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# On in-out splitting of incident fields and the far-field behaviour of Herglotz wavefunctions

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It is easy to write down entire solutions of the Helmholtz equation: Examples are plane waves and Herglotz wavefunctions. We are interested in the far-field behaviour of these solutions motivated by the following question: When is it legitimate to split the far field of such an entire solution into the sum of an incoming spherical wave and an outgoing spherical wave? We review the relevant literature (there are disjoint physical and mathematical threads), and then we answer the question for Herglotz wavefunctions, using a combination of the 2-dimensional method of stationary phase and some explicit examples.

**KEYWORDS**
Helmholtz equation, method of stationary phase, spherical waves

## 1 | INTRODUCTION

Let $u$ be a *regular wavefunction*. These are source-free solutions of the Helmholtz equation, $(\nabla^2 + k^2)u = 0$, everywhere in 3-dimensional space; the suppressed time dependence is $\mathrm{e}^{-\mathrm{i}\omega t}$, $k = \omega/c$ is real and positive, and $c$ is the constant speed of sound. This work began with the following question: when is it legitimate to write

$$u(\mathbf{r}) \sim g_{\text{out}}(\hat{\mathbf{r}})\,\frac{\mathrm{e}^{\mathrm{i}kr}}{\mathrm{i}kr} + g_{\text{inc}}(\hat{\mathbf{r}})\,\frac{\mathrm{e}^{-\mathrm{i}kr}}{\mathrm{i}kr} \quad \text{as} \quad r \to \infty? \tag{1}$$

Here, $\mathbf{r}$ is the position vector of a typical point with respect to an arbitrary origin $O$, $r = |\mathbf{r}|$, $\hat{\mathbf{r}} = \mathbf{r}/r$, and $g_{\text{out}}$ and $g_{\text{inc}}$ are functions of direction $\hat{\mathbf{r}}$ only. The first term on the right-hand side represents an *outgoing* spherical wave (because the time dependence is $e^{-\mathrm{i}\omega t}$), and the second term represents an *incoming* spherical wave.

The Sommerfeld radiation condition ensures that scattered fields are purely outgoing; for such fields, $g_{\text{inc}} \equiv 0$ and $g_{\text{out}}(\hat{\mathbf{r}})$ is the *far-field pattern*. Moreover, a purely outgoing field cannot be regular everywhere (unless it is identically zero).[1, corollary 3.9] Thus, the main interest of (1) lies in its application to *incident* fields.

The formula (1) has a long history. In fact, there is a physics thread (Section 2) and a mathematics thread (Section 3), with little interaction. We start with the physics thread and a paper by Gerjuoy and Saxon.[2] After giving some simple counterexamples to (1) (such as a plane wave), we critique an attempt by Gerjuoy and Saxon to extend (1) to plane waves. This leads to a preliminary discussion of a lemma due to Jones,[3] which gives far-field approximations for certain integral representations of regular wavefunctions. These are superpositions of plane waves propagating in all directions $\hat{\mathbf{s}}$,

$$u_q(\mathbf{r}) = \int_{\Omega} q(\hat{\mathbf{s}})\,\mathrm{e}^{\mathrm{i}k\mathbf{r}\cdot\hat{\mathbf{s}}}\,\mathrm{d}\Omega(\hat{\mathbf{s}}), \tag{2}$$

where $\Omega$ is the unit sphere and $q$ is a given function defined on $\Omega$. The *Jones Lemma* gives the far-field behaviour of $u_q$ when $q$ is smooth. Indeed, it gives an answer to our question: $u(\mathbf{r})$ does have the behaviour (1) if $u$ can be written as $u_q$ with a smooth $q$.

  

We pick up the mathematics thread in Section 3, which is concerned with $u_q$ when $q$ is square integrable, $q \in L^2(\Omega)$. Then $u_q$ is known as a *Herglotz wavefunction*; these have been studied since the 1950s, but they became more familiar later because of their prominent role in inverse scattering theory.[4, section 3.3] Herglotz knew that his wavefunctions may not satisfy (1) in every direction $\hat{\mathbf{r}}$, although they do satisfy a weaker averaged version of (1); see (17) below.

The far-field behaviour of integrals such as (2) can be investigated using the 2-dimensional method of stationary phase.[5, section 8.46] This method is summarised in Section 4, and the Jones Lemma is recovered. As we are interested in violations of (1) by $u_q$, we examine two simple choices for $q$ in Section 5. In the first, $q$ is piecewise constant with a discontinuity around the equator of the sphere $\Omega$ (Section 5.1). We find that $u_q$ does satisfy (1) in all directions except along the axis perpendicular to the equatorial plane. The second example (Section 5.2) is more complicated in that $q$ has an integrable singularity around the equator; the singularity strength can be varied, so that $q \in L^2(\Omega)$ or $q \notin L^2(\Omega)$; this difference is shown to have a profound effect on the far-field behaviour.

