \to 0$ as $r \to \infty$. As an alternative, we could have considered a reduced problem for $w - A_0 - B_0 \log r = \tilde{u}$, say, so that $\tilde{u} \to 0$ as $r \to 0$. It turns out that this approach leads to exactly the same integral equation for v; see Appendix A.1.

4.4 *Solving the integral equation*

To simplify the integral equation (4.22), let $\beta = \alpha/\pi$ and put

$$t = T^\beta, \quad r = R^\beta, \quad a = A^\beta \quad \text{and} \quad b = B^\beta. \tag{4.24}$$

As $dt = \beta T^{\beta-1}\, dT$, we obtain

$$\frac{\beta}{\pi}\int_A^B T^{\beta-1}\, v(T^\beta)\log|T-R|\, dT = q(R^\beta), \quad A < R < B. \tag{4.25}$$

The constraint (4.4) becomes

$$\frac{1}{\pi}\int_A^B T^{\beta-1}\, v(T^\beta)\, dT = B_\infty - B_0. \tag{4.26}$$

Next, map the interval $A < T < B$ to $-1 < \tau < 1$, using

$$T = \lambda\tau + \mu, \quad \text{with} \quad \lambda = (B-A)/2, \quad \mu = (B+A)/2. \tag{4.27}$$

Similarly, put $R = \lambda\rho + \mu$. As $T - R = \lambda(\tau - \rho)$ and $dT = \lambda\, d\tau$, (4.25) becomes

$$\frac{\beta\lambda}{\pi}\int_{-1}^1 T^{\beta-1}\, v(T^\beta)\log\{\lambda|\tau-\rho|\}\, d\tau = q(R^\beta), \quad -1 < \rho < 1, \tag{4.28}$$

whereas (4.26) becomes

$$\frac{\lambda}{\pi}\int_{-1}^1 T^{\beta-1}\, v(T^\beta)\, d\tau = B_\infty - B_0.$$

Using this relation in (4.28) gives

$$\beta(B_\infty - B_0)\log\lambda + \frac{\beta\lambda}{\pi}\int_{-1}^1 T^{\beta-1}\, v(T^\beta)\log|\tau-\rho|\, d\tau = q(R^\beta), \quad -1 < \rho < 1.$$

To simplify the notation, put

$$V(\tau) = \beta\lambda T^{\beta-1} v(T^\beta) \quad \text{and} \quad Q(\rho) = q(R^\beta) - \beta(B_\infty - B_0)\log\lambda, \tag{4.29}$$

whence

$$\frac{1}{\pi}\int_{-1}^1 V(\tau)\log|\tau-\rho|\, d\tau = Q(\rho), \quad -1 < \rho < 1, \tag{4.30}$$

with

$$\frac{1}{\pi}\int_{-1}^1 V(\tau)\, d\tau = \beta(B_\infty - B_0). \tag{4.31}$$

To solve (4.30), we expand using Chebyshev polynomials, $T_n(x)$,

$$V(\tau) = \frac{1}{\sqrt{1-\tau^2}}\sum_{n=0}^\infty V_n T_n(\tau), \quad Q(\rho) = \sum_{m=0}^\infty Q_m T_m(\rho). \tag{4.32}$$

By definition, $T_n(\cos\vartheta) = \cos n\vartheta$, and we have orthogonality,

$$\int_{-1}^1 T_n(\tau)T_m(\tau)\frac{d\tau}{\sqrt{1-\tau^2}} = \begin{cases} 0, & n \neq m, \\ \pi, & n = m = 0, \\ \pi/2, & n = m \neq 0. \end{cases} \tag{4.33}$$

However, the main reason for using T_n is the expansion

$$\log|\tau-\rho| = -\log 2 - \sum_{n=1}^{\infty} \frac{2}{n}\, T_n(\tau)\, T_n(\rho).$$

(This can be proved by rewriting the sum using $\rho = \cos\vartheta$ and $\tau = \cos\varphi$, followed by use of (**5**, 1.314.3) and (**5**, 1.441.2).)