## 2 | PHYSICS THREAD

The physics thread seems to start in 1954 with a statement in a paper by Gerjuoy and Saxon[2, p1448]: "At infinity [the incident field] is composed of incoming and outgoing spherical waves. Hence [as the scattered field is outgoing and] neglecting terms of order $1/r^2$ [the total field $p$ satisfies]

$$\lim_{r\to\infty} p(r\hat{\mathbf{r}}) = F_1(\hat{\mathbf{r}})\frac{\mathrm{e}^{ikr}}{r} + F_2(\hat{\mathbf{r}})\frac{\mathrm{e}^{-ikr}}{r}, \tag{3}$$

where" $F_1$ and $F_2$ are related by a so-called scattering matrix. Similarly, in 1955, Saxon[7, p287] wrote "We now decompose the asymptotic solution for $\mathbf{r} = r\hat{\mathbf{r}}$, $r \to \infty$, into incoming and outgoing waves along $\mathbf{r}$; that is, we write" Equation 1, without further ado. See also Saxon.[8, eq. 2]

One feature of (1) is that its right-hand side $\to 0$ as $r \to \infty$. This property is not enjoyed by the simplest incident field, a plane wave, defined by

$$u_{\mathrm{pw}}(\mathbf{r}) = \exp(\mathrm{i}k\mathbf{r}\cdot\hat{\mathbf{s}}), \tag{4}$$

for which $|u_{\mathrm{pw}}(\mathbf{r})| = 1$ everywhere. (The unit vector $\hat{\mathbf{s}}$ gives the direction of propagation.)

Another feature of (1) is that its right-hand side is $O(r^{-1})$ as $r \to \infty$. For an example that does not have this behaviour, take an axisymmetric Bessel beam,

$$u_{\mathrm{bb}}(\mathbf{r}) = \exp(\mathrm{i}kr\cos\theta\cos\beta)J_0(kr\sin\theta\sin\beta),$$

where $r$ and $\theta$ are spherical polar coordinates, $\beta$ is a real parameter, and $J_0$ is a Bessel function. We see that $u_{\mathrm{bb}}$ decays as $(kr)^{-1/2}$ in all directions $\hat{\mathbf{r}}$ except along the $z$-axis ($\theta = 0, \pi$); here, we have used the Digital Library of Mathematical Functions.[9, 10.7.8]

For another example, we have

$$u(\mathbf{r}) = J_0(kr\sin\theta)\,r\cos\theta,$$

which grows with $r$ in all directions except in the plane $\theta = \pi/2$.

Gerjuoy and Saxon[2] try to fit incident plane waves into (1). Removing the scattered waves from their equations (30) and (31) leads to the following calculation,

$$\begin{aligned}
\exp(\mathrm{i}k\mathbf{r}\cdot\hat{\mathbf{s}}) &= 4\pi\sum_{n,m}\mathrm{i}^n j_n(kr)Y_n^m(\hat{\mathbf{r}})\overline{Y_n^m(\hat{\mathbf{s}})} \\
&\simeq \frac{2\pi}{\mathrm{i}kr}\sum_{n,m}Y_n^m(\hat{\mathbf{r}})\overline{Y_n^m(\hat{\mathbf{s}})}\left\{\mathrm{e}^{\mathrm{i}kr} - (-1)^n\mathrm{e}^{-\mathrm{i}kr}\right\} \\
&= 2\pi\,\delta(\hat{\mathbf{s}} - \hat{\mathbf{r}})\frac{\mathrm{e}^{\mathrm{i}kr}}{\mathrm{i}kr} - 2\pi\,\delta(\hat{\mathbf{s}} + \hat{\mathbf{r}})\frac{\mathrm{e}^{-\mathrm{i}kr}}{\mathrm{i}kr},
\end{aligned} \tag{5}$$

where $j_n$ is a spherical Bessel function and $Y_n^m$ is a spherical harmonic. The first equality is a standard result,[10, eq. 4.40] the second step makes use of the standard asymptotic approximation[9, 10.52.3]

$$j_n(w) \sim \frac{1}{w}\sin\left(w - \frac{n\pi}{2}\right) = \frac{(-\mathrm{i})^n}{2\mathrm{i}w}\left\{\mathrm{e}^{\mathrm{i}w} - (-1)^n\mathrm{e}^{-\mathrm{i}w}\right\} \quad \text{as } w \to \infty \tag{6}$$

(although this is not stated in Gerjuoy and Saxon[2]), and the third defines the “spherical delta function” by Jackson[11, eq. 3.56]

$$\delta(\hat{\mathbf{s}} - \hat{\mathbf{r}}) = \sum_{n,m} Y_n^m(\hat{\mathbf{r}})\overline{Y_n^m(\hat{\mathbf{s}})}. \tag{7}$$

Gerjuoy and Saxon[2, eq. 32] go on to assert that (5) is a “special case” of (1), with $g_{\mathrm{out}}(\hat{\mathbf{r}}) = 2\pi\delta(\hat{\mathbf{s}} - \hat{\mathbf{r}})$ and $g_{\mathrm{inc}}(\hat{\mathbf{r}}) = -2\pi\delta(\hat{\mathbf{s}} + \hat{\mathbf{r}}) = -g_{\mathrm{out}}(-\hat{\mathbf{r}})$. For another formal derivation of (5), see Nieto-Vesperinas and Wolf.[12, appendix A] Roman[13, p163] writes “The asymptotic form of $e^{ikz}$ is obtained from

$$\mathrm{e}^{ikz} = \sum_{n=0}^{\infty}(2n+1)\mathrm{i}^n j_n(kr)P_n(\cos\theta) \tag{8}$$

and (6) as”

$$\mathrm{e}^{ikz} \sim \sum_{n=0}^{\infty}(2n+1)\mathrm{i}^n P_n(\cos\theta)\frac{\sin(kr - n\pi/2)}{kr}. \tag{9}$$

Here, $P_n$ is a Legendre polynomial, and (8) is a special case of the first equality in (5).

The derivation of (5) given above is open to two criticisms. The first is that the series (7) is divergent. Nevertheless, it is customary to take its meaning to be such that

$$Y_n^m(\hat{\mathbf{r}}) = \int_\Omega \delta(\hat{\mathbf{s}} - \hat{\mathbf{r}})Y_n^m(\hat{\mathbf{s}})\,\mathrm{d}\Omega(\hat{\mathbf{s}}),$$

where $\Omega$ is the unit sphere, or, more generally,

$$\int_\Omega F(\hat{\mathbf{s}})\delta(\hat{\mathbf{s}} - \hat{\mathbf{r}})\,\mathrm{d}\Omega(\hat{\mathbf{s}}) = F(\hat{\mathbf{r}}), \tag{10}$$

where $F$ is an arbitrary continuous function defined on $\Omega$.