Substituting these expansions in (4.30), making use of orthogonality, gives

$$-V_0 \log 2 = Q_0, \quad -n^{-1}\, V_n = Q_n, \quad n = 1, 2, \ldots, \tag{4.34}$$

whereas (4.31) gives $V_0 = \beta(B_\infty - B_0)$. Equating the two expressions for V_0 gives

$$Q_0 = \beta(B_0 - B_\infty)\log 2, \tag{4.35}$$

and this can be used to determine A_∞: the coefficient Q_0 is related to Q by $(4.32)_2$, Q is related to q by $(4.29)_2$, and q is related to A_∞ by (4.23). We also have $(4.15)_1$,

$$A_0 - A_\infty = \frac{1}{\pi}\int_{-1}^{1} V(\tau)\log(\lambda\tau+\mu)\,d\tau. \tag{4.36}$$

To evaluate this integral, we expand the logarithmic term. We have

$$\log(\lambda\tau+\mu) = \log\mu + \log(1+\Omega\tau) \quad \text{with} \quad 0 < \Omega = \lambda/\mu < 1,$$

noting from (4.27) that $\mu > \lambda > 0$. We also have (**5**, 1.514)

$$-\sum_{n=1}^{\infty} \frac{2}{n}(-\kappa)^n \cos n\vartheta = \log(1+\kappa^2+2\kappa\cos\vartheta) \quad \text{for } |\kappa| < 1.$$

Setting $\Omega = 2\kappa/(1+\kappa^2)$, we solve for κ and obtain

$$\kappa = \frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}} = \frac{b^\gamma - a^\gamma}{b^\gamma + a^\gamma} \quad \text{with} \quad \gamma = \frac{1}{2\beta} = \frac{\pi}{2\alpha}, \tag{4.37}$$

after use of (4.27) and (4.24). Thus

$$\log(\lambda\tau+\mu) = -\log(2\kappa/\lambda) - \sum_{n=1}^{\infty} \frac{2}{n}(-\kappa)^n T_n(\tau) \tag{4.38}$$

using $\log\mu - \log(1+\kappa^2) = -\log(2\kappa/\lambda)$.

Substituting (4.38) in (4.36), together with $(4.32)_1$, (4.33) and $(4.34)_2$, gives

$$A_0 - A_\infty = -V_0 \log(2\kappa/\lambda) - \sum_{n=1}^{\infty} \frac{1}{n}(-\kappa)^n V_n$$

$$= \beta(B_0 - B_\infty)\log(2\kappa/\lambda) + \sum_{n=1}^{\infty}(-\kappa)^n Q_n. \tag{4.39}$$

This completes the solution of the problem: given B_0, B_∞ and $w_0(r)$ for $a < r < b$, we are able to calculate $w(r, \theta)$ uniquely. An example follows.

4.5 *An example*

As a simple example, take $w_0(r) = W_0 + W_1 r$, where W_0 and W_1 are constants. Then

$$q(r) = W_0 + W_1 r - A_\infty - B_0 \log r,$$

$$q(R^\beta) = W_0 - A_\infty + W_1 R^\beta - \beta B_0 \log R,$$

$$Q(\rho) = W_0 - A_\infty - \beta B_0 \log(\lambda\rho + \mu) - \beta(B_\infty - B_0)\log\lambda + W_1(\lambda\rho + \mu)^\beta.$$

Next, we must expand $Q(\rho)$ as $(4.32)_2$. To expand $\log(\lambda\rho + \mu)$, we can use (4.38). For the last term, we have $(\lambda\rho + \mu)^\beta = \mu^\beta(1 + \Omega\rho)^\beta$ together with the expansion

$$(1 + \Omega\rho)^\beta = L_0(\beta) + 2\sum_{n=1}^{\infty} L_n(\beta)\, T_n(\rho), \tag{4.40}$$

where

$$L_n(\beta) = (-\Omega)^n \frac{(-\beta)_n}{2^n\, n!} F\left(\frac{n-\beta}{2}, \ \frac{n-\beta+1}{2}; \ n+1; \ \Omega^2\right), \tag{4.41}$$

$(a)_n = \Gamma(a + n)/\Gamma(a)$ is the Pochhammer symbol and $F(a, b; c; z)$ is the Gauss hypergeometric function (see Appendix B.1). Thus

$$Q_0 = W_0 - A_\infty + \beta B_0 \log 2\kappa - \beta B_\infty \log\lambda + W_1 \mu^\beta L_0(\beta), \tag{4.42}$$

$$Q_n = (2/n)\beta B_0(-\kappa)^n + 2W_1\mu^\beta L_n(\beta), \quad n = 1, 2, \ldots. \tag{4.43}$$

The function $V(\tau)$ is then given by the Chebyshev expansion $(4.32)_1$ with coefficients defined by (4.34).