The second criticism stems from the use of (6): that asymptotic expansion holds as $w \to \infty$ for fixed $n$ and so it cannot be substituted into an infinite series with respect to $n$ (without further justification). Indeed, it was this step that led to the divergent series in (7).

Related to this criticism is the peculiar nature of the ubiquitous formula (9); it is found in many books apart from Roman,[13] such as Joachain,[14, eq. 4.58] Sakurai,[15, eq. (7.6.8)] and Gonis and Butler.[16, eq. (3.40)] Often, the $\sim$ is replaced by $\overrightarrow{r \to \infty}$. Although it is difficult to assign a meaning to (9), some of the related calculations could be reworked after first subtracting the incident plane wave.

Now, it turns out that we can bypass the second criticism, but only if we already believe that (1) is valid! Thus, given a regular wavefunction $u$, *suppose* that (1) is true and *suppose* that we can write

$$g_\alpha(\hat{\mathbf{r}}) = \sum_{n,m} g_{n,\alpha}^m Y_n^m(\hat{\mathbf{r}}), \quad \alpha = \text{out, inc}.$$

We know that any regular wavefunction $u$ has an expansion

$$u(\mathbf{r}) = \sum_{n,m} d_n^m j_n(kr)Y_n^m(\hat{\mathbf{r}}).$$

Then, as the spherical harmonics are orthonormal, (1) gives

$$d_n^m j_n(kr) = g_{n,\mathrm{out}}^m\frac{\mathrm{e}^{\mathrm{i}kr}}{\mathrm{i}kr} + g_{n,\mathrm{inc}}^m\frac{\mathrm{e}^{-\mathrm{i}kr}}{\mathrm{i}kr} + o(r^{-1}), \quad r \to \infty.$$

As $n$ is fixed, we can use the asymptotic approximation (6) for large $r$ on the left; comparison of the exponential terms then gives

$$g^m_{n,\mathrm{out}} = \frac{1}{2}(-\mathrm{i})^n d^m_n, \quad g^m_{n,\mathrm{inc}} = -\frac{1}{2}\,\mathrm{i}^n d^m_n, \tag{11}$$

and these determine $g_{\mathrm{out}}$ and $g_{\mathrm{inc}}$. In particular, as the spherical harmonics satisfy[10, eq. 3.8]

$$Y^m_n(-\hat{\mathbf{r}}) = (-1)^n Y^m_n(\hat{\mathbf{r}}), \tag{12}$$

we find that $g_{\mathrm{inc}}(\hat{\mathbf{r}}) = -g_{\mathrm{out}}(-\hat{\mathbf{r}})$. If we apply this method to the plane wave (4), we find $d^m_n = 4\pi\mathrm{i}^n\overline{Y^m_n(\hat{\mathbf{s}})}$, and then we are led back to the divergent series (7).

But brushing aside any anxieties about its derivation, how should the formula (5) be interpreted? The presence of the delta functions implies that $u_{\mathrm{pw}}(\mathbf{r}) \sim 0$ as $r \to \infty$ for $\hat{\mathbf{r}} \neq \pm\hat{\mathbf{s}}$, which is clearly false. In Nieto-Vesperinas and Wolf,[12, footnote 9] the authors write: "A plane homogeneous wave (4) formally has the asymptotic behaviour (5) as $kr \to \infty$ … Hence a plane wave provides both incoming and outgoing contributions at infinity." A similar difficulty arises when trying to interpret (9).[13, eq. 3-57]

Another possibility is to multiply (5) by a smooth "test function" $q(\hat{\mathbf{s}})$ followed by integration over $\Omega$, giving

$$u_q(\mathbf{r}) \equiv \int_\Omega q(\hat{\mathbf{s}})\,\mathrm{e}^{\mathrm{i}k\mathbf{r}\cdot\hat{\mathbf{s}}}\,\mathrm{d}\Omega(\hat{\mathbf{s}}) \simeq 2\pi\, q(\hat{\mathbf{r}})\,\frac{\mathrm{e}^{\mathrm{i}kr}}{\mathrm{i}kr} - 2\pi\, q(-\hat{\mathbf{r}})\,\frac{\mathrm{e}^{-\mathrm{i}kr}}{\mathrm{i}kr}, \tag{13}$$

after using (10). In fact, this formula gives a rigorous asymptotic far-field estimate of $u_q(\mathbf{r})$ (replace $\simeq$ by $=$ with an error that is $o(r^{-1})$ as $r \to \infty$) *provided* $q$ is smooth and bounded. This result is due to Jones[3]; his proof uses the 2-dimensional method of stationary phase and requires that $q$ be twice differentiable. The Jones Lemma is stated by Born and Wolf[17, appendix XII] but without the crucial smoothness requirement on $q$. Indeed, if $q$ is not smooth and bounded, then $u_q$ need not behave as in (13); examples will be given in Section 5. But first, we introduce Herglotz wavefunctions, and then (Section 4) we recall and apply the 2-dimensional method of stationary phase.