Substituting for Q_0 from (4.42) in (4.35) gives

$$A_\infty = W_0 + W_1\mu^\beta L_0(\beta) + \beta B_0 \log\kappa - \beta B_\infty \log(\lambda/2)$$

and then (4.39) gives

$$\begin{aligned} A_0 &= A_\infty + \beta(B_0 - B_\infty)\log(2\kappa/\lambda) + \sum_{n=1}^{\infty}(-\kappa)^n Q_n \\ &= W_0 + W_1\mu^\beta L_0(\beta) + \beta B_0 \log(2\kappa^2/\lambda) - \beta B_\infty \log\kappa \\ &\quad + 2\beta B_0 \sum_{n=1}^{\infty}\frac{\kappa^{2n}}{n} + 2W_1\mu^\beta \sum_{n=1}^{\infty}(-\kappa)^n L_n(\beta), \end{aligned}$$

in which the first sum is equal to $-\log(1-\kappa^2)$.

Further terms in the expansions for small or large values of r can be calculated. For example, the third term in the expansion near $r = 0$ is given by (4.16) in which

$$\mathcal{C} = -\frac{1}{\pi}\int_{-1}^{1}(\lambda\tau + \mu)^{-\beta} V(\tau)\,d\tau = -\mu^{-\beta}\sum_{n=0}^{\infty} L_n(-\beta)\,V_n,$$

using $(4.32)_1$, (4.33) and (4.41). In this formula, V_n is related to Q_n by (4.34), and then Q_n is given by (4.42) and (4.43) for the linear choice of $w_0(r)$.

5. Elasticity problems

5.1 *Anti-plane contact problems*

Anti-plane strain elasticity problems reduce to solving Laplace's equation for w, the out-of-plane displacement component. Anti-plane contact problems are not of great interest except as a vehicle for developing techniques. However, one feature is noteworthy: w should be bounded everywhere in the wedge, so that $B_0 = B_\infty = 0$.

Singh *et al.* (**6**) have considered a related contact problem in which the Neumann boundary at $\theta = \alpha$, (4.1), is replaced by a Dirichlet condition, $w(r, \alpha) = 0$ for $r > 0$. This is a simpler problem because the Mellin transform of w exists for $-\gamma < \sigma < \gamma$; see (2.2) and $(4.37)_2$.

Another class of problems involving Laplace's equation inside a wedge concerns surface water waves interacting with a plane beach. For these problems, the Neumann boundary condition at $\theta = 0$, (4.2), is replaced by a Robin condition,

$$Kw - \frac{1}{r}\frac{\partial w}{\partial \theta} = 0,$$

where K is a positive constant. Problems of this kind have been attacked using Mellin transforms by Ehrenmark and others; for example, see (**7**), where the role played by solutions with logarithmic singularities at $r = 0$ is emphasised.

5.2 *Plane-strain problems and discussion*

As noted in the Introduction, Tranter (**1**) used Mellin transforms for plane-strain elasticity problems with tractions prescribed on the two sides of the wedge; see also (**8**, §49), (**9**, §4.4), (**10**) and (**11**).

There are also many papers dealing with composite wedges, where two (or more) wedges are welded together to form a larger wedge; see (**12**) and references therein. Inevitably, these studies involve the so-called Williams eigenfunctions and their relatives; for lengthy reviews, covering theory, applications and history, see (**13–15**). In particular, much is known about the behaviour as $r \to 0$ and as $r \to \infty$.

Contact problems for a single elastic wedge have received some attention (**16–19**) but detailed studies for composite wedges await further work.

Acknowledgements

I thank John Berger and Ed Garboczi for discussions about contact problems in composite wedges. I also thank the referees for their comments and for citing the paper by Williams (**3**).