## 3 | MATHEMATICS THREAD: HERGLOTZ WAVEFUNCTIONS

Recalling the formula (13), write

$$u(\mathbf{r}) = \int_\Omega q(\hat{\mathbf{s}})\,\mathrm{e}^{\mathrm{i}k\mathbf{r}\cdot\hat{\mathbf{s}}}\,\mathrm{d}\Omega(\hat{\mathbf{s}}). \tag{14}$$

This defines a regular wavefunction $u$ for any integrable $q$ as a superposition of plane waves (4) propagating in all directions $\hat{\mathbf{s}}$. This type of integral representation goes back to Whittaker.[18] It is also called a plane-wave expansion[19, p90 and eq. 6.34] or an angular spectrum representation.[20, eq. 2.30] It has been said (erroneously) that "the plane waves form a complete set into which any incident wave can be expanded" in the form (14)[19, p247]; here, the word "any" is too broad. (Approximation of regular wavefunctions by plane waves is a topic of current research; for example, see Moiola et al.[21])

Instead of trying to write a given wavefunction in the form (14), we focus on properties of $u$ given $q$. Colton and Kress[4, definition 3.18] call $u$ a *Herglotz wavefunction* when $q$ is square integrable, $q \in L^2(\Omega)$. Herglotz was interested in the growth of regular wavefunctions. Specifically, we have the following result[4, theorem 3.30]:

**Theorem 1.** *A regular wavefunction u has the growth property*

$$\sup_{R>0} \frac{1}{R} \int_{B_R} |u(\mathbf{r})|^2\,\mathrm{d}V(\mathbf{r}) < \infty \tag{15}$$

*if and only if it is a Herglotz wavefunction, that is, if and only if there is a function $q \in L^2(\Omega)$ such that u can be represented in the form (14). In (15), $B_R$ is a ball of radius R.*

This theorem is due to Hartman and Wilcox.[22, theorem 4] We note that the growth condition (15) is satisfied if $ru(\mathbf{r})$ is bounded everywhere.

Now, given $q \in L^2(\Omega)$, we can write

$$q(\hat{\mathbf{s}}) = \sum_{n,m} q_n^m Y_n^m(\hat{\mathbf{s}}) \quad \text{with} \quad q_n^m = \int_\Omega q(\hat{\mathbf{s}})\overline{Y_n^m(\hat{\mathbf{s}})}\,\mathrm{d}\Omega(\hat{\mathbf{s}}).$$

Then substituting in (14) gives

$$u(\mathbf{r}) = 4\pi \sum_{n,m} \mathrm{i}^n q_n^m j_n(kr) Y_n^m(\hat{\mathbf{r}}).$$

If we insert the asymptotic approximation (6), "there results formally an approximation for $u(\mathbf{r})$ for large $r$ given by"[23, p251]

$$U(\mathbf{r}) = \frac{2\pi}{\mathrm{i}kr} \sum_{n,m} q_n^m \left\{ \mathrm{e}^{\mathrm{i}kr} - (-1)^n \mathrm{e}^{-\mathrm{i}kr} \right\} Y_n^m(\hat{\mathbf{r}}) = 2\pi\, q(\hat{\mathbf{r}}) \frac{\mathrm{e}^{\mathrm{i}kr}}{\mathrm{i}kr} - 2\pi\, q(-\hat{\mathbf{r}}) \frac{\mathrm{e}^{-\mathrm{i}kr}}{\mathrm{i}kr}, \tag{16}$$

where we have used (12). Note that $U(\mathbf{r})$ is the same as the right-hand side of (13). The plausible "formal" derivation just given might lead one to hope that $u(\mathbf{r}) \sim U(\mathbf{r})$ as $r \to \infty$, but "Herglotz has pointed out (in 1945) that, in general, one does not have $u(\mathbf{r}) - U(\mathbf{r}) = o(1)$ as $r \to \infty$ even under the boundedness condition" (15).[23, p252] In fact, although we may not have $u(r\hat{\mathbf{r}}) - U(r\hat{\mathbf{r}}) = o(1)$ as $r \to \infty$ for each choice of direction $\hat{\mathbf{r}}$, a weaker result is available[23, eq. 26;22, eq. 2.5]:

$$\frac{1}{R} \int_{B_R} |u(\mathbf{r}) - U(\mathbf{r})|^2 \,\mathrm{d}V(\mathbf{r}) \to 0 \quad \text{as } R \to \infty. \tag{17}$$

The fact that the density function $q(\hat{\mathbf{s}})$ appearing in (14) also appears as $q(\pm\hat{\mathbf{r}})$ in (16) is a known property of Herglotz wavefunctions; see Hartman and Wilcox.[22, lemma 4.1 and theorem 4]

## 4 | TWO-DIMENSIONAL METHOD OF STATIONARY PHASE

One way to find the far-field behaviour of regular wavefunctions defined by (14) is to use the method of stationary phase (MSP). Start by introducing spherical polar coordinates so that

$$\hat{\mathbf{r}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \quad \hat{\mathbf{s}} = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha).$$

Then, with $q(\hat{\mathbf{s}}) = q(\alpha, \beta)$, (14) becomes

$$u(\mathbf{r}) = u(r, \theta, \phi) = \int_0^\pi \int_{-\pi}^\pi q(\alpha, \beta)\, \mathrm{e}^{\mathrm{i}kr\,\Phi(\alpha,\beta)} \sin\alpha \,\mathrm{d}\beta\,\mathrm{d}\alpha, \tag{18}$$

where $\Phi(\alpha, \beta) = \sin\theta \sin\alpha \cos(\beta - \phi) + \cos\theta\cos\alpha$.

Let us begin with a special case, $\theta = 0$. Then $\Phi = \cos\alpha$ and

$$u(r, 0, \phi) = \int_0^\pi q_0(\alpha)\, \mathrm{e}^{\mathrm{i}kr\cos\alpha} \sin\alpha \,\mathrm{d}\alpha = \frac{\mathrm{i}}{kr} \int_0^\pi q_0(\alpha) \frac{\mathrm{d}}{\mathrm{d}\alpha} \mathrm{e}^{\mathrm{i}kr\cos\alpha} \,\mathrm{d}\alpha,$$

where $q_0(\alpha) = \int_{-\pi}^{\pi} q(\alpha, \beta)\,\mathrm{d}\beta$.

Assuming that $q_0$ is differentiable, we can integrate by parts to give

$$u(r, 0, \phi) = 2\pi\, q(\hat{\mathbf{z}}) \frac{\mathrm{e}^{\mathrm{i}kr}}{\mathrm{i}kr} - 2\pi\, q(-\hat{\mathbf{z}}) \frac{\mathrm{e}^{-\mathrm{i}kr}}{\mathrm{i}kr} + \frac{1}{\mathrm{i}kr} \int_0^\pi q_0'(\alpha)\, \mathrm{e}^{\mathrm{i}kr\cos\alpha} \,\mathrm{d}\alpha \tag{19}$$

using $q_0(0) = 2\pi q(\hat{\mathbf{z}})$ and $q_0(\pi) = 2\pi q(-\hat{\mathbf{z}})$, where $\hat{\mathbf{z}}$ is a unit vector along the $z$-axis ($\theta = 0$). For the remaining integral, we can use the 1-dimensional MSP. This shows that the dominant contributions come from the stationary-phase (SP) points, which are at $\alpha = 0$ and $\alpha = \pi$. If $q_0'(0)$ and $q_0'(\pi)$ are finite, the integral is $O(r^{-1/2})$ as $r \to \infty$, and so the last term in (19) is asymptotically negligible. Hence,

$$u(r, 0, \phi) \sim 2\pi\, q(\hat{\mathbf{z}}) \frac{\mathrm{e}^{\mathrm{i}kr}}{\mathrm{i}kr} - 2\pi\, q(-\hat{\mathbf{z}}) \frac{\mathrm{e}^{-\mathrm{i}kr}}{\mathrm{i}kr} \quad \text{as } r \to \infty. \tag{20}$$

Similarly, when $\theta = \pi$,

$$u(r, \pi, \phi) \sim 2\pi\, q(-\hat{\mathbf{z}})\, \frac{\mathrm{e}^{\mathrm{i}kr}}{\mathrm{i}kr} - 2\pi\, q(\hat{\mathbf{z}})\, \frac{\mathrm{e}^{-\mathrm{i}kr}}{\mathrm{i}kr} \quad \text{as } r \to \infty. \tag{21}$$

Having disposed of the cases $\theta = 0$ and $\theta = \pi$, suppose now that $0 < \theta < \pi$. We use the 2-dimensional MSP.[5, section 8.4] The SP points are given by solving

$$\frac{\partial \Phi}{\partial \alpha} = \sin\theta \cos\alpha \cos(\beta - \phi) - \cos\theta \sin\alpha = 0, \tag{22}$$

$$\frac{\partial \Phi}{\partial \beta} = -\sin\theta \sin\alpha \sin(\beta - \phi) = 0. \tag{23}$$

Equation 23 gives $\sin\alpha = 0$ or $\sin(\beta - \phi) = 0$; of these options, the first is irrelevant because the integrand in (18) vanishes when $\sin\alpha = 0$. Hence, $\beta = \phi$ or $\beta = \phi \pm \pi$ (whichever satisfies $|\beta| < \pi$). Then, from (22), we obtain $\alpha = \theta$ when $\beta = \phi$ and $\alpha = \pi - \theta$ when $\beta = \phi \pm \pi$. In other words, there is one SP point at $\hat{\mathbf{s}} = \hat{\mathbf{r}}$ and one at $\hat{\mathbf{s}} = -\hat{\mathbf{r}}$.

If $(\alpha_0, \beta_0)$ is an (isolated) SP point, it contributes

$$\frac{2\pi}{kr}\, q(\alpha_0, \beta_0) \frac{\sin\alpha_0}{\sqrt{|\det A|}} \exp\left\{\mathrm{i}kr\Phi(\alpha_0, \beta_0) + \mathrm{i}(\pi/4)\,\mathrm{sig}A\right\} \tag{24}$$

to the asymptotic behaviour of $u(\mathbf{r})$, (18), as $r \to \infty$.[5, eq. 8.4.44] Here, the $2\times 2$ matrix $A$ has entries $A_{ij} = \partial^2\Phi/\partial\xi_i\partial\xi_j$ evaluated at $(\alpha_0, \beta_0)$, $\xi_1 = \alpha$, $\xi_2 = \beta$ and sig $A$ is the signature of $A$, equal to the number of positive eigenvalues minus the number of negative eigenvalues. Some calculation gives

$$A(\hat{\mathbf{s}}) = A(\alpha_0, \beta_0) = \begin{pmatrix} -\sin\theta \sin\alpha_0 \cos(\beta_0 - \phi) - \cos\theta\cos\alpha_0 & -\sin\theta\cos\alpha_0\sin(\beta_0 - \phi) \\ -\sin\theta\cos\alpha_0\sin(\beta_0 - \phi) & -\sin\theta\sin\alpha_0\cos(\beta_0 - \phi) \end{pmatrix},$$

$$A(\hat{\mathbf{r}}) = \begin{pmatrix} -1 & 0 \\ 0 & -\sin^2\theta \end{pmatrix} = -A(-\hat{\mathbf{r}}), \quad \mathrm{sig}A(\hat{\mathbf{r}}) = -2, \quad \mathrm{sig}A(-\hat{\mathbf{r}}) = 2.$$