References

1. C. J. Tranter, The use of the Mellin transform in finding the stress distribution in an infinite wedge, *Quart. J. Mech. Appl. Math.* **1** (1948) 125–130.
2. A. H. England, *Complex Variable Methods in Elasticity* (Dover, New York 2003).
3. W. E. Williams, Two-dimensional mixed boundary-value problems in a wedge-shaped region, *Proc. Camb. Phil. Soc.* **64** (1968) 503–505.
4. B. Noble, *Methods Based on the Wiener–Hopf Technique*, 2nd edn (Chelsea, New York 1988)
5. I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, 5th edn (Academic Press, San Diego 1994).
6. B. M. Singh, J. Rokne and R. S. Dhaliwal, Contact problem for a wedge-shaped elastic solid in antiplane shear stress distribution, *Q. Appl. Math.* **63** (2005) 545–551.
7. U. T. Ehrenmark, Overconvergence of the near-field expansion for linearized waves normally incident on a sloping beach, *SIAM J. Appl. Math.* **49** (1989) 799–815.
8. I. N. Sneddon, *Fourier Transforms* (McGraw-Hill, New York 1951).
9. C. J. Tranter, *Integral Transforms in Mathematical Physics*, 3rd edn (Methuen, London 1966).
10. W. J. Harrington and T. W. Ting, Stress boundary-value problems for infinite wedges, *J. Elast.* **1** (1971) 65–81.
11. C. Atkinson and R. V. Craster, Theoretical aspects of fracture mechanics, *Prog. Aerospace Sci.* **31** (1995) 1–83.
12. P. A. Martin, On Green's function for a bimaterial elastic half-plane, *Int. J. Solids & Struct.* **40** (2003) 2101–2119.
13. V. V. Meleshko, Selected topics in the history of the two-dimensional biharmonic problem, *Appl. Mech. Rev.* **56** (2003) 33–85.
14. G. B. Sinclair, Stress singularities in classical elasticity – I: Removal, interpretation, and analysis, *Appl. Mech. Rev.* **57** (2004) 251–298.
15. G. B. Sinclair, Stress singularities in classical elasticity – II: Asymptotic identification, *Appl. Mech. Rev.* **57** (2004) 385–439.
16. F. Erdogan and K. Arin, Fracture and contact problems for an elastic wedge, *J. Elast.* **6** (1976) 57–71.
17. F. Erdogan and G. D. Gupta, Contact and crack problems for an elastic wedge, *Int. J. Eng. Sci.* **14** (1976) 155–164.

18. V. M. Aleksandrov and D. A. Pozharskii, On a contact problem for an elastic wedge, *J. Appl. Math. & Mech.* **52** (1988) 506–511.
19. M. T. Hanson and L. M. Keer, Mechanics of edge effects on frictionless contacts, *Int. J. Solids & Struct.* **32** (1995) 391–405.

APPENDIX A

A.1 *Reduced problem with zero at the origin*

Instead of starting from (4.7), define

$$\tilde{u}(r,\theta) = w(r,\theta) - B_0 \log r - A_0, \tag{A.1}$$

so that $\nabla^2 \tilde{u} = 0$ in the wedge, together with $\partial\tilde{u}/\partial\theta = 0$ at $\theta = \alpha$ and $\partial\tilde{u}/\partial\theta = rf(r)$ at $\theta = 0$, where f is defined by (4.12). We have $\tilde{u} \to 0$ as $r \to 0$ and

$$\tilde{u}(r,\theta) = -\mathcal{A} - \mathcal{B}\log r + o(1) \quad \text{as } r \to \infty, \tag{A.2}$$

with $\mathcal{A}$ and $\mathcal{B}$ defined by (4.10). Finally, we have

$$\tilde{u}(r,0) = \tilde{u}_0(r), \quad a < r < b, \tag{A.3}$$

where

$$\tilde{u}_0(r) = w_0(r) - B_0 \log r - A_0. \tag{A.4}$$

To find $\tilde{u}$, we write

$$\tilde{u}(r,\theta) = \frac{1}{2\pi i}\int_{\mathrm{Br}_-} g(z)\, r^{-z} \cos(z(\theta-\alpha))\, dz, \tag{A.5}$$

where the contour is just to the *left* of $z = 0$ and $g(z)$ is given by (4.14). Moving the contour to the right, the double pole at $z = 0$ leads to (A.2) with $\mathcal{A}$ and $\mathcal{B}$ related to v by (4.15).