Summing the contributions from both SP points, noting that $\Phi(\pm\hat{\mathbf{r}}) = \pm 1$, we obtain

$$u(\mathbf{r}) \sim 2\pi\, q(\hat{\mathbf{r}})\, \frac{\mathrm{e}^{\mathrm{i}kr}}{\mathrm{i}kr} - 2\pi\, q(-\hat{\mathbf{r}})\, \frac{\mathrm{e}^{-\mathrm{i}kr}}{\mathrm{i}kr} \quad \text{as } r \to \infty. \tag{25}$$

We note that (25) agrees with the results for $\theta = 0 (\hat{\mathbf{r}} = \hat{\mathbf{z}})$ and for $\theta = \pi (\hat{\mathbf{r}} = -\hat{\mathbf{z}})$, (20) and (21), respectively. In other words, (25) holds in all directions $\hat{\mathbf{r}}$. This result is known as the *Jones Lemma*,[3] for smooth functions $q$. It shows that Herglotz wavefunctions with smooth density functions $q$ do satisfy (1), with $g_{\mathrm{out}}(\hat{\mathbf{r}}) = 2\pi\, q(\hat{\mathbf{r}})$ and $g_{\mathrm{inc}}(\hat{\mathbf{r}}) = -2\pi\, q(-\hat{\mathbf{r}})$.

## 4.1 | Discussion

The contribution from the SP point, (24), assumes that $q(\alpha, \beta)$ is smooth in a neighbourhood of $(\alpha_0, \beta_0)$: It arises from a *local* analysis near the SP point. This contrasts with the analysis leading to (20) and (21), where a 1-dimensional integral over $0 < \alpha < \pi$ was integrated by parts. Indeed, suppose we had tried to use the 2-dimensional MSP when $\theta = 0$; then (22) gives $\alpha = 0$ and $\alpha = \pi$, whereas (23) is satisfied identically: We have 2 *lines* of stationary phase.

One could argue that this difficulty is spurious: For we could have chosen spherical polar coordinates on $\Omega$, $(\alpha', \beta')$ with polar axis aligned with $\mathbf{r}$ so that $\alpha' = 0$ is in the direction of $\hat{\mathbf{r}}$ and $\hat{\mathbf{r}} \cdot \hat{\mathbf{s}} = \cos\alpha'$. Then, writing $q(\hat{\mathbf{s}}) = q(\alpha', \beta'; \hat{\mathbf{r}})$ and defining $\tilde{q}_0(\alpha') = \int_{-\pi}^{\pi} q(\alpha', \beta'; \hat{\mathbf{r}})\,\mathrm{d}\beta'$, (14) becomes

$$u(\mathbf{r}) = \int_0^{\pi} \tilde{q}_0(\alpha')\, \mathrm{e}^{\mathrm{i}kr\cos\alpha'} \sin\alpha'\, \mathrm{d}\alpha'.$$

Hence, if $\tilde{q}_0(\alpha')$ is sufficiently smooth, we can integrate by parts as we did above on the way to (20); the result is (25) again.

The argument just used, in which we chose spherical polar coordinate with respect to $\hat{\mathbf{r}}$, is not very convenient if the function $q$ is not smooth. For example, suppose that $q(\alpha, \beta)$ is not smooth across the curve $\alpha = \pi/2$, the equator around the unit sphere $\Omega$. Then, because the 2-dimensional MSP uses local arguments, we obtain exactly the estimate (25) for all directions $\hat{\mathbf{r}}$ that do not pass through the equator $(\alpha = \pi/2)$; the directions $\hat{\mathbf{r}} = \pm\hat{\mathbf{z}} (\alpha = 0, \pi)$ may also require a special treatment. We explore these potential deviations from (25) in Section 5 by studying two specific choices for $q$.

## 5 | TWO EXAMPLES

We investigate two examples of (18) in which $q$ is not smooth. For simplicity, we suppose that $q$ is axisymmetric, $q(\hat{\mathbf{s}}) = q(\alpha)$. It follows that $u$ is axisymmetric, so that (18) becomes

$$u(\mathbf{r}) = u(r,\theta) = \int_0^\pi \int_{-\pi}^\pi q(\alpha)\mathrm{e}^{\mathrm{i}kr\,\Phi(\alpha,\beta)} \sin\alpha\,\mathrm{d}\beta\,\mathrm{d}\alpha, \tag{26}$$

where $\Phi(\alpha,\beta) = \sin\theta\sin\alpha\cos\beta + \cos\theta\cos\alpha$. In particular, at the origin

$$u(0,\theta) = 2\pi \int_0^\pi q(\alpha)\sin\alpha\,\mathrm{d}\alpha. \tag{27}$$

In both examples, $q(\alpha)$ is smooth except at the equator ($\alpha = \pi/2$). Hence, (25) gives the far field for $0 < \theta < \pi/2$ and for $\pi/2 < \theta < \pi$.

In our first example (Section 5.1), $q$ is piecewise constant, with $q = 1$ on one half of $\Omega$ and $q = 0$ on the other half. In our second example (Section 5.2), $q$ is singular around the equator.

### 5.1 | Piecewise-constant $q$

We consider a piecewise-constant $q \in L^2(\Omega)$, with $q(\alpha) = 1$ for $0 < \alpha < \pi/2$ and $q = 0$ for $\pi/2 < \alpha < \pi$. Equation 26 reduces to

$$u(r,\theta) = \int_0^{\pi/2} \int_{-\pi}^\pi \mathrm{e}^{\mathrm{i}kr\,\Phi(\alpha,\beta)} \sin\alpha\,\mathrm{d}\beta\,\mathrm{d}\alpha. \tag{28}$$

The Jones Lemma (25) gives

$$u(r,\theta) \sim \frac{2\pi}{\mathrm{i}kr}\mathrm{e}^{\mathrm{i}kr} \quad \text{as } r \to \infty \quad \text{for } 0 < \theta < \pi/2, \tag{29}$$

$$u(r,\theta) \sim -\frac{2\pi}{\mathrm{i}kr}\mathrm{e}^{-\mathrm{i}kr} \quad \text{as } r \to \infty \quad \text{for } \pi/2 < \theta < \pi. \tag{30}$$

These results may have been anticipated: Plane waves are sent out from a hemisphere in all directions, so we expect to see an outgoing spherical wave on one side (29) and an incoming spherical wave on the other side (30).

To see what happens when $\theta = \pi/2$, we recall that the SP points are at $\hat{\mathbf{s}} = \pm\hat{\mathbf{r}}$ (see below (23), but the integration is over $0 < \alpha < \pi/2$. So when $0 < \theta < \pi/2$, $q(-\hat{\mathbf{r}}) = 0$ and the SP point at $\hat{\mathbf{s}} = \hat{\mathbf{r}}$ contributes, whereas when $\pi/2 < \theta < \pi$, $q(\hat{\mathbf{r}}) = 0$ and the SP point at $\hat{\mathbf{s}} = -\hat{\mathbf{r}}$ contributes. When $\theta = \pi/2$, both SP points contribute, but their contributions are halved[5, eq. 8.4.46] because they are on the boundary of the domain integration. Hence, for $\theta = \pi/2$, we obtain the average of (29) and (30). In fact, $u(r,\pi/2)$ can be found exactly:

$$u(r,\pi/2) = \int_0^{\pi/2} \int_{-\pi}^\pi \mathrm{e}^{\mathrm{i}kr\sin\alpha\sin\beta} \sin\alpha\,\mathrm{d}\alpha = 2\pi \int_0^{\pi/2} J_0(kr\sin\alpha)\sin\alpha\,\mathrm{d}\alpha = 2\pi\,\frac{\sin kr}{kr},$$

where we have used the *Table of Integrals, Series, and Products*.[24, 6.683.8] This result is precisely the average of (29) and (30).

Finally, on the axis, we have

$$u(r,0) = 2\pi \int_0^{\pi/2} \mathrm{e}^{\mathrm{i}kr\cos\alpha} \sin\alpha\,\mathrm{d}\alpha = \frac{2\pi}{\mathrm{i}kr}\left(\mathrm{e}^{\mathrm{i}kr} - 1\right),$$

$$u(r,\pi) = 2\pi \int_0^{\pi/2} \mathrm{e}^{-\mathrm{i}kr\cos\alpha} \sin\alpha\,\mathrm{d}\alpha = \frac{2\pi}{\mathrm{i}kr}\left(1 - \mathrm{e}^{-\mathrm{i}kr}\right).$$

These exact results do not agree with (25) because the terms $\pm 2\pi/(ikr)$ are missed. They do give the correct result at the origin, $2\pi$, in agreement with (27). The deviation from (20) can be understood readily: In arriving at (19), we integrated by parts, giving 2 endpoint contributions, and here, those endpoints are at $\alpha = 0$ and $\pi/2$, not $\alpha = 0$ and $\pi$.

We conclude that, for the particular discontinuous choice of $q$, the estimate (25) is valid except along the $z$-axis ($\theta = 0, \pi$).

## 5.2 | Singular *q*

Suppose $q(\alpha) = |\cos\alpha|^{\mu-1}$ with $0 < \mu < 1$: $q$ has an integrable singularity at the equator of $\Omega$, $\alpha = \pi/2$. Also, $q \in L^2(\Omega)$ when $\mu > \frac{1}{2}$. The corresponding wavefunction is given by (26); as $u(r,\theta) = u(r, \pi-\theta)$, we can suppose that $0 \le \theta \le \pi/2$ without loss of generality.

We consider the cases $\theta = 0$, $\theta = \pi/2$, and $0 < \theta < \pi/2$, separately. When $\theta = 0$, (26) gives

$$\begin{aligned} u(r,0) &= 2\pi \int_0^{\pi} \mathrm{e}^{ikr\cos\alpha} |\cos\alpha|^{\mu-1} \sin\alpha \,\mathrm{d}\alpha = 2\pi \int_{-1}^{1} |t|^{\mu-1} \mathrm{e}^{ikrt}\,\mathrm{d}t \\ &= 4\pi \int_0^1 t^{\mu-1} \cos(krt)\,\mathrm{d}t = \frac{4\pi}{(kr)^{\mu}} \int_0^{kr} x^{\mu-1} \cos x \,\mathrm{d}x \\ &\sim \frac{4\pi}{(kr)^{\mu}} \int_0^{\infty} x^{\mu-1} \cos x\,\mathrm{d}x = \frac{4\pi}{(kr)^{\mu}} \Gamma(\mu) \cos\frac{\mu\pi}{2}, \qquad r \to \infty, \end{aligned} \tag{31}$$

where $0 < \mu < 1$, $\Gamma$ is the gamma function and we have used the *Table of Integrals, Series, and Products*.[24, 3.761.9] Note how $u(r,0)$ decays with $r$. Note also that the condition on $\mu$ was used in the asymptotic step above, $\int_0^{kr} \sim \int_0^{\infty}$; if $\mu > 1$ (making $q$ non-singular), we could integrate by parts leading to a result in agreement with (20).