From (4.14), (A.3) and (A.5), we have

$$\tilde{u}_0(r) = \frac{1}{2\pi i}\int_{\mathrm{Br}_-} g(z)\, r^{-z}\cos(z\alpha)\, dz = \int_a^b \widetilde{K}(r/t)\, v(t)\, dt, \quad a < r < b, \tag{A.6}$$

where the kernel is defined by

$$\widetilde{K}(\tau) = \frac{1}{2\pi i}\int_{\mathrm{Br}_-} \frac{\cos z\alpha}{z \sin z\alpha}\, \tau^{-z}\, dz = \frac{1}{\alpha}\log\tau + K(\tau),$$

and K is defined by (4.18). Substituting in (A.6), together with (4.21) and (A.4), gives

$$w_0(r) - B_0\log r - A_0 = -\frac{1}{\alpha}\int_a^b v(t)\log t\, dt + \frac{1}{\pi}\int_a^b v(t)\log\left|t^{\pi/\alpha} - r^{\pi/\alpha}\right| dt.$$

But, using (4.15), the first integral is equal to $\alpha(A_0 - A_\infty)$, and so we obtain the same integral equation as before, namely (4.22).

APPENDIX B

B.1 *A Chebyshev expansion*

From the binomial expansion (**5**, 9.121.1), we have

$$(1+\Omega\rho)^{\beta} = F(-\beta, b; b; -\Omega\rho) = \sum_{n=0}^{\infty} \frac{(-\beta)_n}{n!}(-\Omega\rho)^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-\beta)_{2n}}{(2n)!}(\Omega\rho)^{2n} - \sum_{n=1}^{\infty} \frac{(-\beta)_{2n-1}}{(2n-1)!}(\Omega\rho)^{2n-1}, \tag{B.1}$$

where $F(a, b; c; z)$ is the Gauss hypergeometric function and $(-\beta)_m = \Gamma(m-\beta)/\Gamma(-\beta)$ is Pochhammer's symbol. Put $\rho = \cos\vartheta$ and use (**5**, 1.320)

$$\cos^{2n}\vartheta = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}}\sum_{m=1}^{n}\binom{2n}{n-m}\cos 2m\vartheta,$$

$$\cos^{2n-1}\vartheta = \frac{1}{2^{2n-2}}\sum_{m=1}^{n}\binom{2n-1}{n-m}\cos(2m-1)\vartheta.$$

After substitution in (B.1), change the order of summation; the result is (4.40).

In detail, consider the first sum in (B.1). It becomes

$$\sum_{n=1}^{\infty} \frac{(-\beta)_{2n}}{(2n)!}\frac{\Omega^{2n}}{2^{2n}}\binom{2n}{n} + 2\sum_{m=1}^{\infty} L_{2m}T_{2m}(\rho), \tag{B.2}$$

where

$$L_{2m} = \sum_{n=m}^{\infty} \frac{(-\beta)_{2n}}{(2n)!}\frac{\Omega^{2n}}{2^{2n}}\binom{2n}{n-m} = \sum_{n=m}^{\infty} \frac{(-\beta)_{2n}}{(n-m)!\,(n+m)!}\frac{\Omega^{2n}}{2^{2n}}$$

$$= \frac{\Omega^{2m}}{2^{2m}}\sum_{p=0}^{\infty} \frac{(-\beta)_{2p+2m}}{p!\,(p+2m)!}\frac{\Omega^{2p}}{2^{2p}} = \Omega^{2m}\frac{(-\beta)_{2m}}{2^{2m}}\sum_{p=0}^{\infty}\frac{(2m-\beta)_{2p}}{2^{2p}\,(p+2m)!}\frac{\Omega^{2p}}{p!}$$

$$= \Omega^{2m}\frac{(-\beta)_{2m}}{2^{2m}\,(2m)!}\sum_{p=0}^{\infty}\frac{(m-\beta/2)_p(m+[1-\beta]/2)_p}{(2m+1)_p}\frac{\Omega^{2p}}{p!}. \tag{B.3}$$

Here, we have used $(-\beta)_{2p+2m} = (2m-\beta)_{2p}(-\beta)_{2m}$ and $(-\beta)_{2n} = 2^{2n}(-\beta/2)_n([1-\beta]/2)_n$; the second of these made use of the duplication formula for $\Gamma(2z)$ (**5**, 8.335.1). The remaining sum in (B.3) is

$$F\left(\frac{2m-\beta}{2}, \frac{2m-\beta+1}{2}; 2m+1; \Omega^2\right).$$

For L_0, combine the 1 on the right-hand side of (B.1) with the first sum in (B.2). Similar calculations, and also for L_{2m-1}, yield (4.40) with (4.41).