On the equatorial plane, where $\theta = \pi/2$, we can calculate $u$ exactly. Thus

$$\begin{aligned} u(r,\pi/2) &= \int_0^{\pi} \int_{-\pi}^{\pi} \mathrm{e}^{ikr\sin\alpha\sin\beta} |\cos\alpha|^{\mu-1} \sin\alpha\,\mathrm{d}\beta\,\mathrm{d}\alpha \\ &= 2\pi \int_0^{\pi} J_0(kr\sin\alpha) |\cos\alpha|^{\mu-1} \sin\alpha\,\mathrm{d}\alpha = 4\pi \int_0^{\pi/2} J_0(kr\sin\alpha)(\cos\alpha)^{\mu-1} \sin\alpha\,\mathrm{d}\alpha \\ &= 4\pi \int_0^1 J_0(krt)(1-t^2)^{\mu/2-1} t\,\mathrm{d}t = 2\pi\Gamma(\mu/2) \frac{J_{\mu/2}(kr)}{(kr/2)^{\mu/2}} \end{aligned} \tag{32}$$

using the *Table of Integrals, Series, and Products*.[24, 6.567.1] This shows decay as $r^{-(\mu+1)/2}$ with $r$. We note that (32) gives the correct result at $r = 0$; from (27), this value is $4\pi/\mu$.

Suppose next that $0 < \theta < \pi$. It is then natural to partition $\Omega$ into 2 hemispheres, $\Omega_+ (0 \le \alpha < \pi/2)$ and $\Omega_- (\pi/2 < \alpha \le \pi)$, followed by an application of the 2-dimensional MSP to the integrals over $\Omega_\pm$. In standard applications, the boundary of the domain of integration makes a smaller contribution than that coming from interior SP points,[5, p. 349] but this statement may require modification when the integrand is singular at the boundary.

With this in mind, we partition $\Omega$ into 2 spherical caps, $0 \le \alpha < \pi/2 - \delta$ and $\pi/2 + \delta < \alpha \le \pi$, and a narrow band around the equator, $\pi/2 - \delta < \alpha < \pi/2 + \delta$, where $0 < \delta \ll 1$. Suppose that $\theta$ is not near $\pi/2$, so that $\alpha = \theta$ is not in the band.

The 2-dimensional MSP shows that the spherical caps contribute terms of the expected form, $\mathrm{e}^{\pm ikr}/(\mathrm{ikr})$; see (24).

From the band, we have

$$\begin{aligned} I &\equiv \int_{\pi/2-\delta}^{\pi/2+\delta} \int_{-\pi}^{\pi} \mathrm{e}^{ikr\Phi(\alpha,\beta)} |\cos\alpha|^{\mu-1} \sin\alpha\,\mathrm{d}\beta\,\mathrm{d}\alpha \\ &= \int_{-\delta}^{\delta} \int_{-\pi}^{\pi} \mathrm{e}^{-ikr\cos\theta\sin\alpha'} \mathrm{e}^{ikr\sin\theta\sin\beta\cos\alpha'} |\sin\alpha'|^{\mu-1} \cos\alpha'\,\mathrm{d}\beta\,\mathrm{d}\alpha'. \end{aligned}$$

In the second exponent, use the approximation $\cos\alpha' \simeq 1$. Then the integral with respect to $\beta$ evaluates to $2\pi J_0(kr\sin\theta)$, whence, following the derivation of (31),

$$\begin{aligned} I &\simeq 2\pi J_0(kr\sin\theta)\int_{-\delta}^{\delta} \mathrm{e}^{-\mathrm{i}kr\cos\theta\sin\alpha'}|\sin\alpha'|^{\mu-1}\cos\alpha'\,\mathrm{d}\alpha' \\ &= 4\pi J_0(kr\sin\theta)\int_0^{\sin\delta} t^{\mu-1}\cos(krt\cos\theta)\,\mathrm{d}t = 4\pi\frac{J_0(kr\sin\theta)}{(kr\cos\theta)^{\mu}}\int_0^{kr\cos\theta\sin\delta} x^{\mu-1}\cos x\,\mathrm{d}x \\ &\sim 4\pi\frac{J_0(kr\sin\theta)}{(kr\cos\theta)^{\mu}}\Gamma(\mu)\cos\frac{\mu\pi}{2}, \quad r\to\infty, \quad 0<\mu<1. \end{aligned} \tag{33}$$

This estimate holds for $0<\theta<\pi$ with $\theta$ bounded away from $\pi/2$. Evidently, the estimate is not uniform in $\theta$ as it fails at $\theta=\pi/2$ (and recall that we have the exact result at $\theta=\pi/2$, (32). It does agree with (31) when $\theta=0$.

Approximating $J_0$ in (33) for large arguments,[9, 10.7.8] we see decay as $r^{-\mu-1/2}$. We conclude that when $\frac{1}{2}<\mu<1$, the dominant contributions come from the spherical caps (via the points of stationary phase); recall that for these values of $\mu$, $q\in L^2(\Omega)$. For smaller values of $\mu$, $0<\mu<\frac{1}{2}$, the dominant contribution to the far field comes from the equatorial singularity at $\alpha=\pi/2$.

## 6 | CONCLUSION

We have seen that regular wavefunctions do not always behave as (1). When attention is restricted to Herglotz wavefunctions (14) with $q\in L^2(\Omega)$, it appears that (1) is typical, although it may be violated in some directions $\hat{\mathbf{r}}$. When $q$ is smooth (twice differentiable), the behaviour given in (1) is correct (Jones Lemma). When $q$ is integrable with $q\notin L^2(\Omega)$, the example in Section 5.2 shows that the far-field behaviour can be more complicated. Thus, plane-wave representations do not always behave as (1) in the far field, despite claims to the contrary.[20, p154]

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